Equivalence of F with a Sub-Theory of Peano Arithmetic

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The system called **F** is essentially a sub-theory of Frege Arithmetic without the *ad infinitum* assumption that there is always a next number. In a series of papers (*Systems for a Foundation of Arithmetic*, *True" Arithmetic Can Prove Its Own Consistency* and *Proving Quadratic Reciprocity*) it was shown that **F** proves a large number of basic arithmetic truths, such as the Euclidean Algorithm, Unique Prime Factorization (i.e. the Fundamental Law of Arithmetic), and Quadratic Reciprocity, indeed a sizable amount of arithmetic. In particular, **F** proves some (but not all) of the Peano Axioms; that is, **F** proves the axioms of a sub-theory - call it **FPA** - of second-order Peano-Arithmetic. This short technical note will demonstrate that the converse also holds, in the following sense. **F** has the same language as second-order Peano Arithmetic except that, in addition, it has a two-place predicate symbol "M". Then it is possible to provide a definition, indeed a reasonable definition, for "M" such that **FPA** proves all the axioms of **F**. So **F** and **FPA** effectively have the same proof-theoretic strength. In particular **FPA**, which lacks the Successor Axiom stating that every natural number has a successor, is able to prove the Euclidean Algorithm, Unique Prime Factorization, and Quadratic Reciprocity, indeed (again) a sizable amount of arithmetic.

Abbreviations

$x \in P$ for Px
Dom(R) for $\{x : \exists y\ Rx,y\}$
Im(R) for $\{y : \exists x\ Rx,y\}$
IsFunction(R) for $\forall x \forall y \forall z\ (Rx,y\ \&\ Rx,z \Rightarrow y = z)$
Is1-1(R) for $\forall x \forall y \forall z\ (Rx,y\ \&\ Rz,y \Rightarrow x = z)$
$P \equiv Q$ for $\forall x\ (Px \Leftrightarrow Qx)$
$P \subseteq Q$ for $\forall x\ (Px \Rightarrow Qx)$
$P \cup Q$ for $\{z : Pz \vee Qz\}$ or $\{y,z : Py,z \vee Qy,z\}$ or $\{x,y,z : Px,y,z \vee Qx,y,z\}$
$P \setminus Q$ for $\{z : Pz\ \&\ \neg Qz\}$
$P \sim Q$ for $\exists R$ (*IsFunction*(R) & *Is1-1*(R) & *Dom*(R) $\equiv P$ & *Im*(R) $\equiv Q$)
$\{a\}$ for $\{z : z = a\}$
ϕ for $\{z : \neg z = z\}$

Introduction

We will be working in predicative second-order logic. That is, comprehension is restricted to predicative (arithmetic) predicates.

The language of second-order Peano Arithmetic consists of a first-order constant 0, a second-order one-place predicate N (representing the natural numbers), and a second-order two-place predicate σ (representing succession). The theory of Peano Arithmetic contains these axioms:

(PA1) $\mathrm{N}0$
(PA2) $\forall n \forall m \ (\mathrm{N}n \ \& \ \sigma n,m \Rightarrow \mathrm{N}m)$
(PA3) $\forall n \ (\mathrm{N}n \Rightarrow \exists m \ \sigma n,m)$
(PA4) $\forall n \forall m \forall m' \ (\mathrm{N}n \ \& \ \sigma n,m \ \& \ \sigma n,m' \Rightarrow m = m')$
(PA5) $\forall n \forall m \forall n' \ (\mathrm{N}n \ \& \ \mathrm{N}n' \ \& \ \sigma n,m \ \& \ \sigma n',m \Rightarrow n = n')$
(PA6) $\forall n \ (\mathrm{N}n \Rightarrow \neg \ \sigma n,0)$
(PA7) Induction schema. Let ϕ be a well-formed formula (with no appearance of m). Use $\phi[x\backslash y]$ to mean x replaces all (free) instances of y. Suppose $\phi[0\backslash n]$ and $\forall n \forall m \ (\mathrm{N}n \ \& \ \sigma n,m \ \& \ \phi \Rightarrow \phi[m\backslash n])$. Then $\forall n \ (\mathrm{N}n \Rightarrow \phi)$.

Let **FPA** be the sub-theory of Peano Arithmetic which has as axioms (PA4), (PA5), (PA6) and (PA7). Remark that this list does not include (PA3), the Successor Axiom which states that every natural number has a successor.

F uses the same language as Peano Arithmetic, except that in addition it has a two-place predicate symbol M, which takes a first-order symbol in its first argument, and a second-order symbol in its second. “$\mathrm{M}n,P$” can be taken to mean “P is n in number;” i.e. it has the same meaning as Frege Arithmetic’s “$\#P = n$” except that it does not make the totality and functional assumptions which the “#” symbolism pre-supposes. In fact **F** will assume “M” to be functional but not assume totality. **F** contains these axioms:

(F1) $\forall n \forall m \forall P \ (\mathrm{M}n,P \ \& \ \mathrm{M}m,P \Rightarrow n = m)$
(F2) $\forall P \ (\mathrm{M}0,P \Leftrightarrow \neg \ \exists x \ Px)$
(F3) $\forall n \forall m \forall P \forall Q \forall a \ (\mathrm{N}n \ \& \ \sigma n,m \ \& \ \neg Pa \ \& \ \forall x (Qx \Leftrightarrow Px \vee x = a)$
$\Rightarrow (\mathrm{M}n,P \Leftrightarrow \mathrm{M}m,Q))$
(F4) Induction schema. Let ϕ be a well-formed formula (with no appearance of m). Use $\phi[x\backslash y]$ to mean x replaces all (free) instances of y. Suppose $\phi[0\backslash n]$ and $\forall n \forall m \ (\mathrm{N}n \ \& \ \sigma n,m \ \& \ \phi \Rightarrow \phi[m\backslash n])$. Then $\forall n \ (\mathrm{N}n \Rightarrow \phi)$.

It is shown in *Systems for a Foundation of Arithmetic* that **F** proves the axioms (PA4), (PA5), (PA6), and of course all instances of the schema (PA7). I.e. **F** is proof-theoretically stronger than **FPA**. Here we will show the contrary, in the sense that it is possible to provide a definition of "M" such that **FPA** can prove all the axioms of **F**. Hence **FPA** is proof-theoretically stronger than **F**. And thus **FPA** and **F** are proof-theoretically equivalent.

We work in **FPA** and proceed as follows: first, we define addition; then we use addition to define the usual ordering on the natural numbers; next, with this ordering, we define M; and finally we prove the axioms of **F**.

Definitions and Proofs

This is the technical section, where definitions and proofs are spelled out.

Prop 1. $\forall n\,(\mathrm{N}n \Rightarrow \mathrm{N}0)$
Pf:
By induction, with ϕ as $(\mathrm{N}n \Rightarrow \mathrm{N}0)$.

Prop 2. $\forall n\,(\mathrm{N}n \,\&\, \neg\, n = 0 \Rightarrow \exists k\,(\mathrm{N}k \,\&\, \sigma k,n)\,)$, i.e. every non-zero natural number is preceded by a natural number.
Pf:
By induction, with ϕ as
$(\neg\, n = 0 \Rightarrow \exists k\,(\mathrm{N}k \,\&\, \sigma k,n)\,)$.

Prop 3. $\forall n\,(\mathrm{N}n \Rightarrow \neg\, \sigma n,n\,)$
Pf:
By induction, with ϕ as $(\mathrm{N}n \Rightarrow \neg\, \sigma n,n)$.
$\mathrm{N}0 \Rightarrow \neg\, \sigma 0,0$ by (PA6).
Now assume $\mathrm{N}n \,\&\, \sigma n,m \,\&\, \neg\, \sigma n,n$. Suppose to the contrary $\mathrm{N}m \,\&\, \sigma m,m$. Then by (PA5), $m = n$, contradicting the induction hypothesis.

Def 4. Suppose $\exists j\,(\mathrm{N}j \,\&\, \sigma 0,j)$. Then, by (PA4), this j is unique. Use "1" to refer to this number, should it exist.

Corollary 5. Suppose $\exists j\,(\mathrm{N}j \,\&\, \sigma 0,j)$. Then $\neg\, 0 = 1$.

Prop 6. $\forall n \forall k\,(\,\mathrm{N}n \,\&\, \mathrm{N}k \,\&\, \sigma n,k \Rightarrow \exists j\,(\mathrm{N}j \,\&\, \sigma 0,j)\,)$.
Pf:

By induction, with ϕ as
$\forall k\ (Nn \,\&\, Nk \,\&\, \sigma n,k \Rightarrow \exists j\ (Nj \,\&\, \sigma 0,j))$.

Evidently, $\forall k\ (N0 \,\&\, Nk \,\&\, \sigma 0,k \Rightarrow \exists j\ (Nj \,\&\, \sigma 0,j))$.

Assume $Nn \,\&\, \sigma n,m \,\&\, \forall k\ (Nn \,\&\, Nk \,\&\, \sigma n,k \Rightarrow \exists j\ (Nj \,\&\, \sigma 0,j))$. Suppose $Nm \,\&\, Nk \,\&\, \sigma m,k$. Then $Nn \,\&\, Nm \,\&\, \sigma n,m$, so by the induction hypothesis, $\exists j\ (Nj \,\&\, \sigma 0,j)$.

Following John Burgess (*Fixing Frege*) following Dedekind, define:

Def 7. $+(a,b,c)$ if and only if
$Na \,\&\, Nb \,\&\, Nc \,\&$
$\exists R\ (Ra,0,a \,\&\, Ra,b,c \,\&$
$\forall v\ (Ra,0,v \Rightarrow a = v) \,\&$
$\forall u \forall u' \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u',w \Rightarrow \exists v\ (Nv \,\&\, Ra,u,v)) \,\&$
$\forall u \forall u' \forall v \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u,v \,\&\, Ra,u',w \Rightarrow \sigma v,w))$

Prop 8. $\forall a \forall c\ (+(a,0,c) \Rightarrow a = c)$
Pf:

Suppose $+(a,0,c)$. Then $Ra,0,a \,\&\, Ra,0,c \,\&\, \forall v\ (Ra,0,v \Rightarrow a = v))$, for some R. But then $a = c$.

Prop 9. $\forall a\ (Na \Rightarrow +(a,0,a))$
Pf:

Let $R = \{x,y,z : Nx \,\&\, x = z \,\&\, y = 0\}$. Let Na. By *Prop* 1, $N0$. Then $Na \,\&\, N0 \,\&\, Na \,\&\, Ra,0,a \,\&\, Ra,0,a \,\&\, \forall v\ (Ra,0,v \Rightarrow a = v)$. The last two conditions on R hold trivially by (PA6). Hence $+(a,0,a)$.

Prop 10. $\forall a \forall b \forall c \forall d\ (+(a,b,c) \,\&\, +(a,b,d) \Rightarrow c = d)$.
Pf:

By induction, with ϕ as
$\forall b \forall c \forall d\ (+(a,n,c) \,\&\, +(a,n,d) \Rightarrow c = d)$.

Suppose $+(a,0,c) \,\&\, +(a,0,d)$. By *Prop* 8, $a = c$ and $a = d$, hence $c = d$.

Now assume $Nn \,\&\, \sigma n,m \,\&\, \phi$. And suppose $+(a,m,c) \,\&\, +(a,m,d)$. Then

$Na \,\&\, Nm \,\&\, Nc \,\&$
$Ra,0,a \,\&\, Ra,m,c \,\&$
$\forall v\ (Ra,0,v \Rightarrow a = v) \,\&$
$\forall u \forall u' \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u',w \Rightarrow \exists v\ (Nv \,\&\, Ra,u,v)) \,\&$
$\forall u \forall u' \forall v \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u,v \,\&\, Ra,u',w \Rightarrow \sigma v,w))$,

for some R, and

$Na \,\&\, Nm \,\&\, Nc \,\&$
$Sa,0,a \,\&\, Sa,m,d \,\&$

∀v (Sa,0,v ⇒ a = v) &
∀u∀u’∀w (Nu & σu,u’ & Sa,u’,w ⇒ ∃v (Nv & Sa,u,v)) &
∀u∀u’∀v∀w (Nu & σu,u’ & Sa,u,v & Sa,u’,w ⇒ σv,w)),

for some S.

Now Nn & σn,m & Ra,m,c. Hence Nv & Ra,n,v, for some v. So +(a,n,v). Indeed, since Nn & σn,m & Ra,n,v & Ra,m,c, we can conclude that σv,c. Similar reasoning yields +(a,n,y), where σy,d. By the induction hypothesis, v = y. By (PA4) c = d.

Prop 11. Let +(a,b,c) & Nb’ & Nc’ & σb,b’ & & σc,c’. Then +(a,b’,c’).
Pf:

By definition of +,

Na & Nb & Nc &
Ra,0,a & Ra,b,c &
∀v (Ra,0,v ⇒ a = v) &
∀u∀u’∀w (Nu & σu,u’ & Ra,u’,w ⇒ ∃v (Nv & Ra,u,v)) &
∀u∀u’∀v∀w (Nu & σu,u’ & Ra,u,v & Ra,u’,w ⇒ σv,w)),

for some R.

Define R’ to be R ∪ {x,y,z : x = a & y = b’ & z = c’}.

Clearly R’a,0,a.

Clearly R’a,b’,c’.

Suppose R’a,0,v. By (PA6) b’ cannot be 0, so Ra,0,v. Hence a = v.

Suppose Nu & σu,u’ & R’a,u’,w. If u’ = b’ & w = c’, then by (PA5), u = b; but Nc & Ra,b,c, so ∃v (Nv & Ra,u,v)). Otherwise, Ra,u’,w, so ∃v (Nv & Ra,u,v)) and thus ∃v (Nv & R’a,u,v)).

Finally, suppose Nu & σu,u’ & R’a,u,v & R’a,u’,w.

Case 1. u’ = b’ & w = c’.

Then u = b by (PA5), so σu,b’. By *Prop* 3, ¬ u = b’. This forces Ra,u,v, i.e. Ra,b,v. So Ra,b,c & Ra,b,v. Hence +(a,b,c) & +(a,b,v). By *Prop* 10, c = v. From σc,c’ & w = c’, we conclude that σv,w. End 1

Case 2. ¬ (u’ = b’ & w = c’).

This forces Ra,u’,w. If Ra,u,v, then σv,w.

Otherwise, u = b’ & v = c’. Now, Nu & σu,u’ & Ra,u’,w. Thus Ny & Ra,u,y, for some y. Indeed, since Nu & σu,u’ & Ra,u,y & Ra,u’,w, we may conclude that σy,w. But σb,u, so Ny & σb,u & Ra,u,y. Hence Nz & Ra,b,z. Again, we may conclude that σz,y. We may also conclude that +(a,b,z). By *Prop* 10, z = c. So σc,y. By (PA4) y = c’. So y = v. Thus σv,w. End 2.

Since R’ satisfies all the requisite conditions, +(a,b’,c’).

Prop 12. ∀a∀b∀c∀d (Nb & σb,c & +(a,c,d) ⇒ ∃v (σv,d & +(a,b,v)))
Pf:

Suppose Nb & σb,c & +(a,c,d). Then

$Na \,\&\, Nc \,\&\, Nd \,\&$
$\exists R\ (Ra,0,a \,\&\, Ra,c,d \,\&$
$\quad \forall v\ (Ra,0,v \Rightarrow a = v) \,\&$
$\quad \forall u \forall u' \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u',w \Rightarrow \exists v\ (Nv \,\&\, Ra,u,v)) \,\&$
$\quad \forall u \forall u' \forall v \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u,v \,\&\, Ra,u',w \Rightarrow \sigma v,w)\ ).$
Let R be as detailed. Then $Nb \,\&\, \sigma b,c \,\&\, Ra,c,d$, so Ra,b,v, for some v where Nv. Thus $+(a,b,v)$. Since $\forall u \forall u' \forall v \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u,v \,\&\, Ra,u',w \Rightarrow \sigma v,w)$, we may conclude that $\sigma v,d$.

Corollary 13. $\forall a \forall b \forall c \forall d \forall e\ (Nb \,\&\, Ne \,\&\, \sigma b,c \,\&\, \sigma e,d \,\&\, +(a,c,d) \Rightarrow +(a,b,e))$
Pf:
Assume $Nb \,\&\, Ne \,\&\, \sigma b,c \,\&\, \sigma e,d \,\&\, +(a,c,d)$. By *Prop* 12, $\sigma v,d \,\&\, +(a,b,v)$, for some v. By (PA5) $v = e$.

Prop 14. $\forall a\ (Na \Rightarrow +(0,a,a))$
Pf:
By induction, with ϕ as $(Nn \Rightarrow +(0,n,n))$.
$N0 \Rightarrow +(0,0,0)$ by *Prop* 9.
Assume $Nn \,\&\, Nm \,\&\, \sigma n,m \,\&\, \phi$. Then $+(0,n,n)$ by the induction hypothesis, so $+(0,m,m)$ by *Prop* 11.

Prop 15. $\forall a \forall b\ (+(a,b,0) \Rightarrow a = 0 \,\&\, b = 0)$
Pf:
Assume $+(a,b,0)$. And suppose $\neg\, b = 0$. Then $\sigma c,b$, for some c with Nc, by *Prop* 2. By definition of +,
$Na \,\&\, Nb \,\&\, N0 \,\&$
$Ra,0,a \,\&\, Ra,b,0 \,\&$
$\forall v\ (Ra,0,v \Rightarrow a = v) \,\&$
$\forall u \forall u' \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u',w \Rightarrow \exists v\ (Nv \,\&\, Ra,u,v)) \,\&$
$\forall u \forall u' \forall v \forall w\ (Nu \,\&\, \sigma u,u' \,\&\, Ra,u,v \,\&\, Ra,u',w \Rightarrow \sigma v,w)\ ),$
for some R. Then $Nv \,\&\, Ra,c,v$, for some v. Hence $\sigma v,0$, contradicting (PA6).
Thus $b = 0$. By *Prop* 8, $a = 0$.

Prop 16. $\forall a \forall b \forall c \forall d \forall e\ (Nb \,\&\, Ne \,\&\, \sigma b,a \,\&\, \sigma e,d \,\&\, +(a,c,d) \Rightarrow +(b,c,e))$
Pf:
By induction, with ϕ as
$\forall a \forall b \forall d \forall e\ (Nb \,\&\, Ne \,\&\, \sigma b,a \,\&\, \sigma e,d \,\&\, +(a,n,d) \Rightarrow +(b,n,e))$
Assume $Nb \,\&\, Ne \,\&\, \sigma b,a \,\&\, \sigma e,d \,\&\, +(a,0,d)$. Then by *Prop* 8, $a = d$. So by (PA5), $b = e$. Hence by *Prop* 9, $+(b,0,e)$.
Now assume $Nn \,\&\, \sigma n,m \,\&\, \phi$. Suppose $Nb \,\&\, Ne \,\&\, \sigma b,a \,\&\, \sigma e,d \,\&\, +(a,m,d)$. By *Corollary* 13, $+(a,n,e)$. If $e = 0$, then $a = 0$, by *Prop* 15;

but this contradicts (PA6). So $\neg\, e = 0$, and $\mathrm{N}f \;\&\; \sigma f,e$ for some f. By the induction hypothesis $+(b,n,f)$. By *Prop* 11, $+(b,m,e)$.

Prop 17. $\forall a \forall b\ (\mathrm{N}a \;\&\; \mathrm{N}b \;\&\; \sigma a,b \Leftrightarrow +(a,1,b))$
Remark: Since by *Prop* 6, the left-hand side implies $\exists j\ (\mathrm{N}j \;\&\; \sigma 0,j)$, we are justified in the use of "1" on the right-hand side.
Pf:

Assume $\mathrm{N}a \;\&\; \mathrm{N}b \;\&\; \sigma a,b$. $+(a,0,a)$ by *Prop* 9. $\sigma 0,1 \;\&\; \sigma a,b$. By *Prop* 11, $+(a,1,b)$.

Now assume $+(a,1,b)$. Then by definition of +, $\mathrm{N}a \;\&\; \mathrm{N}b$. By *Prop* 12, $\sigma v,b \;\&\; +(a,0,v)$ for some v. By *Prop* 8, $v = a$.

Prop 18. $\forall a \forall a' \forall b \forall b' \forall c\ (\mathrm{N}a \;\&\; \mathrm{N}b' \;\&\; \sigma a,a' \;\&\; \sigma b,b' \;\&\; +(a',b,c) \Rightarrow +(a,b',c))$
Pf:

By induction, with ϕ as
$\forall a \forall a' \forall b' \forall c\ (\mathrm{N}a \;\&\; \mathrm{N}b' \;\&\; \sigma a,a' \;\&\; \sigma n,b' \;\&\; +(a',n,c) \Rightarrow +(a,b',c))$

Assume $\mathrm{N}a \;\&\; \mathrm{N}b' \;\&\; \sigma a,a' \;\&\; \sigma 0,b' \;\&\; +(a',0,c)$. Then $a' = c$ by *Prop* 8 and $+(a,b',a')$ by *Prop* 11. Hence $+(a,b',c)$.

Now assume $\mathrm{N}n \;\&\; \sigma n,m \;\&\; \phi$. Suppose $\mathrm{N}a \;\&\; \mathrm{N}b' \;\&\; \sigma a,a' \;\&\; \sigma m,b' \;\&\; +(a',m,c)$. By *Prop* 12, $\sigma v,c \;\&\; +(a',n,v)$, for some v. By the induction hypothesis $+(a,m,v)$. By *Prop* 11, $+(a,b',c)$.

Prop 19. $\forall a \forall b \forall c\ (+(a,b,c) \;\&\; \neg\, a = 0 \Rightarrow \exists b'(\mathrm{N}b' \;\&\; \sigma b,b'))$
Pf:

By induction, with ϕ as
$\forall b \forall c\ (+(n,b,c) \;\&\; \neg\, n = 0 \Rightarrow \exists b'(\mathrm{N}b' \;\&\; \sigma b,b'))$

The assertion is trivially true when $n = 0$.

Now assume $\mathrm{N}n \;\&\; \sigma n,m \;\&\; \phi$. Suppose $+(m,b,c) \;\&\; \neg\, m = 0$. By *Prop* 15 $\neg\, c = 0$, so by *Prop* 2, $\mathrm{N}d \;\&\; \sigma d,c$ for some c. By *Prop* 18, $+(n,b,d)$. If $n = 0$, then $+(0,b,b)$ by *Prop* 14, and so $b = d$ by *Prop* 10; hence set $b' = c$. Otherwise $\neg\, n = 0$, so by the induction hypothesis $\exists b'(\mathrm{N}b' \;\&\; \sigma b,b')$.

Corollary 20. $\forall a \forall a' \forall b \forall b' \forall c\ (\mathrm{N}a \;\&\; \sigma a,a' \;\&\; +(a',b,c) \Rightarrow \exists b'(\sigma b,b' \;\&\; +(a,b',c))$
Pf:

Assume $\mathrm{N}a \;\&\; \sigma a,a' \;\&\; +(a',b,c)$. By (PA6) $\neg\, a' = 0$. By *Prop* 19, $\mathrm{N}b' \;\&\; \sigma b,b'$ for some b'. By *Prop* 18 $+(a,b',c)$.

Prop 21. (Commutative Law of Addition) $\forall a \forall b \forall c\ (+(a,b,c) \Rightarrow +(b,a,c))$
Pf:

By induction, with ϕ as

$\forall b \forall c\ (+(n,b,c) \Rightarrow +(b,n,c))$

If $+(0,b,c)$, then $+(0,b,b)$ by *Prop* 14. So $b = c$ by *Prop* 10. And thus $+(b,0,c)$ by *Prop* 8.

Assume $\mathrm{N}n$ & $\sigma n,m$ & ϕ. Suppose $+(m,a,c)$. Then $\neg\ m = 0$ by (PA6). By *Prop* 19 $\mathrm{N}a'$ & $\sigma a,a'$ for some a'. By *Prop* 18, $+(n,a',c)$. Since $\neg\ a' = 0$ by (PA6), $\neg\ c = 0$ by *Prop* 15. By *Prop* 2 $\mathrm{N}d$ & $\sigma d,c$ for some d. By *Prop* 11, $+(n,a,d)$. By the induction hypothesis, $+(a,n,d)$. By *Corollary* 13, $+(a,m,c)$.

Prop 22. $\forall a \forall b\ (+(a,b,a) \Rightarrow b = 0)$

Pf:

By induction, with ϕ as

$\forall b\ (+(n,b,n) \Rightarrow b = 0)$

Suppose $+(0,b,0)$. Then by *Prop* 15, $b = 0$.

Now assume $\mathrm{N}n$ & $\sigma n,m$ & ϕ. Suppose $+(m,b,m)$. Then $+(b,m,m)$ by *Prop* 21. So $+(b,n,n)$ by *Prop* 11. So $+(n,b,n)$ by *Prop* 21 again. By the induction hypothesis, $b = 0$.

Def 23. $n < m$ if and only if $\exists k\ (\neg\ k = 0\ \&\ +(n,k,m))$.

Remark that, if $n < m$, then $\mathrm{N}n$ & $\mathrm{N}m$.

Prop 24. $\forall a \forall b\ (\mathrm{N}a\ \&\ \mathrm{N}b\ \&\ \sigma a,b \Rightarrow a < b)$

Pf:

Assume $\mathrm{N}a$ & $\mathrm{N}b$ & $\sigma a,b$. Then by *Prop* 17, $+(a,1,b)$. By *Corollary* 5, $\neg\ 1 = 0$.

Prop 25. $\forall n\ \neg\ n < 0$

Pf:

Suppose $n < 0$. Then $\exists k\ (\neg\ k = 0\ \&\ +(n,k,0))$. But this contradicts *Prop* 15.

Prop 26. $\forall n\ (\mathrm{N}n\ \&\ \neg\ n = 0 \Rightarrow 0 < n)$

Pf:

Assume $\mathrm{N}n$ & $\neg\ n = 0$. Then $+(0,n,n)$ by *Prop* 14. So $0 < n$.

Prop 27. $\forall a \forall b\ (\ a < b \Rightarrow \exists v\ \ (\mathrm{N}v\ \&\ \sigma a,v)\ )$

Pf:

Suppose $a < b$. Then $\neg\ k = 0$ & $+(a,k,b)$ for some k. By *Prop* 21, $+(k,a,b)$. By *Prop* 19 $\exists v(\mathrm{N}v\ \&\ \sigma a,v)$.

Prop 28. $\forall n\ \neg\ n < n$

Pf:

Suppose $n < n$. Then $+(n,k,n)$ for some non-zero k, contradicting *Prop* 22.

Prop 29. $\forall n \forall m\ (Nn \ \&\ Nm \ \&\ \sigma n,m \Rightarrow \forall z((z < n \vee z = n) \Leftrightarrow z < m))$
Pf:

Assume $Nn \ \&\ Nm \ \&\ \sigma n,m$.

Suppose $z < n \vee z = n$. If $z = n$, then $\sigma z,m$, so by *Prop* 24, $z < m$. Otherwise, suppose $z < n$. Then $+(z,k,n)$ for some k. If $z = 0$, then $k = n$ by the usual argument; in which case $+(0,m,m)$, where $\neg\, m = 0$ by (PA6), so $0 < m$ by *Prop* 26. If $\neg\, z = 0$, then $Nk' \ \&\ \sigma k,k'$ for some k'. But then $+(z,k',m)$ by *Prop* 18. By (PA6) $\neg\, k' = 0$, so $z < m$.

Now assume $z < m$. Then $\neg\, k = 0 \ \&\ +(z,k,m)$, for some k. By *Prop* 2, $Nj \ \&\ \sigma j,k$ for some j. By *Prop* 11, $+(z,j,n)$. If $j = 0$, then $z = n$ by *Prop* 8. And if $\neg\, j = 0$, then $z < n$.

Prop 30. $\forall a \forall b\ (\ a < b \Rightarrow \neg\, \sigma b,a\ )$.
Pf:

Suppose $a < b \ \&\ \sigma b,a$. By *Prop 29* $\forall z((z < b \vee z = b) \Leftrightarrow z < a)$. Hence $a < a$, contradicting *Prop* 28.

Prop 31. $\forall n(Nn \Rightarrow \exists P$ s.t. $\forall z(Pz \Leftrightarrow z < n))$.
Pf:

By induction, with ϕ as
$\exists P$ s.t. $\forall z(Pz \Leftrightarrow z < n)$.

For $n = 0$, let $P = \phi$. Then, by *Prop* 25, $\forall z(Pz \Leftrightarrow z < 0)$.

Now assume $Nn \ \&\ \sigma n,m \ \&\ \exists P$ s.t. $\forall z(Pz \Leftrightarrow z < n)$. Define $P' = \{z : Pz \vee z = n\}$. Then by *Prop* 29, $\forall z(P'z \Leftrightarrow z < m)$.

Def 32. Let Nn. Let $[0\ \_\ n)$ be that P (unique up to equivalence) guaranteed by *Prop 31*, so $\forall z([0\ \_\ n)z \Leftrightarrow z < n))$.

Remark:

Prop 33. $\forall n\ (Nn \ \&\ \neg\, n = 0 \Rightarrow \neg\, [0\ \_\ n) \equiv \phi)$

Def 34 $A \sim B$ if and only if $\exists R$ (*IsFunction*(R) & *Is1-1*(R) & *Dom*(R) $\equiv$ A & *Im*(R) $\equiv$ B).

It is easy to prove in predicative second-order logic that $\sim$ is reflexive, symmetric, and transitive; also that $\forall P\ (P \sim \phi \Leftrightarrow P \equiv \phi)$

Prop 35. $\forall n \forall k\ (Nn \ \&\ Nk \ \&\ [0\ \_\ n) \sim [0\ \_\ k) \Rightarrow n = k)$.
Pf:

By induction, with ϕ as
$\forall k\ (\mathrm{N}n\ \&\ \mathrm{N}k\ \&\ [0\ \_\ n) \sim [0\ \_\ k) \Rightarrow n = k)$

Assume $\mathrm{N}0\ \&\ \mathrm{N}k\ \&\ [0\ \_\ 0) \sim [0\ \_\ k)$. Then by *Def* 32 and *Prop* 25, $[0\ \_\ 0) \equiv \phi$. But this forces $[0\ \_\ k) \equiv \phi$. By *Prop* 26, we must have $k = 0$.

Now assume $\mathrm{N}n\ \&\ \sigma n,m\ \&\ \phi$. Suppose $\mathrm{N}m\ \&\ \mathrm{N}k\ \&$ $[0\ \_\ m) \sim [0\ \_\ k)$. By *Prop* 2, $\mathrm{N}j\ \&\ \sigma j,k$ for some j. By *Prop* 29,

$$[0\ \_\ n) \cup \{m\} \sim [0\ \_\ j) \cup \{k\}.$$

By *Prop* 30, $\neg\ m < n\ \&\ \neg\ k < j$, so $[0\ \_\ n)$ and $\{m\}$ are pairwise disjoint, as are $[0\ \_\ j)$ and $\{k\}$. By logic, $[0\ \_\ n) \sim [0\ \_\ j)$. By the induction hypothesis $n = j$. By (PA4) $m = k$.

Def 36. $\mathrm{M}n,P$ if and only if $(n = 0\ \&\ P \equiv \phi) \vee \exists k \exists a(\mathrm{N}k\ \&\ \sigma k,n\ \&\ Pa\ \&$ $(P\backslash\{a\}) \sim [0\ \_\ k))$.

Remark: It would perhaps seem more natural to define $\mathrm{M}n,P$ as

$$P \sim [0\ \_\ n).$$

However, $[0\ \_\ n) \equiv \phi$ whenever $\neg\ \mathrm{N}n$. It would then be possible to have both $\mathrm{M}0,P$ and $\mathrm{M}n,P$ for 0 natural and some non-natural n. But then $\neg\ n = 0$, contrary to (F1).

We need to prove (F1), (F2), and (F3).

Prop 37. (F1) $\forall n \forall m \forall P\ (\ \mathrm{M}n,P\ \&\ \mathrm{M}m,P \Rightarrow n = m\ )$

Pf:

Assume $\mathrm{M}n,P\ \&\ \mathrm{M}m,P$. If $n = 0$, then $P \equiv \phi$. If $\neg\ m = 0$, then $\mathrm{N}k\ \&$ $\sigma k,m\ \&\ Pa\ \&\ (P\backslash\{a\}) \sim [0\ \_\ k)$, for some k,a, contradicting $P \equiv \phi$. So $m = 0$. A similar argument shows that if $m = 0$, then $n = 0$.

Hence we may assume that both $\neg\ n = 0\ \&\ \neg\ m = 0$. Thus $\mathrm{N}k\ \&\ \sigma k,n\ \&\ Pa\ \&\ (P\backslash\{a\}) \sim [0\ \_\ k))$ and $\mathrm{N}j\ \&\ \sigma j,m\ \&\ Pb\ \&$ $(P\backslash\{b\}) \sim [0\ \_\ j))$ for some j,k,a,b. But $(P\backslash\{a\}) \sim (P\backslash\{b\})$, so $[0\ \_\ k) \sim [0\ \_\ j)$. By *Prop* 35 this forces $k = j$. By (PA4) $n = m$.

Prop 38. (F2) $\forall P\ (\ \mathrm{M}0,P \Leftrightarrow \neg\ \exists x\ Px\ )$

Pf:

If $P \equiv \phi$, then by definition $\mathrm{M}0,P$.

On the other hand, suppose $\mathrm{M}0,P$. Then not:

$$\exists k \exists a(\mathrm{N}k\ \&\ \sigma k,0\ \&\ Pa\ \&\ (P\backslash\{a\}) \sim [0\ \_\ k)),$$

by (PA6). This forces $P \equiv \phi$.

Prop 39. (F3) $\forall n \forall m \forall P \forall Q \forall a\ (\ \mathrm{N}n\ \&\ \sigma n,m\ \&\ \neg Pa\ \&\ \forall x(Qx \Leftrightarrow Px \vee x = a)$ $\Rightarrow (\mathrm{M}n,P \Leftrightarrow \mathrm{M}m,Q)\ )$

Pf:

Assume $\mathrm{N}n\ \&\ \sigma n,m\ \&\ \neg Pa\ \&\ \forall x(Qx \Leftrightarrow Px \vee x = a)$. Obviously, Qa.

Suppose Mn,P. If n = 0, then P ≡ ϕ, so Q ≡ {a}, and [0 _ 0) ~ (Q \ {a})), so Mm,Q. Thus suppose ¬ n = 0. Then Nk & σk,n & Pb & (P\{b}) ~ [0 _ k), for some k,b. But then P ~ [0 _ n). So:

Nn & σn,m & Qa & (Q\{a}) ~ [0 _ n)).

Hence Mm,Q.

On the other hand, suppose Mm,Q. Then ¬ m = 0 by *Prop* 38. Hence Nk & σk,m & Qb & (Q\{b}) ~ [0 _ k), for some k,b. By (PA5) n = k. If n = 0, then (Q\{b}) ≡ ϕ by *Prop* 25 so Q ≡ {b} and P ≡ ϕ. But then Mn,P by *Prop* 38. Otherwise ¬ n = 0, i.e. ¬ k = 0. So σj,k & Nj, by *Prop* 2, for some j. Now by logic, P ~ (Q\{b}). So P ~ [0 _ k). By *Prop* 29, [0 _ j) ~ [0 _ k) ∪ {j}, where ¬ [0 _ k)j by *Prop* 30. But then by logic, (P\{c}) ~ [0 _ j), for some c where Pc. Thus Mn,P.

Additional Remark

Recall that in **F** one can define multiplication and thence prove the usual theorems involving multiplication. Since **FPA** can define M and proves the axioms of **F**, it follows that one can define and prove the usual theorems of multiplication in **FPA**. Alternatively, **FPA** is also able to define multiplication directly, as follows:

*(a,b,c) if and only if
Na & Nb & Nc &
∃R (Ra,0,0 & Ra,b,c &
∀v (Ra,0,v ⇒ v = 0) &
∀u∀u'∀w (Nu & σu,u' & Ra,u',w ⇒ ∃v (Nv & Ra,u,v)) &
∀u∀u'∀v∀w (Nu & σu,u' & Ra,u,v & Ra,u',w ⇒ +(a,v,w))

FPA can then prove the usual theorems directly.

Conclusion

While **F** and **FPA** are equivalent proof-theoretically, they do not appear to have the same epistemological status. **F** seems to provide a deeper foundation of arithmetic than **FPA**, through the use of the numbering predicate "M," which connects 0 with the number of nothing, and which relates the sequential relationship with the numbering of one more or one less. That is, **F** has explanatory power which **FPA** does not. Moreover, (PA5) seems to require proof rather than simple assertion; why *should* , after all, predecessors be unique? Nonetheless, it is doubtless comforting that **F** is equivalent proof-theoretically to a sub-theory of Peano Arithmetic, simply because the latter is so familiar to such a wide class of people. It is also surely of interest that a sub-theory of Peano Arithmetic is, like **F**, able to prove a great deal of arithmetic, yet also prove its own consistency.