

SECTION I

!CHAPTER 3 SOME SIMPLE PROPOSITIONS;

- 1 $\vdash \forall x \forall y (x = y \Rightarrow y = x)$ i
- 2 $\vdash \forall x \forall y \forall z (x = y \ \& \ y = z \Rightarrow x = z)$ i
- 3 $\vdash \forall x \forall y (\neg x = y \Rightarrow \neg y = x)$ i
- 4 $\vdash \forall x \forall y (x = y \vee \neg x = y)$ i
- 5 $\vdash \forall x \forall y \forall z \forall a ((x = y \vee z = a) \ \& \ \neg x = y \Rightarrow z = a)$ i
- 6 $\vdash \forall x \forall y \forall z \forall a ((x = y \vee z = a) \ \& \ \neg z = a \Rightarrow x = y)$ i
- 7 $\vdash \forall x \forall y \forall z \forall a ((\neg x = y \vee \neg z = a) \ \& \ x = y \Rightarrow \neg z = a)$ i
- 8 $\vdash \forall x \forall y \forall z \forall a ((\neg x = y \vee \neg z = a) \ \& \ z = a \Rightarrow \neg x = y)$ i
- 9 $\vdash \forall x \forall y \forall z \forall a \forall b \forall c ((x = y \vee z = a \vee b = c) \ \& \ \neg x = y$
 $\Rightarrow z = a \vee b = c)$ i
- 10 $\vdash \forall x \forall y \forall z \forall a \forall b \forall c \forall d \forall e ((x = y \vee z = a \vee b = c \vee d = e)$
 $\ \& \ \neg x = y$
 $\Rightarrow z = a \vee b = c \vee d = e)$ i
- 11 $\vdash \forall P \forall Q \forall x \forall y ((P[x] \vee Q[y]) \ \& \ \neg P[x] \Rightarrow Q[y])$ i
- 12 $\vdash \forall P \forall Q \forall x \forall y ((P[x] \vee Q[y]) \ \& \ \neg Q[y] \Rightarrow P[x])$ i
- 13 $\vdash \forall x \forall y \forall z \forall a (\neg (x = y \vee z = a) \Rightarrow \neg x = y \ \& \ \neg z = a)$ i
- 14 $\vdash \forall x \forall y \forall z \forall a (\neg x = y \ \& \ \neg z = a \Rightarrow \neg (x = y \vee z = a))$ i
- 15 $\vdash \forall P \forall x (P[x] \vee \neg P[x])$ i
- 16 $\vdash \forall x \forall y (x = y \Rightarrow \forall P (P[x] \Leftrightarrow P[y]))$ i
- 17 $\vdash \forall P \forall x \forall y (P[x] \ \& \ \neg P[y] \Rightarrow \neg x = y)$ i
- 18 $\vdash \forall R \forall x \forall y (R[x,y] \vee \neg R[x,y])$ i
- 19 $\vdash \forall P (\forall x P[x] \Rightarrow \neg \exists x \neg P[x])$ i
- 20 $\vdash \forall P (\exists x P[x] \Rightarrow \neg \forall x \neg P[x])$ i
- 21 $\vdash \forall P (\forall x \neg P[x] \Rightarrow \neg \exists x P[x])$ i
- 22 $\vdash \forall P (\exists x \neg P[x] \Rightarrow \neg \forall x P[x])$ i
- 23 $\vdash \forall P (\neg \forall x P[x] \Rightarrow \exists x \neg P[x])$ i
- 24 $\vdash \forall P (\neg \exists x P[x] \Rightarrow \forall x \neg P[x])$ i
- 25 $\vdash \forall P (\neg \forall x \neg P[x] \Rightarrow \exists x P[x])$ i
- 26 $\vdash \forall P (\neg \exists x \neg P[x] \Rightarrow \forall x P[x])$ i
- 27 $\vdash \forall P (\forall x P[x] \Leftrightarrow \neg \exists x \neg P[x])$ i
- 28 $\vdash \forall P (\exists x P[x] \Leftrightarrow \neg \forall x \neg P[x])$ i

SECTION II: MONADIC PREDICATES

!CHAPTER 1 INCLUSION AND EQUIVALENCE;

- 1 $\mathbb{S} \subseteq ; \ P \subseteq Q ; \ \forall x (P[x] \Rightarrow Q[x])$ i
- 2 $\vdash \forall P \forall Q \forall x (P[x] \ \& \ P \subseteq Q \Rightarrow Q[x])$ i
- 3 $\vdash \forall P \forall Q \forall x (\neg Q[x] \ \& \ P \subseteq Q \Rightarrow \neg P[x])$ i
- 4 $\vdash \forall P \ P \subseteq P$ i
- 5 $\vdash \forall P \forall Q \forall R (P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$ i
- 6 $\vdash \forall P \forall Q \forall R \forall S (P \subseteq Q \ \& \ Q \subseteq R \ \& \ R \subseteq S \Rightarrow P \subseteq S)$ i
- 7 $\mathbb{S} \equiv ; \ P \equiv Q ; \ \forall x (P[x] \Leftrightarrow Q[x])$ i
- 8 $\vdash \forall P \forall Q (P \subseteq Q \ \& \ Q \subseteq P \Rightarrow P \equiv Q)$ i
- 9 $\vdash \forall P \ P \equiv P$ i
- 10 $\vdash \forall P \forall Q (P \equiv Q \Rightarrow Q \equiv P)$ i
- 11 $\vdash \forall P \forall Q (P \equiv Q \Rightarrow P \subseteq Q)$ i
- 12 $\vdash \forall P \forall Q (P \equiv Q \Rightarrow Q \subseteq P)$ i

13	$\vdash \forall P \forall Q (P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P)$	i
14	$\vdash \forall P \forall Q (P \subseteq Q \ \& \ Q \subseteq P \Leftrightarrow P \equiv Q)$	i
15	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ Q \equiv R \Rightarrow P \equiv R)$	i
16	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ Q \equiv R \Rightarrow R \equiv P)$	i
17	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ R \equiv Q \Rightarrow P \equiv R)$	i
18	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ R \equiv Q \Rightarrow R \equiv P)$	i
19	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ P \equiv R \Rightarrow Q \equiv R)$	i
20	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ P \equiv R \Rightarrow R \equiv Q)$	i
21	$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$	i
22	$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$	i
23	$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$	i
24	$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$	i
25	$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ Q \equiv R \ \& \ R \equiv S \Rightarrow P \equiv S)$	i
26	$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ Q \equiv R \ \& \ S \equiv R \Rightarrow P \equiv S)$	i
27	$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ R \equiv Q \ \& \ R \equiv S \Rightarrow P \equiv S)$	i
28	$\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ R \equiv Q \ \& \ S \equiv R \Rightarrow P \equiv S)$	i
29	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$	i
30	$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$	i
31	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$	i
32	$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ R \subseteq Q \Rightarrow R \subseteq P)$	i
33	$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \ \& \ P \subseteq R \Rightarrow Q \subseteq S)$	i
34	$\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \ \& \ Q \subseteq S \Rightarrow P \subseteq R)$	i
35	$\vdash \forall P \forall Q \forall x (P[x] \ \& \ P \equiv Q \Rightarrow Q[x])$	i
36	$\vdash \forall P \forall Q \forall x (P[x] \ \& \ Q \equiv P \Rightarrow Q[x])$	i
37	$\vdash \forall P \forall Q \forall x (\neg P[x] \ \& \ P \equiv Q \Rightarrow \neg Q[x])$	i
38	$\vdash \forall P \forall Q \forall x (\neg P[x] \ \& \ Q \equiv P \Rightarrow \neg Q[x])$	i
39	$\vdash \forall P \forall Q \forall R (\neg P \equiv Q \Rightarrow \neg Q \equiv P)$	i
40	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$	i
41	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$	i
42	$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg P \equiv R)$	i
43	$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg Q \equiv R \Rightarrow \neg R \equiv P)$	i
44	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$	i
45	$\vdash \forall P \forall Q \forall R (P \equiv Q \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$	i
46	$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg P \equiv R)$	i
47	$\vdash \forall P \forall Q \forall R (Q \equiv P \ \& \ \neg R \equiv Q \Rightarrow \neg R \equiv P)$	i
48	$\vdash \forall P \forall Q (P \equiv Q \vee \neg P \equiv Q)$	i
49	$\mathfrak{S} \ C ; P \subset Q ; P \subseteq Q \ \& \ \neg P \equiv Q$	i
50	$\vdash \forall P \forall Q (P \subset Q \Rightarrow P \subseteq Q)$	i
51	$\vdash \forall P \forall Q (P \subset Q \Rightarrow \neg P \equiv Q)$	i
52	$\vdash \forall P \forall Q (P \equiv Q \Rightarrow \neg P \subset Q)$	i
53	$\vdash \forall P \forall Q (P \subseteq Q \Rightarrow P \equiv Q \vee P \subset Q)$	i
54	$\vdash \forall P \forall Q (P \equiv Q \vee P \subset Q \Rightarrow P \subseteq Q)$	i
55	$\vdash \forall P \forall Q (P \subseteq Q \Leftrightarrow P \equiv Q \vee P \subset Q)$	i
56	$\vdash \forall P \neg P \subset P$	i
57	$\vdash \forall P \forall Q (P \subset Q \Rightarrow \neg Q \subset P)$	i
58	$\vdash \forall P \forall Q \forall R (P \subset Q \ \& \ Q \subseteq R \Rightarrow P \subset R)$	i

- 59 $\vdash \forall P \forall Q \forall R (P \subseteq Q \ \& \ Q \subseteq R \Rightarrow P \subseteq R)$ i
60 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow \exists x (Q[x] \ \& \ \neg P[x]))$ i
61 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow \exists x Q[x])$ i

!CHAPTER 2 UNIONS;

- 1 $\mathbb{D} \cup ; (P \cup Q) ; ; \{ a : P[a] \vee Q[a] \}$ i
2 $\vdash \forall P \forall Q \forall x ((P \cup Q)[x] \Leftrightarrow P[x] \vee Q[x])$ i
3 $\vdash \forall P \forall Q \forall x ((P \cup Q)[x] \Rightarrow P[x] \vee Q[x])$ i
4 $\vdash \forall P \forall Q \forall x (P[x] \vee Q[x] \Rightarrow (P \cup Q)[x])$ i
5 $\vdash \forall P \forall Q \forall x (P[x] \Rightarrow (P \cup Q)[x])$ i
6 $\vdash \forall P \forall Q \forall x (Q[x] \Rightarrow (P \cup Q)[x])$ i
7 $\vdash \forall P \forall Q \forall x ((P \cup Q)[x] \ \& \ \neg P[x] \Rightarrow Q[x])$ i
8 $\vdash \forall P \forall Q \forall x ((P \cup Q)[x] \ \& \ \neg Q[x] \Rightarrow P[x])$ i
9 $\vdash \forall P \forall Q \forall x (\neg (P \cup Q)[x] \Rightarrow \neg P[x])$ i
10 $\vdash \forall P \forall Q \forall x (\neg (P \cup Q)[x] \Rightarrow \neg Q[x])$ i
11 $\vdash \forall P \forall Q \forall x (\neg (P \cup Q)[x] \Rightarrow \neg P[x] \ \& \ \neg Q[x])$ i
12 $\vdash \forall P \forall Q \ P \subseteq (P \cup Q)$ i
13 $\vdash \forall P \forall Q \ P \subseteq (Q \cup P)$ i
14 $\vdash \forall P \forall Q \forall R (P \subseteq R \ \& \ Q \subseteq R \Rightarrow (P \cup Q) \subseteq R)$ i
15 $\vdash \forall P \forall Q \ (P \cup Q) \subseteq (Q \cup P)$ i
16 $\vdash \forall P \forall Q \ (P \cup Q) \equiv (Q \cup P)$ i
17 $\vdash \forall P \forall Q \forall R ((P \cup Q) \subseteq R \Rightarrow (Q \cup P) \subseteq R)$ i
18 $\vdash \forall P \forall Q \forall R (R \subseteq (P \cup Q) \Rightarrow R \subseteq (Q \cup P))$ i
19 $\vdash \forall P \forall Q \forall R ((P \cup Q) \equiv R \Rightarrow (Q \cup P) \equiv R)$ i
20 $\vdash \forall P \forall Q \forall R (R \equiv (P \cup Q) \Rightarrow R \equiv (Q \cup P))$ i
21 $\vdash \forall P \forall Q \forall R (P \subseteq Q \Rightarrow P \subseteq (Q \cup R))$ i
22 $\vdash \forall P \forall Q \forall R (P \subseteq R \Rightarrow P \subseteq (Q \cup R))$ i
23 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (P \cup Q) \subseteq Q)$ i
24 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (Q \cup P) \subseteq Q)$ i
25 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (P \cup Q) \equiv Q)$ i
26 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (Q \cup P) \equiv Q)$ i
27 $\vdash \forall P (P \cup P) \equiv P$ i
28 $\vdash \forall P \forall Q (P \equiv (P \cup Q) \Rightarrow Q \subseteq P)$ i
29 $\vdash \forall P \forall Q (P \equiv (Q \cup P) \Rightarrow Q \subseteq P)$ i
30 $\vdash \forall P \forall Q ((P \cup Q) \equiv Q \Rightarrow Q \subseteq P)$ i
31 $\vdash \forall P \forall Q ((Q \cup P) \equiv P \Rightarrow Q \subseteq P)$ i
32 $\vdash \forall P \forall Q \forall R \forall S (P \subseteq Q \ \& \ R \subseteq S \Rightarrow (P \cup R) \subseteq (Q \cup S))$ i
33 $\vdash \forall P \forall Q \forall R (P \subseteq Q \Rightarrow (P \cup R) \subseteq (Q \cup R))$ i
34 $\vdash \forall P \forall Q \forall R (P \subseteq Q \Rightarrow (R \cup P) \subseteq (R \cup Q))$ i
35 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \Rightarrow (P \cup R) \equiv (Q \cup S))$ i
36 $\vdash \forall P \forall Q \forall R (P \equiv Q \Rightarrow (P \cup R) \equiv (Q \cup R))$ i
37 $\vdash \forall P \forall Q \forall R (P \equiv Q \Rightarrow (R \cup P) \equiv (R \cup Q))$ i
38 $\vdash \forall P \forall Q \forall R (P \equiv Q \Rightarrow (P \cup R) \equiv (R \cup Q))$ i
39 $\vdash \forall P \forall Q \forall R (Q \equiv P \Rightarrow (P \cup R) \equiv (R \cup Q))$ i
40 $\vdash \forall P \forall Q \forall R \forall S \forall T (P \equiv Q \ \& \ R \equiv S \ \& \ T \equiv (P \cup R) \Rightarrow T \equiv (Q \cup S))$ i
41 $\vdash \forall P \forall Q \forall R \forall S \forall T (P \equiv Q \ \& \ R \equiv S \ \& \ (P \cup R) \equiv T \Rightarrow (Q \cup S) \equiv T)$ i

- 42 $\vdash \forall P \forall Q \forall R \forall S \forall T (Q \equiv P \ \& \ S \equiv R \ \& \ T \equiv (P \cup R) \Rightarrow T \equiv (Q \cup S))$ i
- 43 $\vdash \forall P \forall Q \forall R \forall S \forall T (Q \equiv P \ \& \ S \equiv R \ \& \ (P \cup R) \equiv T \Rightarrow (Q \cup S) \equiv T)$ i
- 44 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ (P \cup R) \equiv S \Rightarrow (Q \cup R) \equiv S)$ i
- 45 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ S \equiv (P \cup R) \Rightarrow S \equiv (Q \cup R))$ i
- 46 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ (R \cup P) \equiv S \Rightarrow (R \cup Q) \equiv S)$ i
- 47 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ S \equiv (R \cup P) \Rightarrow S \equiv (R \cup Q))$ i
- 48 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ (P \cup R) \equiv S \Rightarrow (Q \cup R) \equiv S)$ i
- 49 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ S \equiv (P \cup R) \Rightarrow S \equiv (Q \cup R))$ i
- 50 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ (R \cup P) \equiv S \Rightarrow (R \cup Q) \equiv S)$ i
- 51 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ S \equiv (R \cup P) \Rightarrow S \equiv (R \cup Q))$ i
- 52 $\vdash \forall P \forall Q \forall R ((P \cup Q) \cup R) \subseteq (P \cup (Q \cup R))$ i
- 53 $\vdash \forall P \forall Q \forall R (P \cup (Q \cup R)) \subseteq ((P \cup Q) \cup R)$ i
- 54 $\vdash \forall P \forall Q \forall R ((P \cup Q) \cup R) \equiv (P \cup (Q \cup R))$ i
- 55 $\vdash \forall P \forall Q \forall R (P \cup (Q \cup R)) \equiv ((P \cup Q) \cup R)$ i

!CHAPTER 3 INTERSECTIONS;

- 1 $\mathbb{D} \cap ; (P \cap Q) ; ; \{a : P[a] \ \& \ Q[a]\}$ i
- 2 $\vdash \forall P \forall Q \forall x ((P \cap Q)[x] \Leftrightarrow P[x] \ \& \ Q[x])$ i
- 3 $\vdash \forall P \forall Q \forall x ((P \cap Q)[x] \Rightarrow P[x] \ \& \ Q[x])$ i
- 4 $\vdash \forall P \forall Q \forall x (P[x] \ \& \ Q[x] \Rightarrow (P \cap Q)[x])$ i
- 5 $\vdash \forall P \forall Q \forall x ((P \cap Q)[x] \Rightarrow P[x])$ i
- 6 $\vdash \forall P \forall Q \forall x ((P \cap Q)[x] \Rightarrow Q[x])$ i
- 7 $\vdash \forall P \forall Q \forall x (\neg (P \cap Q)[x] \ \& \ P[x] \Rightarrow \neg Q[x])$ i
- 8 $\vdash \forall P \forall Q \forall x (\neg (P \cap Q)[x] \ \& \ Q[x] \Rightarrow \neg P[x])$ i
- 9 $\vdash \forall P \forall Q \forall x (\neg (P \cap Q)[x] \Rightarrow \neg P[x] \vee \neg Q[x])$ i
- 10 $\vdash \forall P \forall Q (P \cap Q) \subseteq P$ i
- 11 $\vdash \forall P \forall Q (Q \cap P) \subseteq P$ i
- 12 $\vdash \forall P \forall Q \forall R (P \subseteq Q \ \& \ P \subseteq R \Rightarrow P \subseteq (Q \cap R))$ i
- 13 $\vdash \forall P \forall Q (P \cap Q) \subseteq (Q \cap P)$ i
- 14 $\vdash \forall P \forall Q (P \cap Q) \equiv (Q \cap P)$ i
- 15 $\vdash \forall P \forall Q \forall R ((P \cap Q) \subseteq R \Rightarrow (Q \cap P) \subseteq R)$ i
- 16 $\vdash \forall P \forall Q \forall R (R \subseteq (P \cap Q) \Rightarrow R \subseteq (Q \cap P))$ i
- 17 $\vdash \forall P \forall Q \forall R ((P \cap Q) \equiv R \Rightarrow (Q \cap P) \equiv R)$ i
- 18 $\vdash \forall P \forall Q \forall R (R \equiv (P \cap Q) \Rightarrow R \equiv (Q \cap P))$ i
- 19 $\vdash \forall P \forall Q \forall R (\neg (P \cap Q) \equiv R \Rightarrow \neg (Q \cap P) \equiv R)$ i
- 20 $\vdash \forall P \forall Q \forall R (\neg R \equiv (P \cap Q) \Rightarrow \neg R \equiv (Q \cap P))$ i
- 21 $\vdash \forall P \forall Q \forall R (P \subseteq Q \Rightarrow (P \cap R) \subseteq Q)$ i
- 22 $\vdash \forall P \forall Q \forall R (P \subseteq Q \Rightarrow (R \cap P) \subseteq Q)$ i
- 23 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow P \subseteq (P \cap Q))$ i
- 24 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow P \subseteq (Q \cap P))$ i
- 25 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (P \cap Q) \equiv P)$ i
- 26 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (Q \cap P) \equiv P)$ i
- 27 $\vdash \forall P (P \cap P) \equiv P$ i
- 28 $\vdash \forall P \forall Q ((P \cap Q) \equiv Q \Rightarrow Q \subseteq P)$ i
- 29 $\vdash \forall P \forall Q ((P \cap Q) \equiv P \Rightarrow P \subseteq Q)$ i

- 30 $\vdash \forall P \forall Q \forall R \forall S (P \subseteq Q \ \& \ R \subseteq S \Rightarrow (P \cap R) \subseteq (Q \cap S))$ i
- 31 $\vdash \forall P \forall Q \forall R (P \subseteq Q \Rightarrow (P \cap R) \subseteq (Q \cap R))$ i
- 32 $\vdash \forall P \forall Q \forall R (P \subseteq Q \Rightarrow (R \cap P) \subseteq (R \cap Q))$ i
- 33 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ R \equiv S \Rightarrow (P \cap R) \equiv (Q \cap S))$ i
- 34 $\vdash \forall P \forall Q \forall R (P \equiv Q \Rightarrow (P \cap R) \equiv (Q \cap R))$ i
- 35 $\vdash \forall P \forall Q \forall R (P \equiv Q \Rightarrow (R \cap P) \equiv (R \cap Q))$ i
- 36 $\vdash \forall P \forall Q \forall R \forall S \forall T (P \equiv Q \ \& \ R \equiv S \ \& \ T \equiv (P \cap R) \Rightarrow T \equiv (Q \cap S))$ i
- 37 $\vdash \forall P \forall Q \forall R \forall S \forall T (P \equiv Q \ \& \ R \equiv S \ \& \ (P \cap R) \equiv T \Rightarrow (Q \cap S) \equiv T)$ i
- 38 $\vdash \forall P \forall Q \forall R \forall S \forall T (Q \equiv P \ \& \ S \equiv R \ \& \ T \equiv (P \cap R) \Rightarrow T \equiv (Q \cap S))$ i
- 39 $\vdash \forall P \forall Q \forall R \forall S \forall T (Q \equiv P \ \& \ S \equiv R \ \& \ (P \cap R) \equiv T \Rightarrow (Q \cap S) \equiv T)$ i
- 40 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ (P \cap R) \equiv S \Rightarrow (Q \cap R) \equiv S)$ i
- 41 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ S \equiv (P \cap R) \Rightarrow S \equiv (Q \cap R))$ i
- 42 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ (R \cap P) \equiv S \Rightarrow (R \cap Q) \equiv S)$ i
- 43 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \ \& \ S \equiv (R \cap P) \Rightarrow S \equiv (R \cap Q))$ i
- 44 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ (P \cap R) \equiv S \Rightarrow (Q \cap R) \equiv S)$ i
- 45 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ S \equiv (P \cap R) \Rightarrow S \equiv (Q \cap R))$ i
- 46 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ (R \cap P) \equiv S \Rightarrow (R \cap Q) \equiv S)$ i
- 47 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ S \equiv (R \cap P) \Rightarrow S \equiv (R \cap Q))$ i
- 48 $\vdash \forall P \forall Q \forall R ((P \cap Q) \cap R \subseteq (P \cap (Q \cap R)))$ i
- 49 $\vdash \forall P \forall Q \forall R (P \cap (Q \cap R) \subseteq ((P \cap Q) \cap R))$ i
- 50 $\vdash \forall P \forall Q \forall R ((P \cap Q) \cap R \equiv (P \cap (Q \cap R)))$ i
- 51 $\vdash \forall P \forall Q \forall R (P \cap (Q \cup R) \subseteq ((P \cap Q) \cup (P \cap R)))$ i
- 52 $\vdash \forall P \forall Q \forall R ((P \cap Q) \cup (P \cap R) \subseteq (P \cap (Q \cup R)))$ i
- 53 $\vdash \forall P \forall Q \forall R (P \cap (Q \cup R) \equiv ((P \cap Q) \cup (P \cap R)))$ i
- 54 $\vdash \forall P \forall Q \forall R ((P \cup Q) \cap R \equiv ((P \cap R) \cup (Q \cap R)))$ i
- 55 $\vdash \forall P \forall Q \forall R (P \cup (Q \cap R) \subseteq ((P \cup Q) \cap (P \cup R)))$ i
- 56 $\vdash \forall P \forall Q \forall R ((P \cup Q) \cap (P \cup R) \subseteq (P \cup (Q \cap R)))$ i
- 57 $\vdash \forall P \forall Q \forall R (P \cup (Q \cap R) \equiv ((P \cup Q) \cap (P \cup R)))$ i
- 58 $\vdash \forall P \forall Q \forall R ((P \cap Q) \cup R \equiv ((P \cup R) \cap (Q \cup R)))$ i

! CHAPTER 4 COMPLEMENTS ;

- 1 $\mathbb{D} \quad \mathcal{C} \ ; \ (P^{\mathcal{C}}) \ ; \ ; \ \{ a : \neg P[a] \}$ i
- 2 $\vdash \forall P \forall x ((P^{\mathcal{C}}) [x] \Leftrightarrow \neg P[x])$ i
- 3 $\vdash \forall P \forall x ((P^{\mathcal{C}}) [x] \Rightarrow \neg P[x])$ i
- 4 $\vdash \forall P \forall x (\neg P[x] \Rightarrow (P^{\mathcal{C}}) [x])$ i
- 5 $\vdash \forall P \forall x (P[x] \Rightarrow \neg (P^{\mathcal{C}}) [x])$ i
- 6 $\vdash \forall P \forall x (\neg (P^{\mathcal{C}}) [x] \Rightarrow P[x])$ i
- 7 $\vdash \forall P \forall x (\neg (P^{\mathcal{C}}) [x] \Leftrightarrow P[x])$ i
- 8 $\vdash \forall P \forall x \neg (P[x] \ \& \ (P^{\mathcal{C}}) [x])$ i
- 9 $\vdash \forall P \forall x \neg (P \cap (P^{\mathcal{C}})) [x]$ i
- 10 $\vdash \forall P \forall x (P[x] \vee (P^{\mathcal{C}}) [x])$ i
- 11 $\vdash \forall P \forall x (P \cup (P^{\mathcal{C}})) [x]$ i
- 12 $\vdash \forall P \forall Q ((P^{\mathcal{C}}) \subseteq Q \Rightarrow (Q^{\mathcal{C}}) \subseteq P)$ i
- 13 $\vdash \forall P \forall Q (P \subseteq (Q^{\mathcal{C}}) \Rightarrow Q \subseteq (P^{\mathcal{C}}))$ i

- 14 $\vdash \forall P \forall Q ((P^c) \equiv Q \Rightarrow (Q^c) \equiv P)$ i
- 15 $\vdash \forall P \forall Q ((P^c) \equiv Q \Rightarrow P \equiv (Q^c))$ i
- 16 $\vdash \forall P \forall Q (P \equiv (Q^c) \Rightarrow Q \equiv (P^c))$ i
- 17 $\vdash \forall P \forall Q (P \equiv (Q^c) \Rightarrow (P^c) \equiv Q)$ i
- 18 $\vdash \forall P ((P^c)^c) \subseteq P$ i
- 19 $\vdash \forall P P \subseteq ((P^c)^c)$ i
- 20 $\vdash \forall P ((P^c)^c) \equiv P$ i
- 21 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (Q^c) \subseteq (P^c))$ i
- 22 $\vdash \forall P \forall Q ((P^c) \subseteq (Q^c) \Rightarrow Q \subseteq P)$ i
- 23 $\vdash \forall P \forall Q (P \equiv Q \Rightarrow (P^c) \equiv (Q^c))$ i
- 24 $\vdash \forall P \forall Q ((P^c) \equiv (Q^c) \Rightarrow P \equiv Q)$ i
- 25 $\vdash \forall P \forall Q (P \equiv Q \Leftrightarrow (P^c) \equiv (Q^c))$ i
- 26 $\vdash \forall P \forall Q ((P \cap Q)^c) \subseteq ((P^c) \cup (Q^c))$ i
- 27 $\vdash \forall P \forall Q ((P^c) \cup (Q^c)) \subseteq ((P \cap Q)^c)$ i
- 28 $\vdash \forall P \forall Q ((P \cap Q)^c) \equiv ((P^c) \cup (Q^c))$ i
- 29 $\vdash \forall P \forall Q ((P \cup Q)^c) \subseteq ((P^c) \cap (Q^c))$ i
- 30 $\vdash \forall P \forall Q ((P^c) \cap (Q^c)) \subseteq ((P \cup Q)^c)$ i
- 31 $\vdash \forall P \forall Q ((P \cup Q)^c) \equiv ((P^c) \cap (Q^c))$ i

!CHAPTER 5 AN EMPTY PREDICATE;

- 1 $\mathbb{D} \phi ; \phi ; ; \{a : \neg a = a\}$ i
- 2 $\vdash \forall x (\phi[x] \Leftrightarrow \neg x = x)$ i
- 3 $\vdash \forall x \neg \phi[x]$ i
- 4 $\vdash \neg \phi[0]$ i
- 5 $\vdash \forall P (P \equiv \phi \Rightarrow \forall x \neg P[x])$ i
- 6 $\vdash \forall P (P \equiv \phi \Rightarrow \neg \exists x P[x])$ i
- 7 $\vdash \forall P (\exists x P[x] \Rightarrow \neg P \equiv \phi)$ i
- 8 $\vdash \forall P \forall Q (P \equiv \phi \Rightarrow P \subseteq Q)$ i
- 9 $\vdash \forall P \phi \subseteq P$ i
- 10 $\vdash \forall P (P \subseteq \phi \Rightarrow P \equiv \phi)$ i
- 11 $\vdash \forall P \forall Q (Q \equiv \phi \ \& \ P \subseteq Q \Rightarrow P \equiv \phi)$ i
- 12 $\vdash \forall P \forall Q (\neg P \equiv \phi \ \& \ P \subseteq Q \Rightarrow \neg Q \equiv \phi)$ i
- 13 $\vdash \forall P \forall Q (P \subset Q \Rightarrow \neg Q \equiv \phi)$ i
- 14 $\vdash \forall P \neg P \subset \phi$ i
- 15 $\vdash \forall P (\neg \exists x P[x] \Rightarrow P \equiv \phi)$ i
- 16 $\vdash \forall P (\neg P \equiv \phi \Rightarrow \exists x P[x])$ i
- 17 $\vdash \forall P (\forall x \neg P[x] \Rightarrow P \equiv \phi)$ i
- 18 $\vdash \forall P (P \equiv \phi \Leftrightarrow \forall x \neg P[x])$ i
- 19 $\vdash \forall P (P \cup \phi) \equiv P$ i
- 20 $\vdash \forall P (\phi \cup P) \equiv P$ i
- 21 $\vdash \forall P \forall Q (P \equiv \phi \Rightarrow (P \cup Q) \equiv Q)$ i
- 22 $\vdash \forall P \forall Q (P \equiv \phi \Rightarrow (Q \cup P) \equiv Q)$ i
- 23 $\vdash \forall P \forall Q \forall x (P[x] \ \& \ (P \cap Q) \equiv \phi \Rightarrow \neg Q[x])$ i
- 24 $\vdash \forall P \forall Q \forall x (Q[x] \ \& \ (P \cap Q) \equiv \phi \Rightarrow \neg P[x])$ i
- 25 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Rightarrow \neg \exists x (P[x] \ \& \ Q[x]))$ i
- 26 $\vdash \forall P \forall Q (\exists x (P[x] \ \& \ Q[x]) \Rightarrow \neg (P \cap Q) \equiv \phi)$ i
- 27 $\vdash \forall P \forall Q (\forall x (P[x] \ \& \ Q[x] \Rightarrow \mathfrak{F}) \Rightarrow (P \cap Q) \equiv \phi)$ i

- 28 $\vdash \forall P \forall Q (\neg \exists x (P[x] \& Q[x]) \Rightarrow (P \cap Q) \equiv \phi)$ i
 29 $\vdash \forall P \forall Q (\neg (P \cap Q) \equiv \phi \Rightarrow \exists x (P[x] \& Q[x]))$ i
 30 $\vdash \forall P (P \cap \phi) \equiv \phi$ i
 31 $\vdash \forall P (\phi \cap P) \equiv \phi$ i
 32 $\vdash \forall P \forall Q (P \equiv \phi \Rightarrow (P \cap Q) \equiv \phi)$ i
 33 $\vdash \forall P \forall Q (P \equiv \phi \Rightarrow (Q \cap P) \equiv \phi)$ i
 34 $\vdash \forall P \forall Q (\neg (P \cap Q) \equiv \phi \Rightarrow \neg P \equiv \phi)$ i
 35 $\vdash \forall P \forall Q (\neg (Q \cap P) \equiv \phi \Rightarrow \neg P \equiv \phi)$ i
 36 $\vdash \forall P \forall Q \forall R \forall S ((P \cap Q) \equiv \phi \& R \subseteq P \& S \subseteq Q \Rightarrow (R \cap S) \equiv \phi)$ i
 37 $\vdash \forall P \forall Q \forall R ((P \cap Q) \equiv \phi \& R \subseteq P \Rightarrow (R \cap Q) \equiv \phi)$ i
 38 $\vdash \forall P \forall Q \forall R ((P \cap Q) \equiv \phi \& R \subseteq Q \Rightarrow (P \cap R) \equiv \phi)$ i
 39 $\vdash \forall P \forall Q \forall R ((P \cap Q) \equiv \phi \& (P \cap R) \equiv \phi \Rightarrow (P \cap (Q \cup R)) \equiv \phi)$ i
 40 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Rightarrow P \subseteq (Q^c))$ i
 41 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Rightarrow Q \subseteq (P^c))$ i
 42 $\vdash \forall P (P \cap (P^c)) \equiv \phi$ i
 43 $\vdash \forall P \forall Q \forall R (Q \subseteq P \& R \subseteq (P^c) \Rightarrow (Q \cap R) \equiv \phi)$ i
 44 $\vdash \forall P \forall Q \forall R (Q \subseteq P \& R \subseteq (P^c) \Rightarrow (R \cap Q) \equiv \phi)$ i
 45 $\vdash \forall P \forall Q (Q \subseteq (P^c) \Rightarrow (P \cap Q) \equiv \phi)$ i
 46 $\vdash \forall P \forall Q (P \subseteq (Q^c) \Rightarrow (P \cap Q) \equiv \phi)$ i
 47 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Leftrightarrow Q \subseteq (P^c))$ i
 48 $\vdash \forall P \forall Q (Q \subseteq P \Rightarrow ((P^c) \cap Q) \equiv \phi)$ i
 49 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (P \cap (Q^c)) \equiv \phi)$ i

!CHAPTER 6 A UNIVERSAL PREDICATE;

- 1 $\mathbb{D} \quad \mathbf{U} ; \mathbf{U} ; ; \{a : a = a\}$ i
 2 $\vdash \forall x (\mathbf{U}[x] \Leftrightarrow x = x)$ i
 3 $\vdash \forall x \mathbf{U}[x]$ i
 4 $\vdash \forall P (P \equiv \mathbf{U} \Rightarrow \forall x P[x])$ i
 5 $\vdash (\mathbf{U}^c) \equiv \phi$ i
 6 $\vdash (\phi^c) \equiv \mathbf{U}$ i
 7 $\vdash \forall P P \subseteq \mathbf{U}$ i
 8 $\vdash \forall P (\mathbf{U} \subseteq P \Rightarrow P \equiv \mathbf{U})$ i
 9 $\vdash \forall P (\forall x P[x] \Rightarrow P \equiv \mathbf{U})$ i
 10 $\vdash \forall P (\neg \exists x \neg P[x] \Rightarrow P \equiv \mathbf{U})$ i
 11 $\vdash \forall P (\neg P \equiv \mathbf{U} \Rightarrow \exists x \neg P[x])$ i
 12 $\vdash \forall P (P \cup \mathbf{U}) \equiv \mathbf{U}$ i
 13 $\vdash \forall P (\mathbf{U} \cup P) \equiv \mathbf{U}$ i
 14 $\vdash \forall P (P \cap \mathbf{U}) \equiv P$ i
 15 $\vdash \forall P (\mathbf{U} \cap P) \equiv P$ i
 16 $\vdash \forall P (P \cup (P^c)) \equiv \mathbf{U}$ i

!CHAPTER 7 DIFFERENCES;

- 1 $\mathbb{D} \quad \setminus ; (P \setminus Q) ; ; (P \cap (Q^c))$ i
 2 $\vdash \forall P \forall Q \forall x ((P \setminus Q)[x] \Leftrightarrow P[x] \& \neg Q[x])$ i
 3 $\vdash \forall P \forall Q \forall x ((P \setminus Q)[x] \Rightarrow P[x] \& \neg Q[x])$ i
 4 $\vdash \forall P \forall Q \forall x (P[x] \& \neg Q[x] \Rightarrow (P \setminus Q)[x])$ i
 5 $\vdash \forall P \forall Q \forall x ((P \setminus Q)[x] \Rightarrow P[x])$ i

- 6 $\vdash \forall P \forall Q \forall x ((P \setminus Q)[x] \Rightarrow \neg Q[x])$ i
- 7 $\vdash \forall P \forall Q \forall x (\neg P[x] \Rightarrow \neg (P \setminus Q)[x])$ i
- 8 $\vdash \forall P \forall Q \forall x (Q[x] \Rightarrow \neg (P \setminus Q)[x])$ i
- 9 $\vdash \forall P \forall Q \forall x (P[x] \& \neg (P \setminus Q)[x] \Rightarrow Q[x])$ i
- 10 $\vdash \forall P \forall Q \forall x (P[x] \Rightarrow (P \setminus Q)[x] \vee Q[x])$ i
- 11 $\vdash \forall P \forall Q (P \setminus Q \equiv (P \cap (Q^c)))$ i
- 12 $\vdash \forall P \forall Q (P \setminus Q \equiv ((Q^c) \cap P))$ i
- 13 $\vdash \forall P \forall Q (P \setminus Q \subseteq P)$ i
- 14 $\vdash \forall P \forall Q (P \setminus Q \subseteq (Q^c))$ i
- 15 $\vdash \forall P \forall Q \forall R (P \subseteq R \Rightarrow (P \setminus Q) \subseteq R)$ i
- 16 $\vdash \forall P \forall Q \forall R ((Q^c) \subseteq R \Rightarrow (P \setminus Q) \subseteq R)$ i
- 17 $\vdash \forall P \forall Q \forall R ((R^c) \subseteq Q \Rightarrow (P \setminus Q) \subseteq R)$ i
- 18 $\vdash \forall P \forall Q \forall R (R \subseteq Q \Rightarrow (P \setminus Q) \subseteq (R^c))$ i
- 19 $\vdash \forall P \forall Q \forall R (R \subseteq (P \setminus Q) \Rightarrow R \subseteq P)$ i
- 20 $\vdash \forall P \forall Q \forall R (R \subseteq (P \setminus Q) \Rightarrow R \subseteq (Q^c))$ i
- 21 $\vdash \forall P \forall Q \forall R (R \subseteq (P \setminus Q) \Rightarrow Q \subseteq (R^c))$ i
- 22 $\vdash \forall P \forall Q \forall R (R \subseteq (P \setminus Q) \Rightarrow (Q \cap R) \equiv \phi)$ i
- 23 $\vdash \forall P \forall Q \forall R (R \subseteq (P \setminus Q) \Rightarrow (R \cap Q) \equiv \phi)$ i
- 24 $\vdash \forall P \forall Q \forall R (R \subseteq P \& R \subseteq (Q^c) \Rightarrow R \subseteq (P \setminus Q))$ i
- 25 $\vdash \forall P \forall Q \forall R (R \subseteq P \& Q \subseteq (R^c) \Rightarrow R \subseteq (P \setminus Q))$ i
- 26 $\vdash \forall P \forall Q \forall R (R \subseteq P \& Q \subseteq (P \setminus R) \Rightarrow R \subseteq (P \setminus Q))$ i
- 27 $\vdash \forall P \forall Q \forall R \forall S (P \subseteq R \& S \subseteq Q \Rightarrow (P \setminus Q) \subseteq (R \setminus S))$ i
- 28 $\vdash \forall P \forall Q \forall R (P \subseteq R \Rightarrow (P \setminus Q) \subseteq (R \setminus Q))$ i
- 29 $\vdash \forall P \forall Q \forall R (R \subseteq Q \Rightarrow (P \setminus Q) \subseteq (P \setminus R))$ i
- 30 $\vdash \forall P \forall Q \forall R \forall S (P \equiv R \& Q \equiv S \Rightarrow (P \setminus S) \equiv (R \setminus Q))$ i
- 31 $\vdash \forall P \forall Q \forall R \forall S (P \equiv R \& Q \equiv S \Rightarrow (P \setminus Q) \equiv (R \setminus S))$ i
- 32 $\vdash \forall P \forall Q \forall R \forall S (P \equiv R \& Q \equiv S \Rightarrow (R \setminus S) \equiv (P \setminus Q))$ i
- 33 $\vdash \forall P \forall Q \forall R \forall S (P \equiv R \& Q \equiv S \Rightarrow (R \setminus Q) \equiv (P \setminus S))$ i
- 34 $\vdash \forall P \forall Q \forall R (P \equiv R \Rightarrow (P \setminus Q) \equiv (R \setminus Q))$ i
- 35 $\vdash \forall P \forall Q \forall R (P \equiv R \Rightarrow (R \setminus Q) \equiv (P \setminus Q))$ i
- 36 $\vdash \forall P \forall Q \forall R (Q \equiv R \Rightarrow (P \setminus R) \equiv (P \setminus Q))$ i
- 37 $\vdash \forall P \forall Q \forall R (Q \equiv R \Rightarrow (P \setminus Q) \equiv (P \setminus R))$ i
- 38 $\vdash \forall P \forall Q \forall R \forall S \forall T (P \equiv Q \& R \equiv S \& T \equiv (P \setminus R) \Rightarrow T \equiv (Q \setminus S))$ i
- 39 $\vdash \forall P \forall Q \forall R \forall S \forall T (P \equiv Q \& R \equiv S \& (P \setminus R) \equiv T \Rightarrow (Q \setminus S) \equiv T)$ i
- 40 $\vdash \forall P \forall Q \forall R \forall S \forall T (Q \equiv P \& S \equiv R \& T \equiv (P \setminus R) \Rightarrow T \equiv (Q \setminus S))$ i
- 41 $\vdash \forall P \forall Q \forall R \forall S \forall T (Q \equiv P \& S \equiv R \& (P \setminus R) \equiv T \Rightarrow (Q \setminus S) \equiv T)$ i
- 42 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \& (P \setminus R) \equiv S \Rightarrow (Q \setminus R) \equiv S)$ i
- 43 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \& S \equiv (P \setminus R) \Rightarrow S \equiv (Q \setminus R))$ i
- 44 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \& (R \setminus P) \equiv S \Rightarrow (R \setminus Q) \equiv S)$ i
- 45 $\vdash \forall P \forall Q \forall R \forall S (P \equiv Q \& S \equiv (R \setminus P) \Rightarrow S \equiv (R \setminus Q))$ i
- 46 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \& (P \setminus R) \equiv S \Rightarrow (Q \setminus R) \equiv S)$ i
- 47 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \& S \equiv (P \setminus R) \Rightarrow S \equiv (Q \setminus R))$ i
- 48 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \& (R \setminus P) \equiv S \Rightarrow (R \setminus Q) \equiv S)$ i

- 49 $\vdash \forall P \forall Q \forall R \forall S (Q \equiv P \ \& \ S \equiv (R \setminus P) \Rightarrow S \equiv (R \setminus Q))$ i
- 50 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow (P \setminus Q) \equiv \phi)$ i
- 51 $\vdash \forall P (P \setminus P) \equiv \phi$ i
- 52 $\vdash \forall P \forall Q (P \equiv \phi \Rightarrow (P \setminus Q) \equiv \phi)$ i
- 53 $\vdash \forall P \forall Q ((P \setminus Q) \equiv \phi \Rightarrow P \subseteq Q)$ i
- 54 $\vdash \forall P \forall Q (P \subseteq Q \Leftrightarrow (P \setminus Q) \equiv \phi)$ i
- 55 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Rightarrow (P \setminus Q) \equiv P)$ i
- 56 $\vdash \forall P (P \setminus \phi) \equiv P$ i
- 57 $\vdash \forall P \forall Q (Q \equiv \phi \Rightarrow (P \setminus Q) \equiv P)$ i
- 58 $\vdash \forall P \forall Q ((P \setminus Q) \equiv P \Rightarrow (P \cap Q) \equiv \phi)$ i
- 59 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Leftrightarrow (P \setminus Q) \equiv P)$ i
- 60 $\vdash \forall P \forall Q \forall R ((P \setminus Q) \cup R) \equiv ((P \cup R) \setminus (Q \cap (R^c)))$ i
- 61 $\vdash \forall P \forall Q \forall R ((P \setminus Q) \cup R) \equiv ((P \cup R) \setminus (Q \setminus R))$ i
- 62 $\vdash \forall P \forall Q \forall R (Q \subseteq R \Rightarrow ((P \setminus Q) \cup R) \equiv (P \cup R))$ i
- 63 $\vdash \forall P \forall Q ((P \setminus Q) \cup Q) \equiv (P \cup Q)$ i
- 64 $\vdash \forall P \forall Q P \subseteq ((P \setminus Q) \cup Q)$ i
- 65 $\vdash \forall P \forall Q (Q \subseteq P \Rightarrow ((P \setminus Q) \cup Q) \equiv P)$ i
- 66 $\vdash \forall P \forall Q \forall R (R \subseteq P \Rightarrow ((P \setminus Q) \cup R) \equiv (P \setminus (Q \setminus R)))$ i
- 67 $\vdash \forall P \forall Q \forall R ((P \cup Q) \setminus R) \equiv ((P \setminus R) \cup (Q \setminus R))$ i
- 68 $\vdash \forall P \forall Q \forall R ((P \cup Q) \setminus R) \equiv ((Q \cup P) \setminus R)$ i
- 69 $\vdash \forall P \forall Q \forall R (P \subseteq R \Rightarrow ((P \cup Q) \setminus R) \subseteq Q)$ i
- 70 $\vdash \forall P \forall Q \forall R (Q \subseteq R \Rightarrow ((P \cup Q) \setminus R) \subseteq P)$ i
- 71 $\vdash \forall P \forall Q ((P \cup Q) \setminus P) \subseteq Q$ i
- 72 $\vdash \forall P \forall Q ((P \cup Q) \setminus Q) \subseteq P$ i
- 73 $\vdash \forall P \forall Q \forall R ((Q \cap R) \equiv \phi \Rightarrow ((P \cup Q) \setminus R) \equiv ((P \setminus R) \cup Q))$ i
- 74 $\vdash \forall P \forall Q \forall R ((P \cap R) \equiv \phi \Rightarrow ((P \cup Q) \setminus R) \equiv (P \cup (Q \setminus R)))$ i
- 75 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Rightarrow ((P \cup Q) \setminus Q) \equiv P)$ i
- 76 $\vdash \forall P \forall Q ((P \cap Q) \equiv \phi \Rightarrow ((P \cup Q) \setminus P) \equiv Q)$ i
- 77 $\vdash \forall P \forall Q \forall R ((P \cap Q) \equiv \phi \ \& \ (P \cup Q) \equiv R \Rightarrow (R \setminus Q) \equiv P)$ i
- 78 $\vdash \forall P \forall Q \forall R ((P \cap Q) \equiv \phi \ \& \ (P \cup Q) \equiv R \Rightarrow (R \setminus P) \equiv Q)$ i
- 79 $\vdash \forall P \forall Q \forall R (P \setminus (Q \cup R)) \equiv ((P \setminus Q) \setminus R)$ i
- 80 $\vdash \forall P \forall Q \forall R (R \subseteq Q \Rightarrow ((P \setminus Q) \cap R) \equiv \phi)$ i
- 81 $\vdash \forall P \forall Q ((P \setminus Q) \cap Q) \equiv \phi$ i
- 82 $\vdash \forall P \forall Q \forall R ((P \setminus Q) \cap R) \equiv (P \cap (R \setminus Q))$ i
- 83 $\vdash \forall P \forall Q \forall R ((P \setminus Q) \cap R) \equiv ((P \cap R) \setminus Q)$ i
- 84 $\vdash \forall P \forall Q \forall R (P \cap (R \setminus Q)) \equiv ((P \cap R) \setminus Q)$ i
- 85 $\vdash \forall P \forall Q \forall R ((P \cap Q) \setminus R) \subseteq ((P \setminus R) \cap (Q \setminus R))$ i
- 86 $\vdash \forall P \forall Q \forall R ((P \setminus R) \cap (Q \setminus R)) \subseteq ((P \cap Q) \setminus R)$ i
- 87 $\vdash \forall P \forall Q \forall R ((P \cap Q) \setminus R) \equiv ((P \setminus R) \cap (Q \setminus R))$ i
- 88 $\vdash \forall P \forall Q (P \subseteq Q \Rightarrow P \equiv (Q \setminus (Q \setminus P)))$ i
- 89 $\vdash \forall P \forall Q \forall R ((P \setminus Q) \setminus R) \equiv ((P \setminus R)) \setminus Q)$ i
- 90 $\vdash \forall P \forall Q \forall R (R \subseteq Q \Rightarrow (P \setminus Q) \equiv ((P \setminus (Q \setminus R)) \setminus R))$ i

!CHAPTER 8 SINGLETON PREDICATES;

- 1 $\mathbb{D} \quad \bullet \quad ; \quad (a^\bullet) \quad ; \quad ; \quad \{ a : a = a \}$ i
- 2 $\vdash \forall a \forall x ((a^\bullet) [x] \Leftrightarrow x = a)$ i
- 3 $\vdash \forall a \forall x ((a^\bullet) [x] \Rightarrow x = a)$ i

- 4 $\vdash \forall a \forall x (x = a \Rightarrow (a^\bullet)[x])$ i
- 5 $\vdash \forall a (a^\bullet)[a]$ i
- 6 $\vdash \forall x \forall y (\forall P (P[x] \Leftrightarrow P[y]) \Rightarrow x = y)$ i
- 7 $\vdash \forall a \forall x (\neg x = a \Rightarrow \neg (a^\bullet)[x])$ i
- 8 $\vdash \forall a \forall x (\neg a = x \Rightarrow \neg (a^\bullet)[x])$ i
- 9 $\vdash \forall a \forall x (\neg (a^\bullet)[x] \Rightarrow \neg x = a)$ i
- 10 $\vdash \forall a \forall x (\neg (a^\bullet)[x] \Rightarrow \neg a = x)$ i
- 11 $\vdash \forall a \neg (a^\bullet) \equiv \phi$ i
- 12 $\vdash \forall P \forall a ((a^\bullet) \subseteq P \Rightarrow P[a])$ i
- 13 $\vdash \forall P \forall a (P[a] \Rightarrow (a^\bullet) \subseteq P)$ i
- 14 $\vdash \forall P \forall Q \forall a (Q[a] \ \& \ Q \subseteq P \Rightarrow (a^\bullet) \subseteq P)$ i
- 15 $\vdash \forall P \forall a \forall x (P[x] \ \& \ P \subseteq (a^\bullet) \Rightarrow x = a)$ i
- 16 $\vdash \forall P \forall a (\forall x (P[x] \Rightarrow x = a) \Leftrightarrow P \subseteq (a^\bullet))$ i
- 17 $\vdash \forall P \forall a (P \equiv (a^\bullet) \Rightarrow P[a])$ i
- 18 $\vdash \forall P \forall a \forall x (P[x] \ \& \ P \equiv (a^\bullet) \Rightarrow x = a)$ i
- 19 $\vdash \forall P \forall a (P \equiv (a^\bullet) \Rightarrow \forall x (P[x] \Leftrightarrow x = a))$ i
- 20 $\vdash \forall P \forall a (P \equiv (a^\bullet) \Rightarrow P[a] \ \& \ \forall x (P[x] \Rightarrow x = a))$ i
- 21 $\vdash \forall P \forall a (P[a] \ \& \ P \subseteq (a^\bullet) \Rightarrow P \equiv (a^\bullet))$ i
- 22 $\vdash \forall P \forall a (P[a] \ \& \ \forall x (P[x] \Rightarrow x = a) \Rightarrow P \equiv (a^\bullet))$ i
- 23 $\vdash \forall P \forall a (P \subseteq (a^\bullet) \Rightarrow P \equiv \phi \vee P \equiv (a^\bullet))$ i
- 24 $\vdash \forall a \forall b (a = b \Rightarrow (a^\bullet) \equiv (b^\bullet))$ i
- 25 $\vdash \forall a \forall b ((a^\bullet) \equiv (b^\bullet) \Rightarrow a = b)$ i
- 26 $\vdash \forall a \forall x (((a^\bullet)^c)[x] \Leftrightarrow \neg x = a)$ i
- 27 $\vdash \forall P \forall a (\neg P[a] \Rightarrow (a^\bullet) \subseteq (P^c))$ i
- 28 $\vdash \forall P \forall a \forall x ((P \cup (a^\bullet))[x] \Leftrightarrow P[x] \vee x = a)$ i
- 29 $\vdash \forall P \forall Q \forall a (\forall x (Q[x] \Leftrightarrow P[x] \vee x = a) \Rightarrow Q \equiv (P \cup (a^\bullet)))$ i
- 30 $\vdash \forall P \forall Q \forall a (Q \equiv (P \cup (a^\bullet)) \Rightarrow \forall x (Q[x] \Leftrightarrow P[x] \vee x = a))$ i
- 31 $\vdash \forall P \forall a \{x : P[x] \vee x = a\} \equiv (P \cup (a^\bullet))$ i
- 32 $\vdash \forall P \forall a (P \cup (a^\bullet)) [a]$ i
- 33 $\vdash \forall P \forall a ((a^\bullet) \cup P) [a]$ i
- 34 $\vdash \forall P \forall Q \forall a (Q \equiv (P \cup (a^\bullet)) \Rightarrow Q[a])$ i
- 35 $\vdash \forall P \forall a (P[a] \Rightarrow (P \cup (a^\bullet)) \equiv P)$ i
- 36 $\vdash \forall P \forall a ((P \cup (a^\bullet)) \equiv P \vee \neg P[a])$ i
- 37 $\vdash \forall P \forall a \neg (P \cup (a^\bullet)) \equiv \phi$ i
- 38 $\vdash \forall P \forall Q (Q \subset P \Rightarrow \exists a (\neg Q[a] \ \& \ (Q \cup (a^\bullet)) \subseteq P))$ i
- 39 $\vdash \forall P \forall Q \forall a (P \subseteq Q \ \& \ Q \subseteq (P \cup (a^\bullet))$
 $\Rightarrow Q \equiv P \vee Q \equiv (P \cup (a^\bullet)))$ i
- 40 $\vdash \forall P \forall a (\neg P[a] \Rightarrow (P \cap (a^\bullet)) \equiv \phi)$ i
- 41 $\vdash \forall P \forall Q \forall a (\neg P[a] \ \& \ Q \subseteq P \Rightarrow (Q \cap (a^\bullet)) \equiv \phi)$ i
- 42 $\vdash \forall P \forall Q \forall a (\neg P[a] \ \& \ Q \subseteq (a^\bullet) \Rightarrow (Q \cap P) \equiv \phi)$ i
- 43 $\vdash \forall a \forall b (\neg a = b \Rightarrow ((a^\bullet) \cap (b^\bullet)) \equiv \phi)$ i
- 44 $\vdash \forall P \forall a ((P \cap (a^\bullet)) \equiv \phi \Rightarrow \neg P[a])$ i
- 45 $\vdash \forall a \forall b (((a^\bullet) \cap (b^\bullet)) \equiv \phi \Rightarrow \neg a = b)$ i
- 46 $\vdash \forall a \forall b (\neg a = b \Leftrightarrow ((a^\bullet) \cap (b^\bullet)) \equiv \phi)$ i
- 47 $\vdash \forall P \forall a \neg (P \setminus (a^\bullet)) [a]$ i
- 48 $\vdash \forall P \forall a \forall x ((P \setminus (a^\bullet)) [x] \Leftrightarrow P[x] \ \& \ \neg x = a)$ i

- 49 $\vdash \forall P \forall Q \forall a (P[a] \& \neg Q[a] \Rightarrow (a^\bullet) \subseteq (P \setminus Q))$ i
- 50 $\vdash \forall P \forall a (\neg P[a] \Rightarrow (P \setminus (a^\bullet)) \equiv P)$ i
- 51 $\vdash \forall a \forall b (\neg a = b \Rightarrow ((a^\bullet) \setminus (b^\bullet)) \equiv (a^\bullet))$ i
- 52 $\vdash \forall a \forall b (\neg a = b \Rightarrow ((b^\bullet) \setminus (a^\bullet)) \equiv (b^\bullet))$ i
- 53 $\vdash \forall P \forall Q \forall a (\forall x (Q[x] \Leftrightarrow P[x] \& \neg x = a) \Rightarrow Q \equiv (P \setminus (a^\bullet)))$ i
- 54 $\vdash \forall P \forall Q \forall a (Q \equiv (P \setminus (a^\bullet)) \Rightarrow \forall x (Q[x] \Leftrightarrow P[x] \& \neg x = a))$ i
- 55 $\vdash \forall P \forall Q (Q \subseteq P \Rightarrow \exists a (P[a] \& Q \subseteq (P \setminus (a^\bullet))))$ i
- 56 $\vdash \forall P \forall a (P[a] \Rightarrow ((P \setminus (a^\bullet)) \cup (a^\bullet)) \equiv P)$ i
- 57 $\vdash \forall P \forall a (P[a] \Rightarrow P \equiv ((P \setminus (a^\bullet)) \cup (a^\bullet)))$ i
- 58 $\vdash \forall P \forall a (P[a] \Rightarrow \forall x (P[x] \Leftrightarrow (P \setminus (a^\bullet))[x] \vee x = a))$ i
- 59 $\vdash \forall P \forall a (\neg P[a] \Rightarrow P \equiv ((P \cup (a^\bullet)) \setminus (a^\bullet)))$ i
- 60 $\vdash \forall P \forall a (\neg P[a] \Rightarrow ((a^\bullet) \cup P) \setminus (a^\bullet) \equiv P)$ i
- 61 $\vdash \forall P \forall Q \forall a (Q \equiv (P \cup (a^\bullet)) \& \neg P[a] \Rightarrow P \equiv (Q \setminus (a^\bullet)))$ i
- 62 $\vdash \forall a \forall b (\neg a = b \Rightarrow (((a^\bullet) \cup (b^\bullet)) \setminus (a^\bullet)) \equiv (b^\bullet))$ i
- 63 $\vdash \forall a \forall b (\neg a = b \Rightarrow (((b^\bullet) \cup (a^\bullet)) \setminus (a^\bullet)) \equiv (b^\bullet))$ i
- 64 $\vdash \forall P \forall a \forall b (\neg a = b \& P[b]$
 $\Rightarrow ((P \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (P \setminus (a^\bullet)))$ i
- 65 $\vdash \forall P \forall a \forall b (\neg a = b \& P[a]$
 $\Rightarrow ((P \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet))$
 $\equiv (P \setminus (b^\bullet)))$ i
- 66 $\vdash \forall P \forall Q \forall a (Q[a] \& Q \subseteq P$
 $\Rightarrow ((P \setminus Q) \cup (a^\bullet)) \equiv (P \setminus (Q \setminus (a^\bullet))))$ i
- 67 $\vdash \forall P \forall Q \forall a (Q[a] \Rightarrow (P \cup Q) \equiv ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet)))$ i

!CHAPTER 9 DOUBLET, TRIPLET, AND QUADRUPLET PREDICATES;

- 1 $\mathbb{D} \ddagger ; (a \ b \ \ddagger) ; ; \{a : a = a \vee a = b\}$ i
- 2 $\vdash \forall a \forall b \forall x ((a \ b \ \ddagger)[x] \Leftrightarrow x = a \vee x = b)$ i
- 3 $\vdash \forall a \forall b (a \ b \ \ddagger) \equiv ((a^\bullet) \cup (b^\bullet))$ i
- 4 $\vdash \forall a \forall b \forall x (\neg x = a \& \neg x = b \Rightarrow \neg (a \ b \ \ddagger)[x])$ i
- 5 $\mathbb{D} \vee ; (a \ b \ c \ \vee) ; ; \{a : a = a \vee a = b \vee a = c\}$ i
- 6 $\vdash \forall a \forall b \forall c \forall x ((a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c)$ i
- 7 $\vdash \forall a \forall b \forall c (a \ b \ c \ \vee) \equiv ((a \ b \ \ddagger) \cup (c^\bullet))$ i
- 8 $\vdash \forall a \forall b \forall c \forall x (\neg x = a \& \neg x = b \& \neg x = c$
 $\Rightarrow \neg (a \ b \ c \ \vee)[x])$ i
- 9 $\mathbb{D} \forall ; (a \ b \ c \ d \ \forall) ; ; \{a : a = a \vee a = b \vee a = c \vee a = d\}$ i
- 10 $\vdash \forall a \forall b \forall c \forall d \forall x ((a \ b \ c \ d \ \forall)[x]$
 $\Leftrightarrow x = a \vee x = b \vee x = c \vee x = d)$ i
- 11 $\vdash \forall a \forall b \forall c \forall d (a \ b \ c \ d \ \forall) \equiv ((a \ b \ c \ \vee) \cup (d^\bullet))$ i

SECTION III: DYADIC AND TRIADIC PREDICATES

!CHAPTER 1 DYADIC INCLUSION AND EQUIVALENCE;

- 1 $\mathbb{S} \subseteq ; \mathbb{R} \subseteq \mathbb{S} ; \forall x \forall y (R[x,y] \Rightarrow S[x,y])$ i
- 2 $\vdash \forall R \forall S \forall x \forall y (R[x,y] \& R \subseteq S \Rightarrow S[x,y])$ i
- 3 $\vdash \forall R \ R \subseteq R$ i
- 4 $\vdash \forall R \forall S \forall T (R \subseteq S \& S \subseteq T \Rightarrow R \subseteq T)$ i
- 5 $\mathbb{S} \equiv ; \mathbb{R} \equiv \mathbb{S} ; \forall x \forall y (R[x,y] \Leftrightarrow S[x,y])$ i
- 6 $\vdash \forall R \forall S (R \subseteq S \& S \subseteq R \Rightarrow R \equiv S)$ i

- 7 $\vdash \forall R R \equiv R$ i
- 8 $\vdash \forall R \forall S (R \equiv S \Rightarrow S \equiv R)$ i
- 9 $\vdash \forall R \forall S (R \equiv S \Rightarrow R \subseteq S)$ i
- 10 $\vdash \forall R \forall S (R \equiv S \Rightarrow S \subseteq R)$ i
- 11 $\vdash \forall R \forall S (R \equiv S \Rightarrow R \subseteq S \ \& \ S \subseteq R)$ i
- 12 $\vdash \forall R \forall S (R \subseteq S \ \& \ S \subseteq R \Leftrightarrow R \equiv S)$ i
- 13 $\vdash \forall R \forall S \forall T (R \subseteq S \ \& \ S \equiv T \Rightarrow R \subseteq T)$ i
- 14 $\vdash \forall R \forall S \forall T (R \subseteq S \ \& \ T \equiv S \Rightarrow R \subseteq T)$ i
- 15 $\vdash \forall R \forall S \forall T (R \equiv S \ \& \ S \equiv T \Rightarrow R \equiv T)$ i
- 16 $\vdash \forall R \forall S \forall T (R \equiv S \ \& \ S \equiv T \Rightarrow T \equiv R)$ i
- 17 $\vdash \forall R \forall S \forall T (R \equiv S \ \& \ T \equiv S \Rightarrow R \equiv T)$ i
- 18 $\vdash \forall R \forall S \forall T (R \equiv S \ \& \ R \equiv T \Rightarrow S \equiv T)$ i
- 19 $\vdash \forall R \forall S \forall T \forall U (R \equiv S \ \& \ S \equiv T \ \& \ T \equiv U \Rightarrow R \equiv U)$ i
- 20 $\vdash \forall R \forall S \forall x \forall y (R[x,y] \ \& \ R \equiv S \Rightarrow S[x,y])$ i
- 21 $\vdash \forall R \forall S \forall x \forall y (R[x,y] \ \& \ S \equiv R \Rightarrow S[x,y])$ i

!CHAPTER 2 DYADIC UNIONS;

- 1 $\mathbb{D} \ \sqcup ; (R \sqcup S) ; ; \{ a, b : R[a,b] \vee S[a,b] \}$ i
- 2 $\vdash \forall R \forall S \forall x \forall y ((R \sqcup S)[x,y] \Leftrightarrow R[x,y] \vee S[x,y])$ i
- 3 $\vdash \forall R \forall S \forall x \forall y ((R \sqcup S)[x,y] \Rightarrow R[x,y] \vee S[x,y])$ i
- 4 $\vdash \forall R \forall S \forall x \forall y (R[x,y] \vee S[x,y] \Rightarrow (R \sqcup S)[x,y])$ i
- 5 $\vdash \forall R \forall S (R \sqcup S) \subseteq (S \sqcup R)$ i
- 6 $\vdash \forall R \forall S (R \sqcup S) \equiv (S \sqcup R)$ i
- 7 $\vdash \forall R \forall S R \subseteq (R \sqcup S)$ i
- 8 $\vdash \forall R \forall S R \subseteq (S \sqcup R)$ i
- 9 $\vdash \forall R \forall S \forall T (R \subseteq T \ \& \ S \subseteq T \Rightarrow (R \sqcup S) \subseteq T)$ i
- 10 $\vdash \forall R (R \sqcup R) \equiv R$ i
- 11 $\vdash \forall R \forall S \forall T \forall U (R \subseteq S \ \& \ T \subseteq U \Rightarrow (R \sqcup T) \subseteq (S \sqcup U))$ i
- 12 $\vdash \forall R \forall S \forall T \forall U (R \equiv S \ \& \ T \equiv U \Rightarrow (R \sqcup T) \equiv (S \sqcup U))$ i
- 13 $\vdash \forall R \forall S \forall T (R \equiv S \Rightarrow (R \sqcup T) \equiv (S \sqcup T))$ i
- 14 $\vdash \forall R \forall S \forall T (R \equiv S \Rightarrow (T \sqcup R) \equiv (T \sqcup S))$ i
- 15 $\vdash \forall R \forall S \forall T \forall U \forall V (T \equiv (R \sqcup S) \ \& \ R \equiv U \ \& \ S \equiv V \Rightarrow T \equiv (U \sqcup V))$ i
- 16 $\vdash \forall R \forall S \forall T \forall U (T \equiv (R \sqcup S) \ \& \ R \equiv U \Rightarrow T \equiv (U \sqcup S))$ i
- 17 $\vdash \forall R \forall S \forall T \forall U (T \equiv (S \sqcup R) \ \& \ R \equiv U \Rightarrow T \equiv (S \sqcup U))$ i
- 18 $\vdash \forall R \forall S ((R \sqcup S) \sqcup S) \equiv (R \sqcup S)$ i

!CHAPTER 3 RELATIONAL INVERSES;

- 1 $\mathbb{D} \ ^* ; (R^*) ; ; \{ a, b : R[b,a] \}$ i
- 2 $\vdash \forall R \forall x \forall y ((R^*)[x,y] \Leftrightarrow R[y,x])$ i
- 3 $\vdash \forall R \forall x \forall y ((R^*)[x,y] \Rightarrow R[y,x])$ i
- 4 $\vdash \forall R \forall x \forall y (R[y,x] \Rightarrow (R^*)[x,y])$ i
- 5 $\vdash \forall R \forall z (\exists x R[x,y] \Leftrightarrow \exists y (R^*)[z,y])$ i
- 6 $\vdash \forall R \forall S \forall x \forall y ((R^*) \subseteq S \ \& \ R[x,y] \Rightarrow S[y,x])$ i
- 7 $\vdash \forall R \forall S \forall x \forall y (R \subseteq (S^*) \ \& \ R[x,y] \Rightarrow S[y,x])$ i
- 8 $\vdash \forall R \forall S \forall x \forall y ((R^*) \subseteq S \Rightarrow R \subseteq (S^*))$ i
- 9 $\vdash \forall R \forall S \forall x \forall y (R \subseteq (S^*) \Rightarrow (R^*) \subseteq S)$ i
- 10 $\vdash \forall R \forall S \forall x \forall y ((R^*) \equiv S \ \& \ R[x,y] \Rightarrow S[y,x])$ i
- 11 $\vdash \forall R \forall S \forall x \forall y (S \equiv (R^*) \ \& \ R[x,y] \Rightarrow S[y,x])$ i

- 12 $\vdash \forall R \forall S \forall x \forall y ((S^*) \equiv R \ \& \ R[x,y] \Rightarrow S[y,x])$ i
- 13 $\vdash \forall R \forall S \forall x \forall y (R \equiv (S^*) \ \& \ R[x,y] \Rightarrow S[y,x])$ i
- 14 $\vdash \forall R (\forall x \forall y (R[x,y] \Rightarrow R[y,x]) \Rightarrow R \equiv (R^*))$ i
- 15 $\vdash \forall R ((R^*)^*) \subseteq R$ i
- 16 $\vdash \forall R R \subseteq ((R^*)^*)$ i
- 17 $\vdash \forall R ((R^*)^*) \equiv R$ i
- 18 $\vdash \forall R R \equiv ((R^*)^*)$ i
- 19 $\vdash \forall R \forall S (R \subseteq S \Rightarrow (R^*) \subseteq (S^*))$ i
- 20 $\vdash \forall R \forall S (R \equiv S \Rightarrow (R^*) \equiv (S^*))$ i
- 21 $\vdash \forall R \forall S ((R^*) \equiv (S^*) \Rightarrow R \equiv S)$ i
- 22 $\vdash \forall R \forall S (R \equiv S \Leftrightarrow (R^*) \equiv (S^*))$ i
- 23 $\vdash \forall R \forall S ((R^*) \equiv S \Rightarrow (S^*) \equiv R)$ i
- 24 $\vdash \forall R \forall S ((R^*) \equiv S \Rightarrow R \equiv (S^*))$ i
- 25 $\vdash \forall R \forall S (R \equiv (S^*) \Rightarrow S \equiv (R^*))$ i
- 26 $\vdash \forall R \forall S (R \equiv (S^*) \Rightarrow (R^*) \equiv S)$ i
- 27 $\vdash \forall R \forall S ((R \sqcup S)^*) \equiv ((R^*) \sqcup (S^*))$ i
- 28 $\vdash \forall R \forall S ((R^*) \sqcup (S^*)) \equiv ((R \sqcup S)^*)$ i

!CHAPTER 4 A DYADIC EMPTY PREDICATE;

- 1 $\mathbb{D} \ \Phi ; \Phi ; ; \{a,b : \neg a = a\}$ i
- 2 $\vdash \forall x \forall y (\Phi[x,y] \Leftrightarrow \neg x = x)$ i
- 3 $\vdash \forall x \forall y \neg \Phi[x,y]$ i
- 4 $\vdash \forall R (R \equiv \Phi \Rightarrow \forall x \forall y \neg R[x,y])$ i
- 5 $\vdash \forall R (R \equiv \Phi \Rightarrow \neg \exists x \exists y R[x,y])$ i
- 6 $\vdash \forall R \Phi \subseteq R$ i
- 7 $\vdash \forall R (R \subseteq \Phi \Rightarrow R \equiv \Phi)$ i
- 8 $\vdash \forall R (\forall x \forall y \neg R[x,y] \Rightarrow R \equiv \Phi)$ i
- 9 $\vdash \forall R (\neg \exists x \exists y R[x,y] \Rightarrow R \equiv \Phi)$ i
- 10 $\vdash \forall R (R \sqcup \Phi) \equiv R$ i
- 11 $\vdash \forall R (\Phi \sqcup R) \equiv R$ i
- 12 $\vdash \forall R \forall S (S \equiv \Phi \Rightarrow (R \sqcup S) \equiv R)$ i
- 13 $\vdash \forall R \forall S (S \equiv \Phi \Rightarrow (S \sqcup R) \equiv R)$ i
- 14 $\vdash \Phi \equiv (\Phi^*)$ i
- 15 $\vdash \forall R ((R^*) \equiv \Phi \Rightarrow R \equiv \Phi)$ i

!CHAPTER 5 DOMAINS;

- 1 $\mathbb{D} \ D ; (R^D) ; ; \{a : \exists y R[a,y]\}$ i
- 2 $\vdash \forall R \forall x ((R^D)[x] \Leftrightarrow \exists y R[x,y])$ i
- 3 $\vdash \forall R \forall x ((R^D)[x] \Rightarrow \exists y R[x,y])$ i
- 4 $\vdash \forall R \forall x (\exists y R[x,y] \Rightarrow (R^D)[x])$ i
- 5 $\vdash \forall R \forall x \forall y (R[x,y] \Rightarrow (R^D)[x])$ i
- 6 $\vdash \forall R \forall A \forall x \forall y (R[x,y] \ \& \ (R^D) \subseteq A \Rightarrow A[x])$ i
- 7 $\vdash \forall R \forall A \forall x \forall y (R[x,y] \ \& \ (R^D) \equiv A \Rightarrow A[x])$ i
- 8 $\vdash \forall R \forall A \forall x (A[x] \ \& \ A \subseteq (R^D) \Rightarrow \exists y R[x,y])$ i
- 9 $\vdash \forall R \forall A \forall x (A[x] \ \& \ (R^D) \equiv A \Rightarrow \exists y R[x,y])$ i
- 10 $\vdash \forall R \forall A (\forall x (\exists y R[x,y] \Rightarrow A[x]) \Rightarrow (R^D) \subseteq A)$ i
- 11 $\vdash \forall R \forall A (\forall x \forall y (R[x,y] \Rightarrow A[x]) \Rightarrow (R^D) \subseteq A)$ i
- 12 $\vdash \forall R \forall A (\forall x (A[x] \Rightarrow \exists y R[x,y]) \Rightarrow A \subseteq (R^D))$ i

- 13 $\vdash \forall R \forall A (\forall x (\exists y R[x,y] \Leftrightarrow A[x]) \Rightarrow (R^D) \equiv A)$ i
14 $\vdash \forall R \forall S (R \subseteq S \Rightarrow (R^D) \subseteq (S^D))$ i
15 $\vdash \forall R \forall S (R \equiv S \Rightarrow (R^D) \equiv (S^D))$ i
16 $\vdash \forall R \forall S \forall A ((R^D) \equiv A \ \& \ R \equiv S \Rightarrow (S^D) \equiv A)$ i
17 $\vdash \forall R \forall S \forall A \forall B ((R^D) \equiv A \ \& \ R \equiv S \ \& \ A \equiv B \Rightarrow (S^D) \equiv B)$ i
18 $\vdash \forall R \forall S \forall x (((R \sqcup S)^D)[x] \Rightarrow (R^D)[x] \vee (S^D)[x])$ i
19 $\vdash \forall R \forall S ((R \sqcup S)^D \equiv ((R^D) \cup (S^D)))$ i
20 $\vdash \forall R \forall S \forall A \forall B ((R^D) \equiv A \ \& \ (S^D) \equiv B \Rightarrow ((R \sqcup S)^D) \equiv (A \cup B))$ i
21 $\vdash (\Phi^D) \subseteq \phi$ i
22 $\vdash (\Phi^D) \equiv \phi$ i
23 $\vdash \forall R (R \equiv \Phi \Rightarrow (R^D) \equiv \phi)$ i
24 $\vdash \forall R ((R^D) \equiv \phi \Rightarrow R \equiv \Phi)$ i
25 $\vdash \forall R (\forall x \exists y R[x,y] \Rightarrow (R^D) \equiv \mathbb{U})$ i

!CHAPTER 6 IMAGES;

- 1 $\mathbb{D} \quad \mathbb{I} \quad ; \quad (R^I) \quad ; \quad ; \quad \{b : \exists x R[x,b]\}$ i
2 $\vdash \forall R \forall y ((R^I)[y] \Leftrightarrow \exists x R[x,y])$ i
3 $\vdash \forall R \forall y ((R^I)[y] \Rightarrow \exists x R[x,y])$ i
4 $\vdash \forall R \forall y (\exists x R[x,y] \Rightarrow (R^I)[y])$ i
5 $\vdash \forall R \forall x \forall y (R[x,y] \Rightarrow (R^I)[y])$ i
6 $\vdash \forall R \forall y (\neg (R^I)[y] \Rightarrow \neg \exists x R[x,y])$ i
7 $\vdash \forall R \forall x \forall y (R[x,y] \ \& \ \neg (R^I)[a] \Rightarrow \neg y = a)$ i
8 $\vdash \forall R \forall B \forall x \forall y (R[x,y] \ \& \ (R^I) \subseteq B \Rightarrow B[y])$ i
9 $\vdash \forall R \forall B \forall x \forall y (R[x,y] \ \& \ (R^I) \equiv B \Rightarrow B[y])$ i
10 $\vdash \forall R \forall B \forall y (B[y] \ \& \ B \subseteq (R^I) \Rightarrow \exists x R[x,y])$ i
11 $\vdash \forall R \forall B \forall y (B[y] \ \& \ (R^I) \equiv B \Rightarrow \exists x R[x,y])$ i
12 $\vdash \forall R \forall B (\forall y (\exists x R[x,y] \Rightarrow B[y]) \Rightarrow (R^I) \subseteq B)$ i
13 $\vdash \forall R \forall B (\forall y (B[y] \Rightarrow \exists x R[x,y]) \Rightarrow B \subseteq (R^I))$ i
14 $\vdash \forall R \forall B (\forall y (\exists x R[x,y] \Leftrightarrow B[y]) \Rightarrow (R^I) \equiv B)$ i
15 $\vdash \forall R (R^I) \equiv ((R^*)^D)$ i
16 $\vdash \forall R (R^D) \equiv ((R^*)^I)$ i
17 $\vdash \forall R \forall B ((R^I) \equiv B \Rightarrow ((R^*)^D) \equiv B)$ i
18 $\vdash \forall R \forall B (((R^*)^D) \equiv B \Rightarrow (R^I) \equiv B)$ i
19 $\vdash \forall R \forall B ((R^I) \equiv B \Leftrightarrow ((R^*)^D) \equiv B)$ i
20 $\vdash \forall R \forall S (R \subseteq S \Rightarrow (R^I) \subseteq (S^I))$ i
21 $\vdash \forall R \forall S \forall y (S \subseteq R \ \& \ \neg (R^I)[y] \Rightarrow \neg (S^I)[y])$ i
22 $\vdash \forall R \forall S (R \equiv S \Rightarrow (R^I) \equiv (S^I))$ i
23 $\vdash \forall R \forall S \forall B ((R^I) \equiv B \ \& \ R \equiv S \Rightarrow (S^I) \equiv B)$ i
24 $\vdash \forall R \forall S \forall B \forall A ((R^I) \equiv B \ \& \ R \equiv S \ \& \ B \equiv A \Rightarrow (S^I) \equiv A)$ i
25 $\vdash \forall R \forall S ((R \sqcup S)^I \equiv ((R^I) \cup (S^I)))$ i
26 $\vdash \forall R \forall S \forall A \forall B ((R^I) \equiv A \ \& \ (S^I) \equiv B \Rightarrow ((R \sqcup S)^I) \equiv (A \cup B))$ i
27 $\vdash \forall R \forall A \forall B ((R^D) \equiv A \ \& \ (R^I) \equiv B \ \& \ \exists x A[x] \Rightarrow \exists y B[y])$ i
28 $\vdash (\Phi^I) \equiv \phi$ i
29 $\vdash \forall R (R \equiv \Phi \Rightarrow (R^I) \equiv \phi)$ i
30 $\vdash \forall R ((R^I) \equiv \phi \Rightarrow R \equiv \Phi)$ i

!CHAPTER 7 RESTRICTIONS;

- 44 $\vdash \forall R \forall S \forall x (\neg (R^D)[x] \Rightarrow ((R \sqcup S) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I))$ i
- 45 $\vdash \forall R \forall S \forall x (\neg (R^D)[x] \Rightarrow ((S \sqcup R) \uparrow (x^\bullet))^I \equiv ((S \uparrow (x^\bullet))^I))$ i
- 46 $\vdash \forall R \forall A \forall y (((R \uparrow A)^I)[y] \Leftrightarrow \exists z (R[z,y] \& A[z]))$ i
- 47 $\vdash \forall R \forall A \{y : \exists z (R[z,y] \& A[z])\} \equiv ((R \uparrow A)^I)$ i
- 48 $\vdash \forall R \forall x \{y : R[x,y]\} \equiv ((R \uparrow (x^\bullet))^I)$ i
- 49 $\mathbb{D} \lfloor ; (\mathbf{R} \lfloor \mathbf{B}) ; ; \{a,b : \mathbf{R}[a,b] \& \mathbf{B}[b]\}$ i
- 50 $\vdash \forall R \forall B \forall x \forall y ((R \lfloor B)[x,y] \Leftrightarrow R[x,y] \& B[y])$ i
- 51 $\vdash \forall R \forall B (R \lfloor B) \equiv ((R^*) \uparrow B)^*$ i
- 52 $\vdash \forall R \forall B ((R \lfloor B)^*) \equiv ((R^*) \uparrow B)$ i
- 53 $\vdash \forall R \forall A \forall B (\forall x \forall y (R[x,y] \Rightarrow (A[x] \Leftrightarrow B[y]))$
 $\Rightarrow (R \uparrow A) \equiv (R \lfloor B))$ i

!CHAPTER 8 FUNCTIONAL PREDICATES;

- 1 $\mathbb{S} \mathbf{f} ; \mathbf{f} \mathbf{R} ; \forall x \forall y \forall z (\mathbf{R}[x,y] \& \mathbf{R}[x,z] \Rightarrow y = z)$ i
- 2 $\vdash \forall R \forall x \forall y \forall z (\mathbf{f} R \& R[x,y] \& R[x,z] \Rightarrow y = z)$ i
- 3 $\vdash \forall R \forall S (\mathbf{f} R \& S \subseteq R \Rightarrow \mathbf{f} S)$ i
- 4 $\vdash \forall R \forall A (\mathbf{f} R \Rightarrow \mathbf{f} (R \uparrow A))$ i
- 5 $\vdash \forall R \forall S (\mathbf{f} R \& R \equiv S \Rightarrow \mathbf{f} S)$ i
- 6 $\vdash \forall R \forall S (\mathbf{f} R \& S \equiv R \Rightarrow \mathbf{f} S)$ i
- 7 $\vdash \forall R \forall S (\mathbf{f} R \& \mathbf{f} S \& ((R^D) \cap (S^D)) \equiv \phi \Rightarrow \mathbf{f} (R \sqcup S))$ i
- 8 $\vdash \forall R \forall S \forall A \forall B (\mathbf{f} R \& \mathbf{f} S \& (R^D) \equiv A \& (S^D) \equiv B \& (A \cap B) \equiv \phi$
 $\Rightarrow \mathbf{f} (R \sqcup S))$ i
- 9 $\vdash \mathbf{f} \phi$ i
- 10 $\mathbb{S} \mathbf{F} ; \mathbf{R} \mathbf{F} \mathbf{A} ; (R^D) \equiv \mathbf{A} \& \mathbf{f} \mathbf{R}$ i
- 11 $\vdash \forall R \forall S \forall A \forall B (R \mathbf{F} A \& R \equiv S \& A \equiv B \Rightarrow S \mathbf{F} B)$ i
- 12 $\vdash \forall R \forall S \forall A (R \mathbf{F} A \& R \equiv S \Rightarrow S \mathbf{F} A)$ i
- 13 $\vdash \forall R \forall A \forall B (R \mathbf{F} A \& A \equiv B \Rightarrow R \mathbf{F} B)$ i
- 14 $\vdash \forall R \forall A \forall B (R \mathbf{F} A \& B \equiv A \Rightarrow R \mathbf{F} B)$ i
- 15 $\vdash \forall R \forall A (\mathbf{f} R \& A \subseteq (R^D) \Rightarrow (R \uparrow A) \mathbf{F} A)$ i
- 16 $\vdash \forall R \forall S \forall A \forall B (R \mathbf{F} A \& S \mathbf{F} B \& (A \cap B) \equiv \phi$
 $\Rightarrow (R \sqcup S) \mathbf{F} (A \cup B))$ i
- 17 $\vdash \phi \mathbf{F} \phi$ i
- 18 $\vdash \forall R \forall x (\mathbf{f} R \& (R^D)[x] \Rightarrow \exists a R[x,a])$ i
- 19 $\vdash \forall R \forall x (\mathbf{f} R \& (R^D)[x] \Rightarrow \forall y \forall z (R[x,y] \& R[x,z] \Rightarrow y = z))$ i
- 20 $\mathbb{T} \acute{x} ; (\mathbf{R}' \mathbf{x}) ; \mathbf{f} \mathbf{R} \& (R^D)[\mathbf{x}] ; \mathbf{R}[\mathbf{x}, \mathbf{y}]$ i
- 21 $\vdash \forall R \forall x \forall y (\mathbf{f} R \& R[x,y] \Rightarrow y = (R'x))$ i
- 22 $\vdash \forall R \forall x \forall y (\mathbf{f} R \& R[x,y] \Rightarrow (R'x) = y)$ i
- 23 $\vdash \forall R \forall S \forall x (\mathbf{f} R \& S \subseteq R \& (S^D)[x] \Rightarrow (S'x) = (R'x))$ i
- 24 $\vdash \forall R \forall S \forall x (\mathbf{f} R \& S \subseteq R \& (S^D)[x] \Rightarrow (R'x) = (S'x))$ i
- 25 $\vdash \forall R \forall S \forall x (\mathbf{f} R \& (R^D)[x] \& R \equiv S \Rightarrow (R'x) = (S'x))$ i
- 26 $\vdash \forall R \forall S \forall x (\mathbf{f} R \& (R^D)[x] \& S \equiv R \Rightarrow (R'x) = (S'x))$ i
- 27 $\vdash \forall R \forall S \forall x \forall y (R \equiv S \& (R'x) = y \Rightarrow (S'x) = y)$ i
- 28 $\vdash \forall R \forall S \forall x \forall y (R \equiv S \& (S'x) = y \Rightarrow (R'x) = y)$ i
- 29 $\vdash \forall R \forall S \forall x \forall y \forall z (R \equiv S \& (R'x) = y \& (S'x) = z \Rightarrow y = z)$ i
- 30 $\vdash \forall R \forall S (\mathbf{f} R \& (R^D) \equiv (S^D) \& \forall x ((R^D)[x] \Rightarrow (R'x) = (S'x))$ i

- $\Rightarrow R \equiv S$) i
- 31 $\vdash \forall R \forall S \forall x (\mathbf{f} (R \sqcup S) \ \& \ (R^D)[x] \Rightarrow ((R \sqcup S)'x) = (R'x))$ i
- 32 $\vdash \forall R \forall S \forall x (\mathbf{f} (R \sqcup S) \ \& \ (S^D)[x] \Rightarrow ((R \sqcup S)'x) = (S'x))$ i
- 33 $\vdash \forall R \forall S \forall x (\mathbf{f} R \ \& \ \mathbf{f} S \ \& \ ((R^D) \cap (S^D)) \equiv \phi \ \& \ ((R \sqcup S)^D)[x]$
 $\Rightarrow ((R \sqcup S)'x) = (R'x) \vee ((R \sqcup S)'x) = (S'x))$ i
- 34 $\vdash \forall R \forall A \forall x (\mathbf{f} R \ \& \ A \subseteq (R^D) \ \& \ A[x] \Rightarrow ((R \sqcup A)'x) = (R'x))$ i
- !CHAPTER 9 1-1 PREDICATES;**
- 1 **§ 1 ; 1 R ; $\forall x \forall y \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$** i
- 2 $\vdash \forall R \forall x \forall y \forall z (\mathbf{1} R \ \& \ R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$ i
- 3 $\vdash \forall R (\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$ i
- 4 $\vdash \forall R (\mathbf{f} (R^*) \Rightarrow \mathbf{1} R)$ i
- 5 $\vdash \forall R (\mathbf{1} R \Leftrightarrow \mathbf{f} (R^*))$ i
- 6 $\vdash \forall R (\mathbf{f} R \Leftrightarrow \mathbf{1} (R^*))$ i
- 7 $\vdash \forall R (\mathbf{f} R \Rightarrow \mathbf{1} (R^*))$ i
- 8 $\vdash \forall R (\mathbf{1} (R^*) \Rightarrow \mathbf{f} R)$ i
- 9 $\vdash \forall R \forall S (\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S)$ i
- 10 $\vdash \forall R \forall A (\mathbf{1} R \Rightarrow \mathbf{1} (R \sqcup A))$ i
- 11 $\vdash \forall R \forall S (\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S)$ i
- 12 $\vdash \forall R \forall S (\mathbf{1} R \ \& \ S \equiv R \Rightarrow \mathbf{1} S)$ i
- 13 $\vdash \forall R \forall S (\mathbf{1} R \ \& \ (R^*) \equiv S \Rightarrow \mathbf{f} S)$ i
- 14 $\vdash \forall R \forall S (\mathbf{1} R \ \& \ (S^*) \equiv R \Rightarrow \mathbf{f} S)$ i
- 15 $\vdash \forall R \forall S (\mathbf{f} R \ \& \ (R^*) \equiv S \Rightarrow \mathbf{1} S)$ i
- 16 $\vdash \forall R \forall S (\mathbf{f} R \ \& \ (S^*) \equiv R \Rightarrow \mathbf{1} S)$ i
- 17 $\vdash \forall R \forall A \forall B (\mathbf{1} R \ \& \ (A \cap B) \equiv \phi$
 $\Rightarrow (((R \sqcup A)^I) \cap ((R \sqcup B)^I)) \equiv \phi)$ i
- 18 $\vdash \mathbf{1} \Phi$ i
- 19 **§ 1 ; R 1 B ; $(R^I) \equiv B \ \& \ \mathbf{1} R$** i
- 20 $\vdash \forall R \forall B (R \mathbf{1} B \Rightarrow (R^*) \mathbf{F} B)$ i
- 21 $\vdash \forall R \forall B ((R^*) \mathbf{F} B \Rightarrow R \mathbf{1} B)$ i
- 22 $\vdash \forall R \forall B (R \mathbf{1} B \Leftrightarrow (R^*) \mathbf{F} B)$ i
- 23 $\vdash \forall R \forall A (R \mathbf{F} A \Rightarrow (R^*) \mathbf{1} A)$ i
- 24 $\vdash \forall R \forall A ((R^*) \mathbf{1} A \Rightarrow R \mathbf{F} A)$ i
- 25 $\vdash \forall R \forall A (R \mathbf{F} A \Leftrightarrow (R^*) \mathbf{1} A)$ i
- 26 $\vdash \forall R \forall S \forall A \forall B (R \mathbf{1} A \ \& \ R \equiv S \ \& \ A \equiv B \Rightarrow S \mathbf{1} B)$ i
- 27 $\vdash \forall R \forall S \forall A (R \mathbf{1} A \ \& \ R \equiv S \Rightarrow S \mathbf{1} A)$ i
- 28 $\vdash \forall R \forall A \forall B (R \mathbf{1} A \ \& \ A \equiv B \Rightarrow R \mathbf{1} B)$ i
- 29 $\vdash \forall R (\mathbf{1} R \Rightarrow R \mathbf{1} (R^I))$ i
- 30 $\vdash \forall R \forall A (\mathbf{1} R \Rightarrow (R \sqcup A) \mathbf{1} ((R \sqcup A)^I))$ i
- 31 $\vdash \forall R \forall S \forall A \forall B (R \mathbf{1} A \ \& \ S \mathbf{1} B \ \& \ (A \cap B) \equiv \phi$
 $\Rightarrow (R \sqcup S) \mathbf{1} (A \cup B))$ i
- 32 $\vdash \Phi \mathbf{1} \phi$ i
- 33 $\vdash \forall R \forall y (\mathbf{1} R \ \& \ (R^I)[y] \Rightarrow \exists a R[a,y])$ i
- 34 $\vdash \forall R \forall y (\mathbf{1} R \ \& \ (R^I)[y] \Rightarrow \forall x \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z))$ i
- 35 **T ` ; $(R`y)$; 1 R & $(R^I)[y]$; $R[x,y]$** i
- 36 $\vdash \forall R \forall x \forall y (\mathbf{1} R \ \& \ R[x,y] \Rightarrow x = (R`y))$ i
- 37 $\vdash \forall R \forall x \forall y (\mathbf{1} R \ \& \ R[x,y] \Rightarrow (R`y) = x)$ i

- 38 $\vdash \forall R \forall S \forall Y (\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[Y] \Rightarrow (S`Y) = (R`Y))$ i
39 $\vdash \forall R \forall S \forall Y (\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[Y] \Rightarrow (R`Y) = (S`Y))$ i
40 $\vdash \forall R \forall S \forall Y (\mathbf{1} R \ \& \ (R^I)[Y] \ \& \ R \equiv S \Rightarrow (R`Y) = (S`Y))$ i
41 $\vdash \forall R \forall S \forall Y (\mathbf{1} R \ \& \ (R^I)[Y] \ \& \ S \equiv R \Rightarrow (R`Y) = (S`Y))$ i
42 $\vdash \forall R \forall x (\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^D)[x] \Rightarrow (R`(R`x)) = x)$ i
43 $\vdash \forall R \forall y (\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[y] \Rightarrow (R`(R`y)) = y)$ i
44 $\vdash \forall R \forall x \forall y (\mathbf{1} R \ \& \ (R`x) = (R`y) \Rightarrow x = y)$ i
45 $\vdash \forall R \forall x \forall y (\mathbf{f} R \ \& \ (R`x) = (R`y) \Rightarrow x = y)$ i
46 $\vdash \forall R \forall x (\mathbf{f} R \ \& \ (R^D)[x] \Rightarrow ((R^*)`x) = (R`x))$ i
47 $\vdash \forall R \forall y (\mathbf{1} R \ \& \ (R^I)[y] \Rightarrow ((R^*)`y) = (R`y))$ i

!CHAPTER 10 COMPOSITION;

- 1 $\mathbb{D} \ \circ ; (R \ \circ \ S) ; ; \{a, b : \exists z (R[a, z] \ \& \ S[z, b])\}$ i
2 $\vdash \forall R \forall S \forall x \forall y ((R \ \circ \ S)[x, y] \Leftrightarrow \exists z (R[x, z] \ \& \ S[z, y]))$ i
3 $\vdash \forall R \forall S \forall x \forall y ((R \ \circ \ S)[x, y] \Rightarrow \exists z (R[x, z] \ \& \ S[z, y]))$ i
4 $\vdash \forall R \forall S \forall x \forall y (\exists z (R[x, z] \ \& \ S[z, y]) \Rightarrow (R \ \circ \ S)[x, y])$ i
5 $\vdash \forall R \forall S \forall x \forall y \forall z (R[x, z] \ \& \ S[z, y] \Rightarrow (R \ \circ \ S)[x, y])$ i
6 $\vdash \forall R \forall S \forall T \forall U (R \subseteq T \ \& \ S \subseteq U \Rightarrow (R \ \circ \ S) \subseteq (T \ \circ \ U))$ i
7 $\vdash \forall R \forall S \forall T \forall U (R \equiv T \ \& \ S \equiv U \Rightarrow (R \ \circ \ S) \equiv (T \ \circ \ U))$ i
8 $\vdash \forall R \forall S \forall T (S \equiv T \Rightarrow (R \ \circ \ S) \equiv (R \ \circ \ T))$ i
9 $\vdash \forall R \forall S \forall T (S \equiv T \Rightarrow (S \ \circ \ R) \equiv (T \ \circ \ R))$ i
10 $\vdash \forall R \forall S \forall T (R \ \circ \ (S \ \circ \ T) \equiv ((R \ \circ \ S) \ \circ \ T))$ i
11 $\vdash \forall R \forall S ((R \ \circ \ S)^* \equiv ((S^*) \ \circ \ (R^*)))$ i
12 $\vdash \forall R \forall S (((S^*) \ \circ \ (R^*)) \equiv ((R \ \circ \ S)^*))$ i
13 $\vdash \forall R (R \ \circ \ \Phi \equiv \Phi)$ i
14 $\vdash \forall R (\Phi \ \circ \ R \equiv \Phi)$ i
15 $\vdash \forall R \forall S \forall x \forall y (R[x, y] \ \& \ (S^D)[y] \Rightarrow ((R \ \circ \ S)^D)[x])$ i
16 $\vdash \forall R \forall S ((R \ \circ \ S)^I \equiv (((S^*) \ \circ \ (R^*))^D))$ i
17 $\vdash \forall R \forall S ((R \ \circ \ S)^D \subseteq (R^D))$ i
18 $\vdash \forall R \forall S ((R^I) \subseteq (S^D) \Rightarrow ((R \ \circ \ S)^D) \equiv (R^D))$ i
19 $\vdash \forall R \forall S \forall A \forall B ((R^D) \equiv A \ \& \ (S^D) \equiv B \ \& \ (R^I) \subseteq B$
 $\Rightarrow ((R \ \circ \ S)^D) \equiv A)$ i
20 $\vdash \forall R \forall S \forall A ((R \ \circ \ S) \lceil A \equiv ((R \lceil A) \ \circ \ S))$ i
21 $\vdash \forall R \forall S \forall A ((R \ \circ \ S) \lfloor A \equiv (R \ \circ \ (S \lfloor A)))$ i
22 $\vdash \forall R \forall S (\mathbf{f} R \ \& \ \mathbf{f} S \Rightarrow \mathbf{f} (R \ \circ \ S))$ i
23 $\vdash \forall R \forall S \forall x (\mathbf{f} S \ \& \ (S^D)[(R`x)] \Rightarrow ((R \ \circ \ S)`x) = (S`(R`x)))$ i
24 $\vdash \forall R \forall S \forall A \forall B (R \ \mathbf{F} \ A \ \& \ S \ \mathbf{F} \ B \ \& \ (R^I) \subseteq B \Rightarrow (R \ \circ \ S) \ \mathbf{F} \ A)$ i
25 $\vdash \forall R \forall S \forall B \forall C (R \ \mathbf{1} \ B \ \& \ S \ \mathbf{1} \ C \ \& \ (S^D) \subseteq B \Rightarrow (R \ \circ \ S) \ \mathbf{1} \ C)$ i
26 $\vdash \forall R \forall S \forall A \forall B \forall C (R \ \mathbf{F} \ A \ \& \ R \ \mathbf{1} \ B \ \& \ S \ \mathbf{F} \ B \ \& \ S \ \mathbf{1} \ C$
 $\Rightarrow (R \ \circ \ S) \ \mathbf{F} \ A \ \& \ (R \ \circ \ S) \ \mathbf{1} \ C)$ i

!CHAPTER 11 IDENTITY PREDICATES;

- 1 $\mathbb{D} \ \mathbf{E} ; \ \mathbf{E} ; ; \{a, b : a = b\}$ i
2 $\vdash \forall x \forall y (\mathbf{E}[x, y] \Leftrightarrow x = y)$ i
3 $\vdash \forall x \forall y (\mathbf{E}[x, y] \Rightarrow x = y)$ i

- 4 $\vdash \forall x \forall y (x = y \Rightarrow \mathbf{E}[x,y])$ i
- 5 $\vdash \forall x \mathbf{E}[x,x]$ i
- 6 $\vdash \forall x \forall y (\mathbf{E}[x,y] \Rightarrow \mathbf{E}[y,x])$ i
- 7 $\vdash \mathbf{E} \equiv (\mathbf{E}^*)$ i
- 8 $\vdash (\mathbf{E}^*) \equiv \mathbf{E}$ i
- 9 $\vdash (\mathbf{E}^D) \equiv \mathbf{U}$ i
- 10 $\vdash \mathbf{f} \mathbf{E}$ i
- 11 $\vdash \mathbf{E} \mathbf{F} \mathbf{U}$ i
- 12 $\vdash \forall A (\mathbf{E} \lceil A) \equiv (\mathbf{E} \lfloor A)$ i
- 13 $\vdash \forall A ((\mathbf{E} \lceil A)^*) \equiv (\mathbf{E} \lceil A)$ i
- 14 $\vdash \forall R (\mathbf{E} \circ R) \equiv R$ i
- 15 $\vdash \forall R (R \circ \mathbf{E}) \equiv R$ i
- 16 $\mathbf{D} \ \mathbf{I} \ ; \ (\mathbf{I}A) \ ; \ ; \ (\mathbf{E} \lceil A)$ i
- 17 $\vdash \forall A \forall x \forall y ((\mathbf{I}A)[x,y] \Leftrightarrow x = y \ \& \ A[x])$ i
- 18 $\vdash \forall x \forall y ((\mathbf{I}A)[x,y] \Rightarrow x = y \ \& \ A[x])$ i
- 19 $\vdash \forall x \forall y (x = y \ \& \ A[x] \Rightarrow (\mathbf{I}A)[x,y])$ i
- 20 $\vdash \forall A \forall x \forall y ((\mathbf{I}A)[x,y] \Rightarrow x = y)$ i
- 21 $\vdash \forall A \forall B (A \subseteq B \Rightarrow (\mathbf{I}A) \subseteq (\mathbf{I}B))$ i
- 22 $\vdash \forall A \forall B (A \equiv B \Rightarrow (\mathbf{I}A) \equiv (\mathbf{I}B))$ i
- 23 $\vdash \forall A ((\mathbf{I}A)^*) \equiv (\mathbf{I}A)$ i
- 24 $\vdash \forall A ((\mathbf{I}A)^D) \equiv A$ i
- 25 $\vdash \forall A \mathbf{f} (\mathbf{I}A)$ i
- 26 $\vdash \forall A (\mathbf{I}A) \mathbf{F} A$ i
- 27 $\vdash \forall A (\mathbf{I}A) \mathbf{1} A$ i
- 28 $\vdash \forall R \forall A ((R^D) \subseteq A \Rightarrow ((\mathbf{I}A) \circ R) \equiv R)$ i
- 29 $\vdash \forall R \forall S \forall A ((R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} R \Rightarrow (R \circ S) \equiv (\mathbf{I}A))$ i
- 30 $\vdash \forall R \forall S \forall A (R \equiv (S^*) \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} R \Rightarrow (R \circ S) \equiv (\mathbf{I}A))$ i
- 31 $\vdash \forall R \forall S \forall A ((R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ \mathbf{f} R \Rightarrow (S \circ R) \equiv (\mathbf{I}A))$ i
- 32 $\vdash \forall R \forall S \forall A (R \equiv (S^*) \ \& \ (R^I) \equiv A \ \& \ \mathbf{f} R \Rightarrow (S \circ R) \equiv (\mathbf{I}A))$ i
- 33 $\vdash \forall R \forall S \forall T (R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} R$
 $\Rightarrow ((R \circ S) \circ T) \equiv T)$ i
- 34 $\vdash \forall R \forall S \forall T (T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} T$
 $\Rightarrow (R \circ (S \circ T)) \equiv R)$ i
- 35 $\vdash \forall R \forall S \forall T (R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} R$
 $\Rightarrow (R \circ (S \circ T)) \equiv T)$ i
- 36 $\vdash \forall R \forall S \forall T (T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} T$
 $\Rightarrow ((R \circ S) \circ T) \equiv R)$ i

!CHAPTER 12 PAIRING PREDICATES;

- 1 $\mathbf{D} \ \blacksquare \ ; \ (a \ \blacksquare \ b) \ ; \ ; \ \{a,b : a = a \ \& \ b = b\}$ i
- 2 $\vdash \forall a \forall b \forall x \forall y ((a \ \blacksquare \ b)[x,y] \Leftrightarrow x = a \ \& \ y = b)$ i
- 3 $\vdash \forall a \forall b \forall x \forall y ((a \ \blacksquare \ b)[x,y] \Rightarrow x = a \ \& \ y = b)$ i
- 4 $\vdash \forall a \forall b \forall x \forall y (x = a \ \& \ y = b \Rightarrow (a \ \blacksquare \ b)[x,y])$ i
- 5 $\vdash \forall a \forall b (a \ \blacksquare \ b)[a,b]$ i
- 6 $\vdash \forall a \forall b \forall x \forall y (\neg (a \ \blacksquare \ b)[x,y] \Rightarrow \neg x = a \ \vee \ \neg y = b)$ i
- 7 $\vdash \forall R \forall a \forall b (R[a,b] \Rightarrow (a \ \blacksquare \ b) \subseteq R)$ i

- 8 $\vdash \forall R \forall a \forall b ((a \sqsubset b) \subseteq R \Rightarrow R[a,b])$ i
9 $\vdash \forall a \forall b \forall c \forall d (a = c \ \& \ b = d \Rightarrow (a \sqsubset b) \subseteq (c \sqsubset d))$ i
10 $\vdash \forall a \forall b \forall c \forall d ((a \sqsubset b) \equiv (c \sqsubset d) \Rightarrow a = c \ \& \ b = d)$ i
11 $\vdash \forall a \forall b \forall c \forall d (a = c \ \& \ b = d \Rightarrow (a \sqsubset b) \equiv (c \sqsubset d))$ i
12 $\vdash \forall R \forall a \forall b (R[a,b] \ \& \ \forall x \forall y (R[x,y] \Rightarrow x = a \ \& \ y = b) \Rightarrow R \equiv (a \sqsubset b))$ i
13 $\vdash \forall R \forall a \forall b (R \sqcup (a \sqsubset b)) [a,b]$ i
14 $\vdash \forall R \forall S \forall a \forall b (S \equiv (R \sqcup (a \sqsubset b)) \Rightarrow \forall x \forall y (S[x,y] \Leftrightarrow R[x,y] \vee (x = a \ \& \ y = b)))$ i
15 $\vdash \forall R \forall S \forall a \forall b (\forall x \forall y (S[x,y] \Leftrightarrow R[x,y] \vee (x = a \ \& \ y = b)) \Rightarrow S \equiv (R \sqcup (a \sqsubset b)))$ i
16 $\vdash \forall R \forall a \forall b \{ x,y : R[x,y] \vee (x = a \ \& \ y = b) \} \equiv (R \sqcup (a \sqsubset b))$ i
17 $\vdash \forall a \forall b ((a \sqsubset b)^*) \equiv (b \sqsubset a)$ i
18 $\vdash \forall a \forall b ((a \sqsubset b)^D) \equiv (a^\bullet)$ i
19 $\vdash \forall a \forall b ((a \sqsubset b)^I) \equiv (b^\bullet)$ i
20 $\vdash \forall R \forall S \forall a \forall b \forall x (\neg (R^I)[a] \ \& \ S \equiv (R \sqcup (x \sqsubset b)) \ \& \ \neg a = b \Rightarrow \neg (S^I)[a])$ i
21 $\vdash \forall a \forall b \mathbf{f} (a \sqsubset b)$ i
22 $\vdash \forall a \forall b (a \sqsubset b) \mathbf{F} (a^\bullet)$ i
23 $\vdash \forall R \forall a (R \mathbf{F} (a^\bullet) \Rightarrow \exists b R \equiv (a \sqsubset b))$ i
24 $\vdash \forall a \forall b ((a \sqsubset b) \acute{a}) = b$ i
25 $\vdash \forall a \forall b \mathbf{1} (a \sqsubset b)$ i
26 $\vdash \forall a \forall b (a \sqsubset b) \mathbf{1} (b^\bullet)$ i

! CHAPTER 13 CORRESPONDENCES ;

- 1 $\mathfrak{S} \sim ; \mathbf{A} \sim \mathbf{B} ; \exists R (R \mathbf{F} \mathbf{A} \ \& \ R \mathbf{1} \mathbf{B})$ i
2 $\vdash \forall A \forall B (A \equiv B \Rightarrow A \sim B)$ i
3 $\vdash \forall A A \sim A$ i
4 $\vdash \forall A \forall B (A \sim B \Rightarrow B \sim A)$ i
5 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ B \sim C \Rightarrow A \sim C)$ i
6 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ B \sim C \Rightarrow C \sim A)$ i
7 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ C \sim B \Rightarrow A \sim C)$ i
8 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ A \sim C \Rightarrow B \sim C)$ i
9 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ A \equiv C \Rightarrow C \sim B)$ i
10 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ C \equiv A \Rightarrow C \sim B)$ i
11 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ B \equiv C \Rightarrow A \sim C)$ i
12 $\vdash \forall A \forall B \forall C (A \sim B \ \& \ C \equiv B \Rightarrow A \sim C)$ i
13 $\vdash \forall A \forall B \forall C \forall D (A \sim B \ \& \ A \equiv C \ \& \ B \equiv D \Rightarrow C \sim D)$ i
14 $\vdash \forall A \forall B \forall C \forall D (A \sim B \ \& \ C \equiv A \ \& \ D \equiv B \Rightarrow C \sim D)$ i
15 $\vdash \forall A \forall B \forall C \forall D (A \sim B \ \& \ C \sim D \ \& \ (A \cap C) \equiv \phi \ \& \ (B \cap D) \equiv \phi \Rightarrow (A \cup C) \sim (B \cup D))$ i
16 $\vdash \forall R \forall A (A \subseteq (R^D) \ \& \ \mathbf{f} R \ \& \ \mathbf{1} R \Rightarrow A \sim ((R \uparrow A)^I))$ i
17 $\vdash \forall A \forall B \forall C ((A \cup B) \sim C \ \& \ (A \cap B) \equiv \phi \Rightarrow \exists D \exists E ((D \cup E) \equiv C \ \& \ (D \cap E) \equiv \phi \ \& \ A \sim D \ \& \ B \sim E))$ i
18 $\vdash \forall A (A \sim \phi \Rightarrow A \equiv \phi)$ i
19 $\vdash \forall A (\phi \sim A \Rightarrow A \equiv \phi)$ i

- 20 $\vdash \forall A (A \sim \phi \Leftrightarrow A \equiv \phi)$ i
- 21 $\vdash \forall A \forall B (A \sim B \ \& \ A \equiv \phi \Rightarrow B \equiv \phi)$ i
- 22 $\vdash \forall A \forall B (A \sim B \ \& \ B \equiv \phi \Rightarrow A \equiv \phi)$ i
- 23 $\vdash \forall A \forall B (A \sim B \ \& \ \neg A \equiv \phi \Rightarrow \neg B \equiv \phi)$ i
- 24 $\vdash \forall A \forall B (A \sim B \ \& \ \neg B \equiv \phi \Rightarrow \neg A \equiv \phi)$ i
- 25 $\vdash \forall A \forall B (A \sim B \ \& \ \exists x A[x] \Rightarrow \exists x B[x])$ i
- 26 $\vdash \forall a \forall b (a^\bullet \sim (b^\bullet))$ i
- 27 $\vdash \forall A \forall b ((b^\bullet) \sim A \Rightarrow \exists a A \equiv (a^\bullet))$ i
- 28 $\vdash \forall A \forall b (A \sim (b^\bullet) \Rightarrow \exists a A \equiv (a^\bullet))$ i
- 29 $\vdash \forall A \forall B \forall a \forall b (A \sim B \ \& \ \neg A[a] \ \& \ \neg B[b] \Rightarrow (A \cup (a^\bullet)) \sim (B \cup (b^\bullet)))$ i
- 30 $\vdash \forall A \forall a \forall b (\neg A[a] \ \& \ \neg A[b] \Rightarrow (A \cup (a^\bullet)) \sim (A \cup (b^\bullet)))$ i
- 31 $\vdash \forall A \forall B \forall a \forall b ((A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \ \& \ A[a] \ \& \ B[b] \Rightarrow A \sim B)$ i
- 32 $\vdash \forall A \forall a \forall b (A[a] \ \& \ A[b] \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet)))$ i
- 33 $\vdash \forall A \forall B \forall a (A \sim B \ \& \ A[a] \Rightarrow \exists b (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))))$ i
- 34 $\vdash \forall A \forall B \forall a \forall b (A \sim B \ \& \ A[a] \ \& \ B[b] \Rightarrow (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)))$ i
- 35 $\vdash \forall A \forall B \forall a \forall b ((A \cup (a^\bullet)) \sim (B \cup (b^\bullet)) \ \& \ \neg A[a] \ \& \ \neg B[b] \Rightarrow A \sim B)$ i
- 36 $\vdash \forall P \forall Q \forall R \forall S \forall a \forall b (R[a] \ \& \ R \subseteq P \ \& \ S[b] \ \& \ S \subseteq Q \ \& \ (P \setminus (R \setminus (a^\bullet))) \sim (Q \setminus (S \setminus (b^\bullet))) \Rightarrow (P \setminus R) \sim (Q \setminus S))$ i
- !CHAPTER 14 PAIR REMOVAL!**
- 1 $\mathbb{D} \ \iota \ ; \ (R \ \iota \ a \ b) \ ; \ ; \ \{a, b : R[a, b] \ \& \ (\neg a = a \vee \neg b = b) \}$ i
- 2 $\vdash \forall R \forall a \forall b \forall x \forall y ((R \ \iota \ a \ b)[x, y] \Leftrightarrow R[x, y] \ \& \ (\neg x = a \vee \neg y = b))$ i
- 3 $\vdash \forall R \forall a \forall b \forall x \forall y ((R \ \iota \ a \ b)[x, y] \Rightarrow R[x, y] \ \& \ (\neg x = a \vee \neg y = b))$ i
- 4 $\vdash \forall R \forall a \forall b \forall x \forall y (R[x, y] \ \& \ (\neg x = a \vee \neg y = b) \Rightarrow (R \ \iota \ a \ b)[x, y])$ i
- 5 $\vdash \forall R \forall a \forall b \forall x \forall y ((R \ \iota \ a \ b)[x, y] \Rightarrow R[x, y])$ i
- 6 $\vdash \forall R \forall a \forall b \forall x \forall y (R[x, y] \ \& \ \neg x = a \Rightarrow (R \ \iota \ a \ b)[x, y])$ i
- 7 $\vdash \forall R \forall a \forall b \forall x \forall y (R[x, y] \ \& \ \neg y = b \Rightarrow (R \ \iota \ a \ b)[x, y])$ i
- 8 $\vdash \forall R \forall a \forall b \neg (R \ \iota \ a \ b)[a, b]$ i
- 9 $\vdash \forall R \forall a \forall b (R \ \iota \ a \ b) \subseteq R$ i
- 10 $\vdash \forall R \forall S \forall a \forall b (R \subseteq S \Rightarrow (R \ \iota \ a \ b) \subseteq (S \ \iota \ a \ b))$ i
- 11 $\vdash \forall R \forall S \forall a \forall b (R \equiv S \Rightarrow (R \ \iota \ a \ b) \equiv (S \ \iota \ a \ b))$ i
- 12 $\vdash \forall R \forall a \forall b (R[a, b] \Rightarrow R \equiv ((R \ \iota \ a \ b) \sqcup (a \ \blacksquare \ b)))$ i
- 13 $\vdash \forall R \forall u \forall a (\mathbf{1} R \ \& \ R[u, a] \Rightarrow \neg ((R \ \iota \ u \ a)^I)[a])$ i
- 14 $\vdash \forall R \forall u \forall a (\neg ((R \ \iota \ u \ a)^I)[a] \Rightarrow ((R \ \iota \ u \ a)^I) \equiv ((R^I) \setminus (a^\bullet)))$ i
- 15 $\vdash \forall R \forall u \forall a (\mathbf{1} R \ \& \ R[u, a] \Rightarrow ((R \ \iota \ u \ a)^I) \equiv ((R^I) \setminus (a^\bullet)))$ i
- 16 $\vdash \forall R \forall S \forall u \forall v \forall a (\mathbf{1} R \ \& \ \mathbf{1} S \ \& \ R[u, a] \ \& \ S[v, a] \ \& \ (R^I) \equiv (S^I) \Rightarrow ((R \ \iota \ u \ a)^I) \equiv ((S \ \iota \ v \ a)^I))$ i

!CHAPTER 15 SWITCHINGS;

- 1 $\mathbb{D} \quad \alpha \ ; \ (R \ \alpha \ a \ b) \ ; \ ;$
 $\{ a, b : (\neg b = a \ \& \ \neg b = b \ \& \ R[a, b])$
 $\quad \vee (R[a, a] \ \& \ b = b)$
 $\quad \vee (R[a, b] \ \& \ b = a) \}$ i
- 2 $\vdash \forall R \forall a \forall b \forall x \forall y \ ((R \ \alpha \ a \ b)[x, y] \Leftrightarrow (\neg y = a \ \& \ \neg y = b \ \& \ R[x, y])$
 $\quad \vee (R[x, a] \ \& \ y = b)$
 $\quad \vee (R[x, b] \ \& \ y = a))$ i
- 3 $\vdash \forall R \forall a \forall b \forall x \forall y \ ((R \ \alpha \ a \ b)[x, y] \Rightarrow (\neg y = a \ \& \ \neg y = b \ \& \ R[x, y])$
 $\quad \vee (R[x, a] \ \& \ y = b)$
 $\quad \vee (R[x, b] \ \& \ y = a))$ i
- 4 $\vdash \forall R \forall a \forall b \forall x \forall y \ ((\neg y = a \ \& \ \neg y = b \ \& \ R[x, y])$
 $\quad \vee (R[x, a] \ \& \ y = b)$
 $\quad \vee (R[x, b] \ \& \ y = a)$
 $\quad \Rightarrow (R \ \alpha \ a \ b)[x, y])$ i
- 5 $\vdash \forall R \forall a \forall b \ (R \ \alpha \ a \ b) \subseteq (R \ \alpha \ b \ a)$ i
- 6 $\vdash \forall R \forall a \forall b \ (R \ \alpha \ a \ b) \equiv (R \ \alpha \ b \ a)$ i
- 7 $\vdash \forall R \forall a \forall b \forall x \forall y \ (\neg y = a \ \& \ \neg y = b \ \& \ R[x, y] \Rightarrow (R \ \alpha \ a \ b)[x, y])$
i
- 8 $\vdash \forall R \forall a \forall b \forall x \ (R[x, a] \Rightarrow (R \ \alpha \ a \ b)[x, b])$ i
- 9 $\vdash \forall R \forall a \forall b \forall x \ (R[x, b] \Rightarrow (R \ \alpha \ a \ b)[x, a])$ i
- 10 $\vdash \forall R \forall a \forall b \forall x \forall y \ (\neg y = a \ \& \ \neg y = b \ \& \ (R \ \alpha \ a \ b)[x, y]$
 $\quad \Rightarrow R[x, y])$ i
- 11 $\vdash \forall R \forall a \forall b \forall x \ ((R \ \alpha \ a \ b)[x, a] \Rightarrow R[x, b])$ i
- 12 $\vdash \forall R \forall a \forall b \forall x \ ((R \ \alpha \ a \ b)[x, b] \Rightarrow R[x, a])$ i
- 13 $\vdash \forall R \forall S \forall a \forall b \ (R \subseteq S \Rightarrow (R \ \alpha \ a \ b) \subseteq (S \ \alpha \ a \ b))$ i
- 14 $\vdash \forall R \forall S \forall a \forall b \ (R \equiv S \Rightarrow (R \ \alpha \ a \ b) \equiv (S \ \alpha \ a \ b))$ i
- 15 $\vdash \forall R \forall a \forall b \ ((R \ \alpha \ a \ b) \ \alpha \ a \ b) \subseteq R$ i
- 16 $\vdash \forall R \forall a \forall b \ R \subseteq ((R \ \alpha \ a \ b) \ \alpha \ a \ b)$ i
- 17 $\vdash \forall R \forall a \forall b \ ((R \ \alpha \ a \ b) \ \alpha \ a \ b) \equiv R$ i
- 18 $\vdash \forall R \forall a \forall b \ ((R \ \alpha \ a \ b) \ \alpha \ b \ a) \equiv R$ i
- 19 $\vdash \forall R \forall a \forall b \ R \subseteq ((R \ \alpha \ a \ b) \ \alpha \ b \ a)$ i
- 20 $\vdash \forall R \forall S \forall a \forall b \ (R \subseteq (S \ \alpha \ a \ b) \Rightarrow (R \ \alpha \ a \ b) \subseteq S)$ i
- 21 $\vdash \forall R \forall S \forall a \forall b \ ((R \ \alpha \ a \ b) \subseteq S \Rightarrow R \subseteq (S \ \alpha \ a \ b))$ i
- 22 $\vdash \forall R \forall S \forall a \forall b \ ((R \ \alpha \ a \ b) \equiv S \Rightarrow (S \ \alpha \ a \ b) \equiv R)$ i
- 23 $\vdash \forall R \forall a \forall b \ (R \subseteq (R \ \alpha \ a \ b) \Rightarrow (R \ \alpha \ a \ b) \equiv R)$ i
- 24 $\vdash \forall R \forall a \ R \subseteq (R \ \alpha \ a \ a)$ i
- 25 $\vdash \forall R \forall a \ (R \ \alpha \ a \ a) \equiv R$ i
- 26 $\vdash \forall R \forall S \forall a \forall b \forall x \ (((R \sqcup S) \ \alpha \ a \ b)[x, a]$
 $\quad \Leftrightarrow ((R \ \alpha \ a \ b) \sqcup (S \ \alpha \ a \ b))[x, a])$ i
- 27 $\vdash \forall R \forall S \forall a \forall b \forall x \ (((R \sqcup S) \ \alpha \ a \ b)[x, b]$
 $\quad \Leftrightarrow ((R \ \alpha \ a \ b) \sqcup (S \ \alpha \ a \ b))[x, b])$ i
- 28 $\vdash \forall R \forall S \forall a \forall b \ ((R \sqcup S) \ \alpha \ a \ b) \equiv ((R \ \alpha \ a \ b) \sqcup (S \ \alpha \ a \ b))$ i
- 29 $\vdash \forall R \forall S \forall T \forall U \forall V \forall a \forall b \ (R \equiv (S \sqcup T) \ \& \ (S \ \alpha \ a \ b) \equiv U$
 $\quad \ \& \ (T \ \alpha \ a \ b) \equiv V$
 $\quad \Rightarrow (R \ \alpha \ a \ b) \equiv (U \sqcup V))$ i
- 30 $\vdash \forall a \forall b \ (\Phi \ \alpha \ a \ b) \equiv \Phi$ i
- 31 $\vdash \forall R \forall a \forall b \forall y \ (\neg y = a \ \& \ \neg y = b \ \& \ ((R \ \alpha \ a \ b)^I)[y] \Rightarrow (R^I)[y])$

- 32 $\vdash \forall R \forall a \forall b (((R \alpha a b)^I)[a] \Rightarrow (R^I)[b])$ i
- 33 $\vdash \forall R \forall a \forall b (((R \alpha a b)^I)[b] \Rightarrow (R^I)[a])$ i
- 34 $\vdash \forall R \forall a \forall b \forall y (\neg y = a \ \& \ \neg y = b \ \& \ \neg (R^I)[y] \Rightarrow \neg ((R \alpha a b)^I)[y])$ i
- 35 $\vdash \forall R \forall a \forall b (\neg (R^I)[a] \Rightarrow \neg ((R \alpha a b)^I)[b])$ i
- 36 $\vdash \forall R \forall a \forall b (\neg (R^I)[b] \Rightarrow \neg ((R \alpha a b)^I)[a])$ i
- 37 $\vdash \forall R \forall u \forall v \forall a \forall b (R[u,a] \ \& \ R[v,b] \Rightarrow ((R \alpha a b)^I) \equiv (R^I))$ i
- 38 $\vdash \forall a \forall b \forall u (u \cdot a \subseteq ((u \cdot b) \alpha a b))$ i
- 39 $\vdash \forall a \forall b \forall u ((u \cdot b) \alpha a b \subseteq (u \cdot a))$ i
- 40 $\vdash \forall a \forall b \forall u ((u \cdot b) \alpha a b \equiv (u \cdot a))$ i
- 41 $\vdash \forall a \forall b \forall u ((u \cdot a) \alpha a b \equiv (u \cdot b))$ i
- 42 $\vdash \forall a \forall b \forall u \forall v (\neg v = a \ \& \ \neg v = b \Rightarrow (u \cdot v) \subseteq ((u \cdot v) \alpha a b))$ i
- 43 $\vdash \forall a \forall b \forall u \forall v (\neg v = a \ \& \ \neg v = b \Rightarrow ((u \cdot v) \alpha a b) \equiv (u \cdot v))$ i
- 44 $\vdash \forall R \forall a \forall b \forall u ((R \alpha a b) \upharpoonright u a \subseteq ((R \upharpoonright u b) \alpha a b))$ i
- 45 $\vdash \forall R \forall a \forall b \forall u ((R \upharpoonright u b) \alpha a b \equiv ((R \alpha a b) \upharpoonright u a))$ i
- !CHAPTER 16 SIMPLE CARTESIAN PRODUCTS;**
- 1 $\mathbb{D} \ X ; (a \ X \ B) ; ; \{a,b : a = a \ \& \ B[b]\}$ i
- 2 $\vdash \forall a \forall B \forall x \forall y ((a \ X \ B)[x,y] \Leftrightarrow x = a \ \& \ B[y])$ i
- 3 $\vdash \forall a \forall B \forall x \forall y ((a \ X \ B)[x,y] \Rightarrow x = a \ \& \ B[y])$ i
- 4 $\vdash \forall a \forall B \forall x \forall y ((a \ X \ B)[x,y] \Rightarrow x = a)$ i
- 5 $\vdash \forall a \forall B \forall x \forall y ((a \ X \ B)[x,y] \Rightarrow B[y])$ i
- 6 $\vdash \forall a \forall B \forall b (B[b] \Rightarrow (a \ X \ B)[a,b])$ i
- 7 $\vdash \forall a \forall B \forall b ((a \ X \ B)[a,b] \Leftrightarrow B[b])$ i
- 8 $\vdash \forall a \forall b \forall A \forall B (a = b \ \& \ A \equiv B \Rightarrow (a \ X \ A) \subseteq (b \ X \ B))$ i
- 9 $\vdash \forall a \forall b \forall A \forall B (a = b \ \& \ A \equiv B \Rightarrow (a \ X \ A) \equiv (b \ X \ B))$ i
- 10 $\vdash \forall a \forall b \forall A \forall B ((a \ X \ A) \equiv (b \ X \ B) \Rightarrow A \subseteq B)$ i
- 11 $\vdash \forall a \forall b \forall A \forall B ((a \ X \ A) \equiv (b \ X \ B) \Rightarrow A \equiv B)$ i
- 12 $\vdash \forall a \forall B (B \equiv \phi \Rightarrow (a \ X \ B) \equiv \Phi)$ i
- 13 $\vdash \forall a (a \ X \ \phi) \equiv \Phi$ i
- 14 $\vdash \forall a \forall b (a \ X \ \phi) \equiv (b \ X \ \phi)$ i
- 15 $\vdash \forall a \forall b \forall A \forall B (\neg A \equiv \phi \ \& \ (a \ X \ A) \equiv (b \ X \ B) \Rightarrow a = b)$ i
- 16 $\vdash \forall a \forall b \forall A \forall B (\neg A \equiv \phi \Rightarrow (a = b \ \& \ A \equiv B \Leftrightarrow (a \ X \ A) \equiv (b \ X \ B)))$ i
- 17 $\vdash \forall a \forall B ((a \ X \ B)^D \subseteq (a^\bullet))$ i
- 18 $\vdash \forall a \forall B (B \equiv \phi \Rightarrow ((a \ X \ B)^D) \equiv \phi)$ i
- 19 $\vdash \forall a \forall B (\neg B \equiv \phi \Rightarrow ((a \ X \ B)^D) \equiv (a^\bullet))$ i
- 20 $\vdash \forall a \forall B \forall R \forall P ((R^D) \subseteq P \Rightarrow ((R \sqcup (a \ X \ B))^D) \subseteq (P \cup (a^\bullet)))$ i
- 21 $\vdash \forall a \forall B ((a \ X \ B)^I \equiv B)$ i
- 22 $\vdash \forall a \forall B \forall R (R \equiv (a \ X \ B) \Rightarrow (R^I) \equiv B)$ i
- 23 $\vdash \forall a \forall B \forall R ((R \sqcup (a \ X \ B))^I \equiv ((R^I) \cup B))$ i
- 24 $\vdash \forall a \forall B ((a \ X \ B) \upharpoonright (a^\bullet) \equiv (a \ X \ B))$ i
- 25 $\vdash \forall a \forall x \forall B (\neg x = a \Rightarrow ((a \ X \ B) \upharpoonright (x^\bullet)) \equiv \Phi)$ i
- 26 $\vdash \forall a \forall B \forall R (\neg (R^D)[a] \Rightarrow ((R \sqcup (a \ X \ B)) \upharpoonright (a^\bullet)) \equiv (a \ X \ B))$ i

- 27 $\vdash \forall a \forall x \forall B \forall R (\neg x = a$
 $\Rightarrow ((R \sqcup (a X B)) \uparrow (x^\bullet)) \equiv (R \uparrow (x^\bullet)))$ i
- 28 $\vdash \forall a \forall B \mathbf{1} (a X B)$ i
- 29 $\vdash \forall a \forall B (a X B) \mathbf{1} B$ i
- 30 $\vdash \forall a \forall A \forall B \forall R (R \mathbf{1} A \ \& \ (A \cap B) \equiv \phi$
 $\Rightarrow (R \sqcup (a X B)) \mathbf{1} (A \cup B))$ i
- 31 $\vdash \forall a \forall B \forall R (\mathbf{1} R \ \& \ ((R^I) \cap B) \equiv \phi \Rightarrow \mathbf{1} (R \sqcup (a X B)))$ i
- !CHAPTER 17 PROJECTIONS OF TRIADIC PREDICATES;**
- 1 $\mathbb{D} \diamond ; (T \diamond c) ; ; \{a, b : T[a, c, b]\}$ i
- 2 $\vdash \forall T \forall c \forall x \forall y ((T \diamond c)[x, y] \Leftrightarrow T[x, c, y])$ i
- 3 $\vdash \forall T \forall c \forall x \forall y ((T \diamond c)[x, y] \Rightarrow T[x, c, y])$ i
- 4 $\vdash \forall T \forall c \forall x \forall y (T[x, c, y] \Rightarrow (T \diamond c)[x, y])$ i
- 5 $\vdash \forall T \forall c \forall x \forall y (((T \diamond c)^*)[x, y] \Leftrightarrow T[y, c, x])$ i

SECTION IV: THE NATURAL NUMBERS

!CHAPTER 1 COUNTING;

- 1 $\vdash \forall n \forall C (\mathbb{C}[n, C] \Rightarrow \mathbf{1} C)$ i
- 2 $\vdash \forall C (\mathbb{C}[0, C] \Leftrightarrow C \equiv \Phi)$ i
- 3 $\vdash \forall C (\mathbb{C}[0, C] \Rightarrow C \equiv \Phi)$ i
- 4 $\vdash \forall C (C \equiv \Phi \Rightarrow \mathbb{C}[0, C])$ i
- 5 $\vdash \forall C (\mathbb{C}[0, C] \Rightarrow \neg \exists x \exists y C[x, y])$ i
- 6 $\vdash \forall n \forall C \forall x \forall y (\mathbb{C}[n, C] \ \& \ C[x, y] \Rightarrow \neg n = 0)$ i
- 7 $\vdash \forall n \forall C (\mathbb{C}[n, C] \ \& \ \mathbb{C}[0, C] \Rightarrow n = 0)$ i
- 8 $\vdash \forall n \forall C \forall D (\mathbb{C}[n, C] \ \& \ \mathbb{C}[0, D] \ \& \ (C^I) \equiv (D^I) \Rightarrow n = 0)$ i
- 9 $\vdash \forall n \forall m \forall C \forall D \forall a (\omega[n] \ \& \ \sigma[n, m] \ \& \ \neg (C^I)[a] \ \& \ D \equiv (C \sqcup (m \blacksquare a))$
 $\Rightarrow (\mathbb{C}[n, C] \Leftrightarrow \mathbb{C}[m, D]))$ i
- 10 $\vdash \forall n \forall m \forall C \forall D \forall a (\omega[n] \ \& \ \sigma[n, m] \ \& \ \neg (C^I)[a] \ \& \ D \equiv (C \sqcup (m \blacksquare a))$
 $\ \& \ \mathbb{C}[n, C]$
 $\Rightarrow \mathbb{C}[m, D])$ i
- 11 $\vdash \forall n \forall m \forall C \forall D \forall a (\omega[n] \ \& \ \sigma[n, m] \ \& \ \neg (C^I)[a] \ \& \ D \equiv (C \sqcup (m \blacksquare a))$
 $\ \& \ \mathbb{C}[m, D]$
 $\Rightarrow \mathbb{C}[n, C])$ i
- 12 $\vdash \forall n \forall m \forall C \forall a (\omega[n] \ \& \ \sigma[n, m] \ \& \ \mathbb{C}[m, C] \ \& \ C[m, a]$
 $\Rightarrow \mathbb{C}[n, (C \uparrow m a)] \ \& \ \neg ((C \uparrow m a)^I)[a])$ i
- 13 $\vdash \forall n \forall m \forall C \forall a (\omega[n] \ \& \ \sigma[n, m] \ \& \ \mathbb{C}[m, C] \ \& \ C[m, a]$
 $\Rightarrow \mathbb{C}[n, (C \uparrow m a)])$ i
- 14 $\vdash \forall n \forall m \forall C (\omega[n] \ \& \ \sigma[n, m] \ \& \ \mathbb{C}[m, C] \ \& \ \neg m = 0$
 $\Rightarrow \exists a (C[m, a] \ \& \ \mathbb{C}[n, (C \uparrow m a)] \ \& \ \neg ((C \uparrow m a)^I)[a])$ i
- 15 $\vdash \forall n (\omega[n] \ \& \ \neg n = 0 \Rightarrow \exists p (\omega[p] \ \& \ \sigma[p, n]))$ i
- 16 $\vdash \forall n \forall C \forall D \forall a \forall b (\omega[n] \ \& \ \mathbb{C}[n, C] \ \& \ D \equiv (C \alpha a b) \Rightarrow \mathbb{C}[n, D])$ i
- 17 $\vdash \forall n \forall C \forall D (\omega[n] \ \& \ \mathbb{C}[n, C] \ \& \ C \equiv D \Rightarrow \mathbb{C}[n, D])$ i
- 18 $\vdash \forall n \forall C \forall a (\omega[n] \ \& \ \mathbb{C}[n, C] \ \& \ \neg \exists x C[x, a]$
 $\Rightarrow \exists m (\omega[m] \ \& \ \mathbb{C}[m, (C \sqcup (m \blacksquare a))]))$ i
- 19 $\vdash \forall n \forall k \forall C \forall D (\omega[n] \ \& \ \exists p (\omega[p] \ \& \ \sigma[p, k]) \ \& \ \mathbb{C}[n, C] \ \& \ \mathbb{C}[k, D]$
 $\ \& \ (C^I) \equiv (D^I) \Rightarrow n = k)$ i

!CHAPTER 2 AN INTERMEDIATE SUPPORT;

- 1 $\vdash \forall n \forall P (\mathfrak{N}[n, P] \Leftrightarrow \exists C (\mathfrak{C}[n, C] \& (C^I) \equiv P))$ i
- 2 $\vdash \forall n \forall P (\mathfrak{N}[n, P] \Rightarrow \exists C (\mathfrak{C}[n, C] \& (C^I) \equiv P))$ i
- 3 $\vdash \forall n \forall P \forall C (\mathfrak{C}[n, C] \& (C^I) \equiv P \Rightarrow \mathfrak{N}[n, P])$ i
- 4 $\vdash \forall n \forall P (\mathfrak{N}[n, P] \& \mathfrak{N}[0, P] \Rightarrow n = 0)$ i
- 5 $\vdash \forall n \forall k \forall P (\omega[n] \& \exists p (\omega[p] \& \sigma[p, k]) \& \mathfrak{N}[n, P] \& \mathfrak{N}[k, P] \Rightarrow n = k)$ i
- 6 $\vdash \forall P (\mathfrak{N}[0, P] \Leftrightarrow P \equiv \phi)$ i
- 7 $\vdash \forall n \forall m \forall P \forall Q \forall a (\omega[n] \& \sigma[n, m] \& \neg P[a] \& Q \equiv (P \cup (a^\bullet)) \& \mathfrak{N}[n, P] \Rightarrow \mathfrak{N}[m, Q])$ i
- 8 $\vdash \forall n \forall m \forall P \forall Q \forall a (\omega[n] \& \sigma[n, m] \& \neg P[a] \& Q \equiv (P \cup (a^\bullet)) \& \mathfrak{N}[m, Q] \Rightarrow \mathfrak{N}[n, P])$ i
- 9 $\vdash \forall P \forall a \forall n (\omega[n] \& \mathfrak{N}[n, P] \& \neg P[a] \Rightarrow \exists m (\omega[m] \& \mathfrak{N}[m, (P \cup (a^\bullet))]))$ i
- 10 $\vdash \forall n \forall m \forall P (\omega[n] \& \omega[m] \& \mathfrak{N}[n, P] \& \mathfrak{N}[m, P] \Rightarrow n = m)$ i
- 11 $\vdash \forall n \forall m \forall P \forall a (\omega[n] \& \sigma[n, m] \& P[a] \& \mathfrak{N}[m, P] \Rightarrow \mathfrak{N}[n, (P \setminus (a^\bullet))])$ i
- 12 $\vdash \forall n \forall m \forall P \forall a (\omega[n] \& \sigma[n, m] \& \neg P[a] \& \mathfrak{N}[n, P] \Rightarrow \mathfrak{N}[m, (P \cup (a^\bullet))])$ i
- 13 $\vdash \forall n \forall P \forall Q (\mathfrak{N}[n, P] \& P \equiv Q \Rightarrow \mathfrak{N}[n, Q])$ i

!CHAPTER 3 THE NUMBER 0;

- 1 $\vdash \forall P (\mathfrak{N}[0, P] \Rightarrow P \equiv \phi)$ i
- 2 $\vdash \forall P (P \equiv \phi \Rightarrow \mathfrak{N}[0, P])$ i
- 3 $\vdash \forall P (\mathfrak{N}[0, P] \Rightarrow \forall x \neg P[x])$ i
- 4 $\vdash \forall P (\forall x \neg P[x] \Rightarrow \mathfrak{N}[0, P])$ i
- 5 $\vdash \forall P (\mathfrak{N}[0, P] \Rightarrow \neg \exists x P[x])$ i
- 6 $\vdash \forall P (\neg \exists x P[x] \Rightarrow \mathfrak{N}[0, P])$ i
- 7 $\vdash \forall P (\neg P \equiv \phi \Rightarrow \neg \mathfrak{N}[0, P])$ i
- 8 $\vdash \forall P (\neg \mathfrak{N}[0, P] \Rightarrow \neg P \equiv \phi)$ i
- 9 $\vdash \forall P (\exists x P[x] \Rightarrow \neg \mathfrak{N}[0, P])$ i
- 10 $\vdash \forall P (\neg \mathfrak{N}[0, P] \Rightarrow \exists x P[x])$ i
- 11 $\vdash \forall P \forall n (\mathfrak{N}[n, P] \& \neg P \equiv \phi \Rightarrow \neg n = 0)$ i
- 12 $\vdash \forall P \forall n (\mathfrak{N}[n, P] \& \exists x P[x] \Rightarrow \neg n = 0)$ i
- 13 $\vdash \forall P (\mathfrak{N}[0, P] \vee \exists x P[x])$ i
- 14 $\vdash \mathfrak{N}[0, \phi]$ i
- 15 $\vdash \forall P \forall Q (\mathfrak{N}[0, P] \Rightarrow (P \sim Q \Leftrightarrow \mathfrak{N}[0, Q]))$ i
- 16 $\vdash \forall P \forall n (\mathfrak{N}[n, P] \& \forall x \neg P[x] \Rightarrow n = 0)$ i
- 17 $\vdash \forall P \forall n (\mathfrak{N}[n, P] \& \neg n = 0 \Rightarrow \exists x P[x])$ i
- 18 $\vdash \forall P \forall Q \forall n (\mathfrak{N}[n, P] \& \mathfrak{N}[n, Q] \& \exists x P[x] \Rightarrow \exists x Q[x])$ i

!CHAPTER 4 NUMBERS AND CORRESPONDENCES;

- 1 $\vdash \forall n \forall P \forall Q (\omega[n] \& \mathfrak{N}[n, P] \Rightarrow (P \sim Q \Leftrightarrow \mathfrak{N}[n, Q]))$ i
- 2 $\vdash \forall n \forall P \forall Q (\omega[n] \& \mathfrak{N}[n, P] \& P \sim Q \Rightarrow \mathfrak{N}[n, Q])$ i
- 3 $\vdash \forall n \forall P \forall Q (\omega[n] \& \mathfrak{N}[n, P] \& Q \sim P \Rightarrow \mathfrak{N}[n, Q])$ i
- 4 $\vdash \forall n \forall P \forall Q (\omega[n] \& \mathfrak{N}[n, P] \& \mathfrak{N}[n, Q] \Rightarrow P \sim Q)$ i
- 5 $\vdash \forall n \forall P \forall Q (\omega[n] \& \mathfrak{N}[n, P] \& P \equiv Q \Rightarrow \mathfrak{N}[n, Q])$ i

- 6 $\vdash \forall n \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n, P] \ \& \ Q \equiv P \Rightarrow \mathfrak{N}[n, Q])$ i
- 7 $\vdash \forall n \forall m \forall P \forall Q (\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n, P] \ \& \ \mathfrak{N}[m, Q] \ \& \ P \sim Q \Rightarrow n = m)$ i
- 8 $\vdash \forall n \forall R \forall S (\omega[n] \ \& \ \mathfrak{N}[n, (R^D)] \ \& \ R \equiv S \Rightarrow \mathfrak{N}[n, (S^D)])$ i
- 9 $\vdash \forall n \forall R \forall S (\omega[n] \ \& \ \mathfrak{N}[n, (R^D)] \ \& \ S \equiv R \Rightarrow \mathfrak{N}[n, (S^D)])$ i
- 10 $\vdash \forall n \forall R \forall S (\omega[n] \ \& \ \mathfrak{N}[n, (R^I)] \ \& \ R \equiv S \Rightarrow \mathfrak{N}[n, (S^I)])$ i
- 11 $\vdash \forall n \forall R \forall S (\omega[n] \ \& \ \mathfrak{N}[n, (R^I)] \ \& \ S \equiv R \Rightarrow \mathfrak{N}[n, (S^I)])$ i
- 12 $\vdash \forall n \forall P \forall R (\omega[n] \ \& \ \mathfrak{N}[n, ((R \uparrow P)^I)]$
 $\Rightarrow \mathfrak{N}[n, \{y : \exists z (R[z, y] \ \& \ P[z])\}])$ i
- 13 $\vdash \forall n \forall R \forall x (\omega[n] \ \& \ \mathfrak{N}[n, ((R \uparrow (x^\bullet))^I)] \Rightarrow \mathfrak{N}[n, \{y : R[x, y]\}])$ i
- 14 $\vdash \forall n \forall m \forall k \forall A \forall B \forall C$
 $(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n, A] \ \& \ \mathfrak{N}[m, B] \ \& \ \mathfrak{N}[k, (A \cup B)]$
 $\ \& \ \mathfrak{N}[k, C] \ \& \ (A \cap B) \equiv \phi$
 $\Rightarrow \exists Q \exists R (\mathfrak{N}[n, Q] \ \& \ \mathfrak{N}[m, R] \ \& \ (Q \cup R) \equiv C \ \& \ (Q \cap R) \equiv \phi))$ i
- 15 $\vdash \forall n \forall P \forall a \forall b (\omega[n] \ \& \ \mathfrak{N}[n, (P \cup (a^\bullet))] \ \& \ \neg P[a] \ \& \ \neg P[b]$
 $\Rightarrow \mathfrak{N}[n, (P \cup (b^\bullet))])$ i
- 16 $\vdash \forall n \forall P \forall a (\omega[n] \ \& \ \mathfrak{N}[n, P] \Rightarrow \exists m (\omega[m] \ \& \ \mathfrak{N}[m, (P \cup (a^\bullet))]))$ i
- 17 $\vdash \forall n \forall k \forall P \forall Q \forall R \forall S (\omega[n] \ \& \ \mathfrak{N}[n, R] \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P$
 $\ \& \ S \subseteq Q$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$ i
- 18 $\vdash \forall n \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n, P] \ \& \ \mathfrak{N}[n, Q] \ \& \ P \subseteq Q \Rightarrow P \equiv Q)$ i
- !CHAPTER 5 FINITE PREDICATES;**
- 1 $\mathfrak{S} f ; f P ; \exists n (\omega[n] \ \& \ \mathfrak{N}[n, P])$ i
- 2 $\vdash \forall n \forall P (\omega[n] \ \& \ \mathfrak{N}[n, P] \Rightarrow f P)$ i
- 3 $\vdash \forall P \forall Q (f P \ \& \ P \sim Q \Rightarrow f Q)$ i
- 4 $\vdash \forall P \forall Q (f P \ \& \ Q \sim P \Rightarrow f Q)$ i
- 5 $\vdash \forall P \forall Q (f P \ \& \ P \equiv Q \Rightarrow f Q)$ i
- 6 $\vdash \forall P \forall Q (f P \ \& \ Q \equiv P \Rightarrow f Q)$ i
- 7 $\vdash f \phi$ i
- 8 $\vdash \forall P (P \equiv \phi \Rightarrow f P)$ i
- 9 $\vdash \forall P (\neg \exists x P[x] \Rightarrow f P)$ i
- 10 $\vdash \forall P \forall Q \forall R \forall S (f R \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$ i
- 11 $\vdash \forall n \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n, P] \ \& \ Q \subseteq P \Rightarrow f Q)$ i
- 12 $\vdash \forall P \forall Q (f P \ \& \ Q \subseteq P \Rightarrow f Q)$ i
- 13 $\vdash \forall P \forall Q (f (P \cup Q) \Rightarrow f P)$ i
- 14 $\vdash \forall P \forall Q (f (P \cup Q) \Rightarrow f Q)$ i
- 15 $\vdash \forall P \forall Q (f (P \cup Q) \Rightarrow f P \ \& \ f Q)$ i
- 16 $\vdash \forall P \forall a (f P \Rightarrow f (P \cup (a^\bullet)))$ i
- 17 $\vdash \forall a f (a^\bullet)$ i
- 18 $\vdash \forall n \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n, Q] \ \& \ f P \Rightarrow f (P \cup Q))$ i
- 19 $\vdash \forall P \forall Q \forall m \forall k (\omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[m, P] \ \& \ \mathfrak{N}[k, Q]$
 $\Rightarrow \exists n (\omega[n] \ \& \ \mathfrak{N}[n, (P \cup Q)]))$ i
- 20 $\vdash \forall P \forall Q (f P \ \& \ f Q \Rightarrow f (P \cup Q))$ i
- 21 $\vdash \forall P \forall Q \forall R (f P \ \& \ f Q \ \& \ (P \cup Q) \equiv R \Rightarrow f R)$ i

- 22 $\vdash \forall P \forall Q \forall R (f P \ \& \ f Q \ \& \ R \equiv (P \cup Q) \Rightarrow f R)$ i
23 $\vdash \forall a \forall b f (a \ b \ \ddagger)$ i
24 $\vdash \forall a \forall b \forall c f (a \ b \ c \ \forall)$ i
25 $\vdash \forall a \forall b \forall c \forall d f (a \ b \ c \ d \ \forall)$ i
26 $\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P$
 $\Rightarrow \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P))$ i

!CHAPTER 6 INFINITE PREDICATES;

- 1 $\S \ \iota ; \ \iota P ; \ \neg f P$ i
2 $\vdash \forall P (f P \Rightarrow \neg \iota P)$ i
3 $\vdash \forall P (\neg \iota P \Rightarrow f P)$ i
4 $\vdash \forall n \forall P (\iota P \Rightarrow \neg (\omega[n] \ \& \ \mathfrak{N}[n,P]))$ i
5 $\vdash \forall P (\neg \exists n (\omega[n] \ \& \ \mathfrak{N}[n,P]) \Rightarrow \iota P)$ i
6 $\vdash \forall P \forall Q (\iota P \ \& \ P \sim Q \Rightarrow \iota Q)$ i
7 $\vdash \forall P \forall Q (\iota P \ \& \ Q \sim P \Rightarrow \iota Q)$ i
8 $\vdash \forall P \forall Q (\iota P \ \& \ P \equiv Q \Rightarrow \iota Q)$ i
9 $\vdash \forall P \forall Q (\iota P \ \& \ Q \equiv P \Rightarrow \iota Q)$ i
10 $\vdash \forall P (\iota P \Rightarrow \exists x P[x])$ i
11 $\vdash \forall P \forall Q (\iota Q \ \& \ Q \subseteq P \Rightarrow \iota P)$ i
12 $\vdash \forall P \forall Q (\iota P \ \& \ f Q \Rightarrow \iota (P \setminus Q))$ i
13 $\vdash \forall P \forall a (\iota P \Rightarrow \iota (P \setminus (a^\bullet)))$ i
14 $\vdash \forall P \forall Q (\iota P \ \& \ f Q \Rightarrow \exists x (P \setminus Q)[x])$ i
15 $\vdash \forall n \forall P (\omega[n] \ \& \ \iota P \Rightarrow \exists Q (\mathfrak{N}[n,Q] \ \& \ Q \subseteq P))$ i
16 $\vdash \forall n \forall P \forall Q (\omega[n] \ \& \ \iota P \ \& \ f Q$
 $\Rightarrow \exists R (\mathfrak{N}[n,R] \ \& \ R \subseteq P \ \& \ (Q \cap R) \equiv \phi))$ i
17 $\vdash \forall n (\omega[n] \Rightarrow \exists P \exists a (\mathfrak{N}[n,P] \ \& \ \neg P[a])) \Rightarrow \iota \mathbb{U}$ i

!CHAPTER 7 AN INFINITY OF THINGS;

- 1 $\S \ \mathfrak{H} ; \ \mathfrak{H} P ; \ \forall Q (P[Q] \Rightarrow Q \subset P)$ i
2 $\vdash \forall P (\mathfrak{H} P \Rightarrow \neg P[P])$ i
3 $\vdash \mathfrak{H} \phi$ i
4 $\vdash \forall P (\mathfrak{H} P \Rightarrow \neg P \equiv (P \cup (P^\bullet)))$ i
5 $\vdash \forall P (\mathfrak{H} P \Rightarrow \mathfrak{H} (P \cup (P^\bullet)))$ i
6 $\vdash \forall n (\omega[n] \Rightarrow \exists B (\mathfrak{H} B \ \& \ \mathfrak{N}[n,B]))$ i
7 $\vdash \forall n (\omega[n] \Rightarrow \exists P \exists a (\mathfrak{N}[n,P] \ \& \ \neg P[a]))$ i
8 $\vdash \forall n (\omega[n] \Rightarrow \exists P \mathfrak{N}[n,P])$ i
9 $\vdash \forall n (\omega[n] \ \& \ \neg n = 0 \Rightarrow \exists P \exists a (\mathfrak{N}[n,P] \ \& \ P[a]))$ i
10 $\vdash \iota \mathbb{U}$ i
11 $\vdash \forall P (f P \Rightarrow \iota (\mathbb{U} \setminus P))$ i
12 $\vdash \forall P (f P \Rightarrow \exists x \neg P[x])$ i
13 $\vdash \forall P \forall n (\omega[n] \ \& \ \mathfrak{N}[n,P] \Rightarrow \exists x \neg P[x])$ i
14 $\vdash \forall n \forall m (\omega[n] \ \& \ \sigma[n,m]$
 $\Rightarrow \exists P \exists a (\mathfrak{N}[n,P] \ \& \ \neg P[a] \ \& \ \mathfrak{N}[m, (P \cup (a^\bullet))]))$ i
15 $\vdash \forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \Rightarrow \exists P \exists a \mathfrak{N}[m, (P \cup (a^\bullet))])$ i
16 $\vdash \forall n \forall m \forall P (\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ \sigma[n,m]$
 $\Rightarrow \exists a (\neg P[a] \ \& \ \mathfrak{N}[m, (P \cup (a^\bullet))]))$ i
17 $\vdash \forall n \forall P (\omega[n] \ \& \ f P \Rightarrow \exists Q (\mathfrak{N}[n,Q] \ \& \ (P \cap Q) \equiv \phi))$ i
18 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m]$
 $\Rightarrow \exists P \exists Q (\mathfrak{N}[n,P] \ \& \ \mathfrak{N}[m,Q] \ \& \ (P \cap Q) \equiv \phi))$ i

- 19 $\vdash \forall n (\omega[n] \Rightarrow \forall P \exists Q (\mathcal{R}_{[n,Q]} \& (P \subseteq Q \vee Q \subseteq P)))$;
20 $\vdash \forall n \forall P (\omega[n] \Rightarrow \exists Q (\mathcal{R}_{[n,Q]} \& (P \subseteq Q \vee Q \subseteq P)))$;
21 $\vdash \forall n \forall m (\omega[n] \& \omega[m]$
 $\Rightarrow \exists P \exists Q (\mathcal{R}_{[n,P]} \& \mathcal{R}_{[m,Q]} \& (P \subseteq Q \vee Q \subseteq P)))$;

!CHAPTER 8 SUCCESSORS;

- 1 $\vdash \forall n \forall m \forall P \forall a (\omega[n] \& \omega[m] \& \mathcal{R}_{[n,P]} \& \mathcal{R}_{[m,(P \cup (a^\bullet))]} \& \neg P[a]$
 $\Rightarrow \sigma[n,m])$;
2 $\vdash \forall n \forall m (\omega[n] \& \sigma[n,m] \Rightarrow \omega[m])$;
3 $\vdash \forall n (\omega[n] \Rightarrow \exists m \sigma[n,m])$;
4 $\vdash \forall n \forall m \forall k (\omega[n] \& \sigma[n,m] \& \sigma[n,k] \Rightarrow m = k)$;
5 $\vdash \forall n \forall m \forall k (\omega[n] \& \omega[k] \& \sigma[n,m] \& \sigma[k,m] \Rightarrow n = k)$;
6 $\vdash \forall n (\omega[n] \Rightarrow \neg \sigma[n,0])$;
7 $\vdash \forall x (\exists y (\sigma \upharpoonright \omega [x,y] \Leftrightarrow \omega[x]))$;
8 $\vdash ((\sigma \upharpoonright \omega)^D) \equiv \omega$;
9 $\vdash \forall x \forall y \forall z ((\sigma \upharpoonright \omega)[x,y] \& (\sigma \upharpoonright \omega)[x,z] \Rightarrow y = z)$;
10 $\vdash \mathbf{f} (\sigma \upharpoonright \omega)$;
11 $\vdash \forall n (\omega[n] \Rightarrow \mathbf{f} (\sigma \upharpoonright \omega) \& ((\sigma \upharpoonright \omega)^D)[n])$;
12 $\mathbb{D} \quad ' ; (n') ; \omega[n] ; ((\sigma \upharpoonright \omega)'n) \quad ;! (\mathbb{D} \text{ Prop})$;
13 $\vdash \forall n (\omega[n] \Rightarrow \sigma[n,(n')])$;
14 $\vdash \forall n \forall m ((n') = m \Rightarrow \omega[n] \& \sigma[n,m])$;
15 $\vdash \forall n \forall m (m = (n') \Rightarrow \omega[n] \& \sigma[n,m])$;
16 $\vdash \forall n \forall m ((n') = m \Rightarrow \sigma[n,m])$;
17 $\vdash \forall n \forall m (m = (n') \Rightarrow \sigma[n,m])$;
18 $\vdash \forall n \forall m (\omega[n] \& \sigma[n,m] \Rightarrow (n') = m)$;
19 $\vdash \forall n \forall m (\omega[n] \& \sigma[n,m] \Rightarrow m = (n'))$;
20 $\vdash \forall n \forall m ((n') = m \Leftrightarrow (\sigma \upharpoonright \omega)[n,m])$;
21 $\vdash \forall n \forall m \forall P \forall a (\omega[n] \& \omega[m] \& \mathcal{R}_{[n,P]} \& \mathcal{R}_{[m,(P \cup (a^\bullet))]} \& \neg P[a]$
 $\Rightarrow (n') = m)$;
22 $\vdash \forall n \forall P \forall a (\omega[n] \& \mathcal{R}_{[n,P]} \& \neg P[a] \Rightarrow \mathcal{R}_{[(n'),(P \cup (a^\bullet))]})$;
23 $\vdash \forall n (\omega[n] \Rightarrow \exists P \exists a (\mathcal{R}_{[n,P]} \& \neg P[a] \& \mathcal{R}_{[(n'),(P \cup (a^\bullet))]}))$;
24 $\vdash \forall n (\omega[n] \Rightarrow \exists P \exists a \mathcal{R}_{[(n'),(P \cup (a^\bullet))]})$;
25 $\vdash \forall n (\omega[n] \Rightarrow \omega[(n')])$;
26 $\vdash \forall n \forall m ((n') = m \Rightarrow \omega[m])$;
27 $\vdash \forall n \forall m (m = (n') \Rightarrow \omega[m])$;
28 $\vdash \forall n \forall m (\omega[n] \& n = m \Rightarrow (n') = (m'))$;
29 $\vdash \forall n \forall m (\neg \omega[n] \& n = m \Rightarrow \neg (n') = (m'))$;
30 $\vdash \forall n \forall m (\omega[n] \& n = m \Rightarrow (m') = (n'))$;
31 $\vdash \forall n \forall m ((n') = (m') \Rightarrow n = m)$;
32 $\vdash \forall n \neg (n') = 0$;
33 $\vdash \neg \exists n (n') = 0$;
34 $\vdash \forall n (\omega[n] \& \neg n = 0 \Rightarrow \exists m (m') = n)$;
35 $\vdash \forall n \neg (n') = n$;
36 $\vdash \forall n \forall m ((n') = m \Rightarrow \neg n = m)$;

!CHAPTER 9 THE NUMBERS 1 AND 2;

- 1 $\vdash (0') = 1$;
2 $\vdash \omega[1]$;

3 $\vdash \forall a \mathcal{N}[1, (a^\bullet)]$ i
 4 $\vdash \mathcal{N}[1, (0^\bullet)]$ i
 5 $\vdash \neg 0 = 1$ i
 6 $\vdash \neg 1 = 0$ i
 7 $\vdash \neg (0^\bullet)[1]$ i
 8 $\vdash \forall P (\mathcal{N}[1, P] \Rightarrow \exists a P \equiv (a^\bullet))$ i
 9 $\vdash \forall P (\mathcal{N}[1, P] \Rightarrow \exists a (P[a] \& \forall x (P[x] \Rightarrow x = a)))$ i
 10 $\vdash (1') = 2$ i
 11 $\vdash \omega[2]$ i
 12 $\vdash \forall a \forall b (\neg a = b \Rightarrow \mathcal{N}[2, (a b \dagger)])$ i
 13 $\vdash \mathcal{N}[2, (0 1 \dagger)]$ i
 14 $\vdash ((0')') = 2$ i
 15 $\vdash \neg 0 = 2$ i
 16 $\vdash \neg 1 = 2$ i
 17 $\vdash \neg 2 = 0$ i
 18 $\vdash \neg 2 = 1$ i
 19 $\vdash \neg (0 1 \dagger)[2]$ i

!CHAPTER 10 THE NUMBERS 3 AND 4;

1 $\vdash (2') = 3$ i
 2 $\vdash \omega[3]$ i
 3 $\vdash \forall a \forall b \forall c (\neg a = b \& \neg c = a \& \neg c = b \Rightarrow \mathcal{N}[3, (a b c \vee)])$ i
 4 $\vdash \mathcal{N}[3, (0 1 2 \vee)]$ i
 5 $\vdash ((1')') = 3$ i
 6 $\vdash (((0')')') = 3$ i
 7 $\vdash \neg 3 = 0$ i
 8 $\vdash \neg 3 = 1$ i
 9 $\vdash \neg 3 = 2$ i
 10 $\vdash \neg (0 1 2 \vee)[3]$ i
 11 $\vdash (3') = 4$ i
 12 $\vdash \omega[4]$ i
 13 $\vdash \forall a \forall b \forall c \forall d (\neg a = b \& \neg c = a \& \neg c = b \& \neg d = a$
 $\quad \& \neg d = b \& \neg d = c$
 $\quad \Rightarrow \mathcal{N}[4, (a b c d \forall)])$ i
 14 $\vdash \mathcal{N}[4, (0 1 2 3 \forall)]$ i
 15 $\vdash ((2')') = 4$ i
 16 $\vdash (((1')')') = 4$ i
 17 $\vdash (((0')')')') = 4$ i

!CHAPTER 11 k-UNIFORM PREDICATES;

1 $\mathcal{S} \cup ; \mathcal{R} \cup_k P; \omega[k] \& \forall x (P[x] \Rightarrow \mathcal{N}[k, ((\mathcal{R} \lceil (x^\bullet))^\perp)])$ i
 2 $\vdash \forall R \forall P \forall k (R \cup_k P \Rightarrow \omega[k])$ i
 3 $\vdash \forall R \forall S \forall P \forall Q \forall k (R \cup_k P \& Q \subseteq P \& R \equiv S \Rightarrow S \cup_k Q)$ i
 4 $\vdash \forall R \forall P \forall Q \forall k (R \cup_k P \& Q \subseteq P \Rightarrow R \cup_k Q)$ i
 5 $\vdash \forall R \forall P \forall Q \forall k (R \cup_k P \& Q \equiv P \Rightarrow R \cup_k Q)$ i
 6 $\vdash \forall R \forall P \forall Q \forall k (R \cup_k P \& P \equiv Q \Rightarrow R \cup_k Q)$ i
 7 $\vdash \forall R \forall S \forall P \forall k (R \cup_k P \& R \equiv S \Rightarrow S \cup_k P)$ i
 8 $\vdash \forall R \forall P \forall A \forall k (R \cup_k P \& A \subseteq P \Rightarrow (R \lceil A) \cup_k A)$ i
 9 $\vdash \forall R \forall k (\omega[k] \Rightarrow R \cup_k \phi)$ i

- 10 $\vdash \forall P \Phi \cup_0 P$ i
- 11 $\vdash \forall R \forall P \forall k (R \equiv \Phi \ \& \ \neg P \equiv \phi \ \& \ R \cup_k P \Rightarrow k = 0)$ i
- 12 $\vdash \forall B \forall a \forall k (\omega[k] \ \& \ \mathfrak{N}_{[k,B]} \Rightarrow (a \ X \ B) \cup_k (a^\bullet))$ i
- 13 $\vdash \forall R \forall S \forall P \forall Q \forall k (R \cup_k P \ \& \ S \cup_k Q \ \& \ ((S^D) \cap P) \equiv \phi$
 $\ \& \ ((R^D) \cap Q) \equiv \phi$
 $\ \Rightarrow (R \sqcup S) \cup_k (P \cup Q))$ i
- 14 $\vdash \forall R \forall P \forall B \forall a \forall k (R \cup_k P \ \& \ (R^D) \subseteq P \ \& \ \neg P[a] \ \& \ \mathfrak{N}_{[k,B]}$
 $\ \Rightarrow (R \sqcup (a \ X \ B)) \cup_k (P \cup (a^\bullet)))$ i
- 15 $\vdash \forall n \forall k \forall P \forall R (\omega[n] \ \& \ \mathfrak{N}_{[n,P]} \ \& \ R \cup_k P \Rightarrow f ((R \uparrow P)^I))$ i
- 16 $\vdash \forall n \forall j (\omega[n] \ \& \ \omega[j]$
 $\ \Rightarrow \exists P \exists R (\mathfrak{N}_{[n,P]} \ \& \ R \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)))$ i
- 17 $\vdash \forall n \forall j (\omega[n] \ \& \ \omega[j] \Rightarrow \exists P \exists R (\mathfrak{N}_{[n,P]} \ \& \ R \cup_j P \ \& \ \mathbf{1} \ R))$ i
- 18 $\vdash \forall n \forall j (\omega[n] \ \& \ \omega[j]$
 $\ \Rightarrow \exists P \exists R (\mathfrak{N}_{[n,P]} \ \& \ R \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P))$ i

SECTION V: ADDITION, SUBTRACTION, AND MULTIPLICATION

!CHAPTER 1 ADDITION;

!CHAPTER 1 ADDITION;

- 1 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}_{[n,A]} \ \& \ \mathfrak{N}_{[m,B]}$
 $\ \& \ (A \cap B) \equiv \phi$
 $\ \Rightarrow (\oplus[n,m,k] \Leftrightarrow \mathfrak{N}_{[k,(A \cup B)]})$ i
- 2 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}_{[n,A]} \ \& \ \mathfrak{N}_{[m,B]}$
 $\ \& \ (A \cap B) \equiv \phi \ \& \ \oplus[n,m,k]$
 $\ \Rightarrow \mathfrak{N}_{[k,(A \cup B)]})$ i
- 3 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}_{[n,A]} \ \& \ \mathfrak{N}_{[m,B]}$
 $\ \& \ (A \cap B) \equiv \phi \ \& \ \mathfrak{N}_{[k,(A \cup B)]}$
 $\ \Rightarrow \oplus[n,m,k])$ i
- 4 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \oplus[n,m,k]$
 $\ \Rightarrow \exists A \exists B (\mathfrak{N}_{[n,A]} \ \& \ \mathfrak{N}_{[m,B]} \ \& \ (A \cap B) \equiv \phi$
 $\ \& \ \mathfrak{N}_{[k,(A \cup B)]})$ i
- 5 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \exists a (\omega[a] \ \& \ \oplus[n,m,a]))$ i
- 6 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m]$
 $\ \Rightarrow \forall a \forall b ((\omega[a] \ \& \ \oplus[n,m,a]) \ \& \ (\omega[b] \ \& \ \oplus[n,m,b])$
 $\ \Rightarrow a = b))$ i
- 7 $\mathbb{T} + ; (n + m) ; \omega[n] \ \& \ \omega[m] ; (\omega[a] \ \& \ \oplus[n,m,a])$
- 8 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \omega[(n+m)])$ i
- 9 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \oplus[n,m,(n+m)])$ i
- 10 $\vdash \forall n \forall m \forall k ((n + m) = k \Rightarrow \omega[k])$ i
- 11 $\vdash \forall n \forall m \forall k (k = (n + m) \Rightarrow \omega[k])$ i
- 12 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \oplus[n,m,k] \Rightarrow k = (n+m))$ i
- 13 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \oplus[n,m,k] \Rightarrow (n+m) = k)$ i
- 14 $\vdash \forall n \forall m \forall k ((n+m) = k \Rightarrow \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \oplus[n,m,k])$ i
- 15 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \oplus[n,m,k] \Leftrightarrow (n+m) = k)$ i
- 16 $\vdash \forall n \forall m \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}_{[n,A]} \ \& \ \mathfrak{N}_{[m,B]} \ \& \ (A \cap B) \equiv \phi$
 $\ \Rightarrow \mathfrak{N}_{[(n+m),(A \cup B)]})$ i

- 17 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B]$
 $\quad \& \ (A \cap B) \equiv \phi \ \& \ \mathfrak{N}[k, (A \cup B)]$
 $\quad \Rightarrow k = (n+m))$ i
- 18 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m]$
 $\quad \Rightarrow \exists A \exists B (\mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ (A \cap B) \equiv \phi$
 $\quad \quad \& \ \mathfrak{N}[(n+m), (A \cup B)]))$ i
- 19 $\vdash \forall n \forall m \forall k (k = (n+m)$
 $\quad \Rightarrow \exists A \exists B (\mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ (A \cap B) \equiv \phi$
 $\quad \quad \& \ \mathfrak{N}[k, (A \cup B)]))$ i
- 20 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \exists A \exists B (\mathfrak{N}[n,A] \ \& \ \mathfrak{N}[(n+m),B] \ \& \ A \subseteq B))$ i
- 21 $\vdash \forall n \forall m \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\quad \Rightarrow \exists k ((k+n) = m \ \& \ \mathfrak{N}[k, (B \setminus A)]))$ i
- 22 $\vdash \forall n \forall m \forall A (\mathfrak{N}[(n+m),A]$
 $\quad \Rightarrow \exists Q \exists R (\mathfrak{N}[n,Q] \ \& \ \mathfrak{N}[m,R] \ \& \ (Q \cup R) \equiv A$
 $\quad \quad \& \ (Q \cap R) \equiv \phi))$ i

!CHAPTER 2 BASIC LAWS OF ADDITION;

- 1 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (n + k) = (m + k))$ i
- 2 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (k + n) = (k + m))$ i
- 3 $\vdash \forall n \forall m \forall k \forall j (\omega[j] \ \& \ (n + m) = k \Rightarrow ((n + m) + j) = (k + j))$ i
- 4 $\vdash \forall n \forall m \forall k \forall j (\omega[j] \ \& \ (n + m) = k \Rightarrow (j + (n + m)) = (j + k))$ i
- 5 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow (n + m) = (m + n))$ i
- 6 $\vdash \forall n \forall m \forall k ((n + m) = k \Rightarrow (m + n) = k)$ i
- 7 $\vdash \forall n \forall m \forall k (k = (n + m) \Rightarrow k = (m + n))$ i
- 8 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\quad \Rightarrow ((n + m) + k) = ((m + n) + k))$ i
- 9 $\vdash \forall n \forall m \forall k \forall j (((n + m) + k) = j \Rightarrow ((m + n) + k) = j)$ i
- 10 $\vdash \forall n \forall m \forall k \forall j (j = ((n + m) + k) \Rightarrow j = ((m + n) + k))$ i
- 11 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\quad \Rightarrow (n + (m + k)) = (n + (k + m)))$ i
- 12 $\vdash \forall n \forall m \forall k \forall j ((n + (m + k)) = j \Rightarrow (n + (k + m)) = j)$ i
- 13 $\vdash \forall n \forall m \forall k \forall j (j = (n + (m + k)) \Rightarrow j = (n + (k + m)))$ i
- 14 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\quad \Rightarrow (n + (m + k)) = ((n + m) + k))$ i
- 15 $\vdash \forall n \forall m \forall k \forall j ((n + (m + k)) = j \Rightarrow ((n + m) + k) = j)$ i
- 16 $\vdash \forall n \forall m \forall k \forall j (((n + m) + k) = j \Rightarrow (n + (m + k)) = j)$ i
- 17 $\vdash \forall n \forall m \forall k \forall j (j = (n + (m + k)) \Rightarrow j = ((n + m) + k))$ i
- 18 $\vdash \forall n \forall m \forall k \forall j (j = ((n + m) + k) \Rightarrow j = (n + (m + k)))$ i
- 19 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\quad \Rightarrow ((n + m) + k) = ((n + k) + m))$ i
- 20 $\vdash \forall n \forall m \forall k \forall j (((n + m) + k) = j \Rightarrow ((n + k) + m) = j)$ i
- 21 $\vdash \forall n \forall m \forall k \forall j (j = ((n + m) + k) \Rightarrow j = ((n + k) + m))$ i
- 22 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\quad \Rightarrow (n + (m + k)) = (m + (n + k)))$ i
- 23 $\vdash \forall n \forall m \forall k \forall j ((n + (m + k)) = j \Rightarrow (m + (n + k)) = j)$ i
- 24 $\vdash \forall n \forall m \forall k \forall j (j = (n + (m + k)) \Rightarrow j = (m + (n + k)))$ i

- 25 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + n) + m))$ i
- 26 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (m + (k + n)))$ i
- 27 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + k) = ((k + m) + n))$ i
- 28 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n + (m + k)) = (k + (m + n)))$ i
- 29 $\vdash \forall n \forall m \forall k \forall j (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j] \Rightarrow ((n + m) + (k + j)) = ((n + k) + (m + j)))$ i
- 30 $\vdash \forall n (\omega[n] \Rightarrow n = (n + 0))$ i
- 31 $\vdash \forall n (\omega[n] \Rightarrow n = (0 + n))$ i
- 32 $\vdash \forall n (\omega[n] \Rightarrow (n + 0) = n)$ i
- 33 $\vdash \forall n (\omega[n] \Rightarrow (0 + n) = n)$ i
- 34 $\vdash \forall n \forall m ((n + 0) = m \Rightarrow n = m)$ i
- 35 $\vdash \forall n \forall m (n = (m + 0) \Rightarrow n = m)$ i
- 36 $\vdash \forall n \forall m (n = (n + m) \Rightarrow m = 0)$ i
- 37 $\vdash \forall n \forall m (n = (m + n) \Rightarrow m = 0)$ i
- 38 $\vdash \forall n \forall m ((n + m) = n \Rightarrow m = 0)$ i
- 39 $\vdash \forall n \forall m ((m + n) = n \Rightarrow m = 0)$ i
- 40 $\vdash \forall n (\omega[n] \Rightarrow (n') = (n + 1))$ i
- 41 $\vdash \forall n (\omega[n] \Rightarrow (n') = (1 + n))$ i
- 42 $\vdash \forall n (\omega[n] \Rightarrow (n + 1) = (n'))$ i
- 43 $\vdash \forall n (\omega[n] \Rightarrow (1 + n) = (n'))$ i
- 44 $\vdash \forall n \forall m ((n') = m \Rightarrow (n + 1) = m)$ i
- 45 $\vdash \forall n \forall m ((n + 1) = m \Rightarrow (n') = m)$ i
- 46 $\vdash \forall n \forall m ((n') = m \Leftrightarrow (n + 1) = m)$ i
- 47 $\vdash \forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \Rightarrow (n + 1) = m)$ i
- 48 $\vdash \forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1))$ i
- 49 $\vdash \forall n (\omega[n] \ \& \ \neg n = 0 \Rightarrow \exists m (m + 1) = n)$ i
- 50 $\vdash \forall n (\omega[n] \Rightarrow n = 0 \vee \exists m (m + 1) = n)$ i
- 51 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow (n + (m')) = ((n + m)'))$ i
- 52 $\vdash \forall n \forall m \forall k ((n + (m')) = k \Rightarrow ((n + m)') = k)$ i
- 53 $\vdash \forall n \forall m ((n + m) = 0 \Rightarrow m = 0)$ i
- 54 $\vdash \forall n \forall m ((n + m) = 0 \Rightarrow n = 0)$ i
- 55 $\vdash \forall n \forall m ((n + m) = 0 \Rightarrow n = 0 \ \& \ m = 0)$ i
- 56 $\vdash \forall n \forall m (m = (n + 1) \Rightarrow \neg m = 0)$ i
- 57 $\vdash \forall n \neg (n + 1) = 0$ i
- 58 $\vdash \forall n \forall P (\mathcal{N}[(n + 1), P] \Rightarrow \exists x P[x])$ i
- 59 $\vdash \forall n \forall m ((n + 1) = (m + 1) \Rightarrow n = m)$ i
- 60 $\vdash \forall j \forall k \forall n ((j + n) = (k + n) \Rightarrow j = k)$ i
- 61 $\vdash \forall j \forall k \forall n ((j + n) = (n + k) \Rightarrow j = k)$ i
- 62 $\vdash \forall j \forall k \forall n ((n + j) = (k + n) \Rightarrow j = k)$ i
- 63 $\vdash \forall j \forall k \forall n ((n + j) = (n + k) \Rightarrow j = k)$ i
- 64 $\vdash \forall P \forall a (\omega[a] \ \& \ P[a] \ \& \ \forall n (P[(a+n)] \Rightarrow P[(a+(n+1))]) \Rightarrow \forall n (\omega[n] \Rightarrow P[(a+n)]))$ i

- 65 $\vdash (0 + 0) = 0$ i
66 $\vdash 0 = (0 + 0)$ i
67 $\vdash (0 + 1) = 1$ i
68 $\vdash (0 + 2) = 2$ i
69 $\vdash (1 + 1) = 2$ i
70 $\vdash (2 + 1) = 3$ i
71 $\vdash (1 + 2) = 3$ i
72 $\vdash (3 + 1) = 4$ i
73 $\vdash (1 + 3) = 4$ i
74 $\vdash (2 + 2) = 4$ i

!CHAPTER 3 INEQUALITY;

- 1 $\mathbb{D} \leq i \leq i ; i \{x, y : \omega[x] \ \& \ \omega[y] \ \& \ \Lambda[x, y]\}$ i
2 $\vdash \forall n \forall m (\leq[n, m] \Leftrightarrow \omega[n] \ \& \ \omega[m] \ \& \ \Lambda[n, m])$ i
3 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \omega[n] \ \& \ \omega[m] \ \& \ \Lambda[n, m])$ i
4 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \Lambda[n, m] \Rightarrow \leq[n, m])$ i
5 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \omega[n] \ \& \ \omega[m])$ i
6 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \omega[n])$ i
7 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \omega[m])$ i
8 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \Lambda[n, m])$ i
9 $\vdash \forall n \forall m \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n, A] \ \& \ \mathfrak{N}[m, B] \ \& \ A \subseteq B$
 $\Rightarrow \leq[n, m])$ i
10 $\vdash \forall n \forall m \forall A \forall B (\leq[n, m] \ \& \ \mathfrak{N}[n, A] \ \& \ \mathfrak{N}[m, B] \ \& \ B \subseteq A \Rightarrow A \equiv B)$ i
11 $\vdash \forall n \forall m \forall A (\leq[n, m] \ \& \ \mathfrak{N}[n, A] \Rightarrow \exists B (\mathfrak{N}[m, B] \ \& \ A \subseteq B))$ i
12 $\vdash \forall n \forall m \forall B (\leq[n, m] \ \& \ \mathfrak{N}[m, B] \Rightarrow \exists A (\mathfrak{N}[n, A] \ \& \ A \subseteq B))$ i
13 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \exists A \exists B (\mathfrak{N}[n, A] \ \& \ \mathfrak{N}[m, B] \ \& \ A \subseteq B))$ i
14 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \exists k (k + n) = m)$ i
15 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \exists k (n + k) = m)$ i
16 $\vdash \forall n \forall m (\exists k (k + n) = m \Rightarrow \leq[n, m])$ i
17 $\vdash \forall n \forall m (\exists k (n + k) = m \Rightarrow \leq[n, m])$ i
18 $\vdash \forall n \forall m (\leq[n, m] \Leftrightarrow \exists k (k + n) = m)$ i
19 $\vdash \forall n \forall m (\leq[n, m] \Leftrightarrow \exists k (n + k) = m)$ i
20 $\vdash \forall n \forall m \forall a (\leq[n, m] \ \& \ \leq[m, a] \Rightarrow \leq[n, a])$ i
21 $\vdash \forall v \forall n \forall m \forall z (\leq[v, n] \ \& \ \leq[n, m] \ \& \ \leq[m, z] \Rightarrow \leq[v, z])$ i
22 $\vdash \forall n \forall m (\leq[n, m] \ \& \ \leq[m, n] \Rightarrow n = m)$ i
23 $\vdash \forall n (\omega[n] \Rightarrow \leq[n, n])$ i
24 $\vdash \forall n (\omega[n] \Rightarrow \leq[0, n])$ i
25 $\vdash \forall n (\leq[n, 0] \Rightarrow n = 0)$ i
26 $\vdash \forall n \forall m (\leq[m, n] \ \& \ \neg m = 0 \Rightarrow \neg n = 0)$ i
27 $\vdash \forall n (\leq[1, n] \Rightarrow \neg n = 0)$ i
28 $\vdash \forall n \forall m \forall z (\leq[n, m] \ \& \ \omega[z] \Rightarrow \leq[(n + z), (m + z)])$ i
29 $\vdash \forall n \forall m \forall z (\leq[n, m] \ \& \ \omega[z] \Rightarrow \leq[(z + n), (z + m)])$ i
30 $\vdash \forall n \forall m \forall u \forall v (\leq[n, m] \ \& \ \leq[u, v] \Rightarrow \leq[(n + u), (m + v)])$ i
31 $\vdash \forall n \forall m \forall z (\leq[n, m] \ \& \ \omega[z] \Rightarrow \leq[n, (m + z)])$ i
32 $\vdash \forall n \forall m \forall z (\leq[n, m] \ \& \ \omega[z] \Rightarrow \leq[n, (z + m)])$ i
33 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \leq[n, (n + m)])$ i
34 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \leq[n, (m + n)])$ i
35 $\vdash \forall n (\omega[n] \Rightarrow \leq[n, (n + 1)])$ i

- 36 $\vdash \forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \Rightarrow \leq[n,m])$ i
- 37 $\vdash \forall n (\omega[n] \Rightarrow \leq[1, (n + 1)])$ i
- 38 $\vdash \forall n (\omega[n] \ \& \ \neg n = 0 \Rightarrow \leq[1,n])$ i
- 39 $\vdash \forall n (\omega[n] \Rightarrow n = 0 \vee \leq[1,n])$ i
- 40 $\vdash \forall n \forall m \forall z (\leq[(m + z), n] \Rightarrow \leq[m,n])$ i
- 41 $\vdash \forall n \forall m \forall z (\leq[(m + z), n] \Rightarrow \leq[z,n])$ i
- 42 $\vdash \forall n \forall m \forall z (\leq[(n + z), (m + z)] \Rightarrow \leq[n,m])$ i
- 43 $\vdash \forall n \forall m \forall z (\leq[(z + n), (z + m)] \Rightarrow \leq[n,m])$ i
- 44 $\vdash \forall n \forall m (\leq[(n + m), n] \Rightarrow m = 0)$ i
- 45 $\vdash \forall n \forall m (\leq[(m + n), n] \Rightarrow m = 0)$ i
- 46 $\vdash \leq[0,0]$ i
- 47 $\vdash \leq[0,1]$ i
- 48 $\vdash \leq[0,2]$ i
- 49 $\vdash \leq[1,1]$ i
- 50 $\vdash \leq[1,2]$ i
- 51 $\vdash \leq[2,2]$ i
- 52 $\vdash \forall n \forall m (\leq[n,m] \ \& \ \neg n = m \Rightarrow \leq[(n+1),m])$ i
- 53 $\vdash \forall n \forall m (\leq[n, (m + 1)] \ \& \ \neg n = (m + 1) \Rightarrow \leq[n,m])$ i
- 54 $\vdash \forall n \forall m (\leq[n, (m + 1)] \Rightarrow \leq[n,m] \vee n = (m + 1))$ i
- 55 $\vdash \forall n \forall m \forall x (\omega[m] \ \& \ \sigma[m,x] \ \& \ \leq[n,x] \Rightarrow \leq[n,m] \vee n = x)$ i
- 56 $\vdash \forall n \forall m (\leq[m,n] \ \& \ \leq[n, (m + 1)] \Rightarrow n = m \vee n = (m + 1))$ i
- 57 $\vdash \forall n \forall m (\leq[(m + 1), n] \ \& \ \leq[n, (m + 2)]$
 $\Rightarrow n = (m + 1) \vee n = (m + 2))$ i
- 58 $\vdash \forall n (\leq[n,1] \Rightarrow n = 0 \vee n = 1)$ i
- 59 $\vdash \forall n (\leq[n,2] \Rightarrow n = 0 \vee n = 1 \vee n = 2)$ i
- 60 $\vdash \forall n (\leq[n,3] \Rightarrow n = 0 \vee n = 1 \vee n = 2 \vee n = 3)$ i
- 61 $\vdash \forall n (\leq[1,n] \ \& \ \leq[n,2] \Rightarrow n = 1 \vee n = 2)$ i
- 62 $\vdash \forall n (\leq[1,n] \ \& \ \leq[n,3] \Rightarrow n = 1 \vee n = 2 \vee n = 3)$ i
- 63 $\vdash \forall n (\leq[1,n] \ \& \ \neg n = 1 \Rightarrow \leq[2,n])$ i
- 64 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \leq[n,m] \vee \leq[m,n])$ i
- 65 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \leq[n,m] \vee \leq[(m+1),n])$ i
- 66 $\vdash \forall n \forall m \forall k (m = (n + 1) \ \& \ \omega[k] \Rightarrow \leq[k,n] \vee \leq[m,k])$ i
- 67 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg \leq[n,m] \Rightarrow \leq[(m+1),n])$ i
- 68 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg \leq[n,m] \Rightarrow \leq[m,n])$ i
- 69 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg \leq[(m+1),n] \Rightarrow \leq[n,m])$ i
- 70 $\vdash \forall n \forall m (\leq[n,m] \ \& \ \neg \leq[(n+1),m] \Rightarrow n = m)$ i
- 71 $\vdash \forall P \forall a (\omega[a] \ \& \ P[a] \ \& \ \forall n (\leq[a,n] \ \& \ P[n] \Rightarrow P[(n+1)])$
 $\Rightarrow \forall n (\leq[a,n] \Rightarrow P[n]))$ i
- 72 $\vdash \forall n (\omega[n]$
 $\Rightarrow \forall P (\forall x (\leq[x,n] \Rightarrow \neg P[x])$
 $\vee \exists x (\leq[x,n] \ \& \ P[x]$
 $\ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[x,y])))$ i
- 73 $\vdash \forall P (\exists x (\omega[x] \ \& \ P[x])$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[x,y])))$ i
- 74 $\vdash \forall n \forall P (\omega[n] \ \& \ \mathfrak{N}[n, (\omega \cap P)] \ \& \ \neg (\omega \cap P) \equiv \phi$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x])))$ i
- 75 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x])))$ i

!CHAPTER 4 STRICT INEQUALITY;

- 1 $\mathbb{D} < ; < ; ; \{n, m : \leq[n, m] \ \& \ \neg n = m\}$ i
- 2 $\vdash \forall x \forall y (<[x, y] \Leftrightarrow \leq[x, y] \ \& \ \neg x = y)$ i
- 3 $\vdash \forall x \forall y (<[x, y] \Rightarrow \leq[x, y] \ \& \ \neg x = y)$ i
- 4 $\vdash \forall x \forall y (\leq[x, y] \ \& \ \neg x = y \Rightarrow <[x, y])$ i
- 5 $\vdash \forall x \forall y (<[x, y] \Rightarrow \leq[x, y])$ i
- 6 $\vdash \forall x \forall y (<[x, y] \Rightarrow \neg x = y)$ i
- 7 $\vdash \forall x \forall y (\neg \leq[x, y] \Rightarrow \neg <[x, y])$ i
- 8 $\vdash \forall x \forall y (\leq[x, y] \Rightarrow <[x, y] \vee x = y)$ i
- 9 $\vdash \forall x \forall y (<[x, y] \Rightarrow \omega[x] \ \& \ \omega[y])$ i
- 10 $\vdash \forall x \forall y (<[x, y] \Rightarrow \omega[x])$ i
- 11 $\vdash \forall x \forall y (<[x, y] \Rightarrow \omega[y])$ i
- 12 $\vdash \forall x \neg <[x, x]$ i
- 13 $\vdash \forall x \neg <[x, 0]$ i
- 14 $\vdash \forall x (<[0, x] \Rightarrow \neg x = 0)$ i
- 15 $\vdash \forall x (\omega[x] \ \& \ \neg x = 0 \Rightarrow <[0, x])$ i
- 16 $\vdash \forall x \forall y \forall z (<[x, y] \ \& \ \leq[y, z] \Rightarrow <[x, z])$ i
- 17 $\vdash \forall x \forall y \forall z (\leq[x, y] \ \& \ <[y, z] \Rightarrow <[x, z])$ i
- 18 $\vdash \forall x \forall y \forall z (<[x, y] \ \& \ <[y, z] \Rightarrow <[x, z])$ i
- 19 $\vdash \forall x \forall y (<[x, y] \ \& \ \leq[y, x] \Rightarrow \mathfrak{F})$ i
- 20 $\vdash \forall x \forall y (<[x, y] \Rightarrow \neg \leq[y, x])$ i
- 21 $\vdash \forall x \forall y (<[x, y] \Rightarrow \neg <[y, x])$ i
- 22 $\vdash \forall x \forall y (\leq[x, y] \Rightarrow \neg <[y, x])$ i
- 23 $\vdash \forall x \forall y (<[x, y] \ \& \ \omega[z] \Rightarrow <[x, (y+z)])$ i
- 24 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg y = 0 \Rightarrow <[x, (x+y)])$ i
- 25 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg y = 0 \Rightarrow <[x, (y+x)])$ i
- 26 $\vdash \forall x (\omega[x] \Rightarrow <[x, (x+1)])$ i
- 27 $\vdash <[0, 1]$ i
- 28 $\vdash \forall x \forall y (<[x, (y+1)] \Rightarrow \leq[x, y])$ i
- 29 $\vdash \forall x \forall y (\leq[x, y] \Rightarrow <[x, (y+1)])$ i
- 30 $\vdash \forall x \forall y (<[x, (y+1)] \Leftrightarrow \leq[x, y])$ i
- 31 $\vdash \forall n \forall m (\omega[n] \ \& \ \sigma[n, m] \Rightarrow <[n, m])$ i
- 32 $\vdash \forall x \forall n \forall m (\leq[x, n] \ \& \ \sigma[n, m] \Rightarrow <[x, m])$ i
- 33 $\vdash \forall x \forall n \forall m (\omega[n] \ \& \ \sigma[n, m] \ \& \ <[x, m] \Rightarrow \leq[x, n])$ i
- 34 $\vdash \forall x \forall y (\leq[(x+1), y] \Rightarrow <[x, y])$ i
- 35 $\vdash \forall x \forall y (\leq[(1+x), y] \Rightarrow <[x, y])$ i
- 36 $\vdash \forall x \forall y (<[x, y] \Rightarrow \leq[(x+1), y])$ i
- 37 $\vdash \forall x \forall y (<[x, y] \Rightarrow \leq[(1+x), y])$ i
- 38 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[1, k] \Rightarrow <[n, (m+k)])$ i
- 39 $\vdash \forall x \forall y \forall z (\leq[(x+y), z] \ \& \ \neg y = 0 \Rightarrow <[x, z])$ i
- 40 $\vdash \forall x \forall y \forall z (<[x, y] \ \& \ \omega[z] \Rightarrow <[(x+z), (y+z)])$ i
- 41 $\vdash \forall x \forall y \forall z (<[x, y] \ \& \ \omega[z] \Rightarrow <[(z+x), (z+y)])$ i
- 42 $\vdash \forall x \forall y \forall z \forall a (<[x, y] \ \& \ \leq[z, a] \Rightarrow <[(x+z), (y+a)])$ i
- 43 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \Rightarrow \leq[x, y] \vee <[y, x])$ i
- 44 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \Rightarrow <[x, y] \vee x = y \vee <[y, x])$ i
- 45 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg <[x, y] \Rightarrow \leq[y, x])$ i

- 46 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg \leq[x,y] \Rightarrow <[y,x])$ i
47 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg x = y \Rightarrow <[x,y] \vee <[y,x])$ i
48 $\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg <[x,y] \ \& \ \neg <[y,x] \Rightarrow x = y)$ i
49 $\vdash \forall P (\exists x (\omega[x] \ \& \ P[x])$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (<[y,x] \Rightarrow \neg P[y])))$ i
50 $\vdash \forall P (\forall y (\omega[y] \Rightarrow \exists x (P[x] \ \& \ <[y,x])) \Rightarrow \iota P)$ i
51 $\vdash \iota \omega$ i

! CHAPTER 5 SUBTRACTION;

- 1 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\Rightarrow (\Theta[m,n,k] \Leftrightarrow \mathfrak{N}[k, (B \setminus A)]))$ i
2 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\ \& \ \Theta[m,n,k]$
 $\Rightarrow \mathfrak{N}[k, (B \setminus A)])$ i
3 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\ \& \ \mathfrak{N}[k, (B \setminus A)]$
 $\Rightarrow \Theta[m,n,k])$ i
4 $\vdash \forall n \forall m \forall k (\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k]$
 $\Rightarrow \exists A \exists B (\mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B \ \& \ \mathfrak{N}[k, (B \setminus A)]))$ i
5 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow \exists a (\omega[a] \ \& \ \Theta[m,n,a]))$ i
6 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow \forall a \forall b ((\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b])$
 $\Rightarrow a = b))$ i
7 $\mathbb{T} - ; (m - n) ; \leq[n,m] ; (\omega[a] \ \& \ \Theta[m,n,a])$ i
8 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow \omega[(m-n)])$ i
9 $\vdash \forall n \forall m \forall k ((m-n) = k \Rightarrow \omega[k])$ i
10 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow \Theta[m,n, (m-n)])$ i
11 $\vdash \forall n \forall m \forall k (\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow k = (m-n))$ i
12 $\vdash \forall n \forall m \forall k (\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow (m-n) = k)$ i
13 $\vdash \forall n \forall m \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\Rightarrow \mathfrak{N}[(m-n), (B \setminus A)])$ i
14 $\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\ \& \ \mathfrak{N}[k, (B \setminus A)]$
 $\Rightarrow k = (m-n))$ i
15 $\vdash \forall n \forall m (\leq[n,m]$
 $\Rightarrow \exists A \exists B (\mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\ \& \ \mathfrak{N}[(m-n), (B \setminus A)]))$ i
16 $\vdash \forall n \forall m \forall k (k = (m-n)$
 $\Rightarrow \exists A \exists B (\mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\ \& \ \mathfrak{N}[k, (B \setminus A)]))$ i
17 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow \exists A \exists B (\mathfrak{N}[m,B] \ \& \ \mathfrak{N}[(m-n), A] \ \& \ A \subseteq B))$ i

! CHAPTER 6 BASIC LAWS OF SUBTRACTION;

- 1 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ n = k \Rightarrow (m - n) = (m - k))$ i
2 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ m = k \Rightarrow (m - n) = (k - n))$ i
3 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow ((m - n) + n) = m)$ i
4 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow (n + (m - n)) = m)$ i
5 $\vdash \forall n \forall m \forall k ((m - n) = k \Rightarrow (n + k) = m)$ i
6 $\vdash \forall n \forall m \forall k ((m - n) = k \Rightarrow (k + n) = m)$ i

- 7 $\vdash \forall n \forall m \forall k ((n + k) = m \Rightarrow (m - n) = k)$ i
- 8 $\vdash \forall n \forall m \forall k ((k + n) = m \Rightarrow (m - n) = k)$ i
- 9 $\vdash \forall n \forall m \forall k ((m - n) = k \Leftrightarrow (n + k) = m)$ i
- 10 $\vdash \forall n \forall m \forall k ((m - n) = k \Leftrightarrow (k + n) = m)$ i
- 11 $\vdash \forall n \forall m \forall k (k = (m - n) \Rightarrow m = (n + k))$ i
- 12 $\vdash \forall n \forall m \forall k (k = (m - n) \Rightarrow m = (k + n))$ i
- 13 $\vdash \forall n \forall m \forall k (m = (n + k) \Rightarrow k = (m - n))$ i
- 14 $\vdash \forall n \forall m \forall k (m = (k + n) \Rightarrow k = (m - n))$ i
- 15 $\vdash \forall n \forall m \forall k ((m - n) = k \Rightarrow (m - k) = n)$ i
- 16 $\vdash \forall n \forall m \forall k ((m - n) = k \Rightarrow \leq[k, m])$ i
- 17 $\vdash \forall n \forall m \forall k ((m - n) = k \ \& \ \neg k = 0 \Rightarrow \neg m = 0)$ i
- 18 $\vdash (0 - 0) = 0$ i
- 19 $\vdash (2 - 1) = 1$ i
- 20 $\vdash (3 - 1) = 2$ i
- 21 $\vdash (3 - 2) = 1$ i
- 22 $\vdash (4 - 1) = 3$ i
- 23 $\vdash (4 - 3) = 1$ i
- 24 $\vdash (4 - 2) = 2$ i
- 25 $\vdash \forall n (\omega[n] \Rightarrow (n - 0) = n)$ i
- 26 $\vdash \forall n (\omega[n] \Rightarrow (n - n) = 0)$ i
- 27 $\vdash \forall n \forall m (n = m \ \& \ \omega[n] \Rightarrow (n - m) = 0)$ i
- 28 $\vdash \forall n \forall m (n = (0 - m) \Rightarrow n = 0)$ i
- 29 $\vdash \forall n \forall m ((m - n) = m \Rightarrow n = 0)$ i
- 30 $\vdash \forall n \forall m (n = (m - 0) \Rightarrow n = m)$ i
- 31 $\vdash \forall n \forall m ((m - n) = 0 \Rightarrow n = m)$ i
- 32 $\vdash \forall n \forall m ((m - n) = 0 \Rightarrow m = n)$ i
- 33 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow (m + (n - n)) = m)$ i
- 34 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow ((n - n) + m) = m)$ i
- 35 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow ((n + m) - m) = n)$ i
- 36 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow ((n + m) - n) = m)$ i
- 37 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow (m - (m - n)) = n)$ i
- 38 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((m + k) - n) = ((m - n) + k))$ i
- 39 $\vdash \forall n \forall m \forall k (\leq[n, k] \ \& \ \omega[m] \Rightarrow ((m + k) - n) = (m + (k - n)))$ i
- 40 $\vdash \forall n \forall m \forall k (\leq[n, k] \ \& \ \leq[n, m] \Rightarrow ((m - n) + k) = (m + (k - n)))$ i
- 41 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (((n + m) + k) - k) = (n + m))$ i
- 42 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (((n + m) + k) - m) = (n + k))$ i
- 43 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (((n + m) + k) - n) = (m + k))$ i
- 44 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((n + m) + (k - k)) = (n + m))$ i
- 45 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow ((k - k) + (n + m)) = (n + m))$ i
- 46 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((m - n) + (k - k)) = (m - n))$ i
- 47 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((k - k) + (m - n)) = (m - n))$ i
- 48 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((m - n) + (k + n)) = (m + k))$ i
- 49 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((m - n) + (n + k)) = (m + k))$ i
- 50 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((k + n) + (m - n)) = (k + m))$ i
- 51 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((n + k) + (m - n)) = (k + m))$ i
- 52 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[k, n] \Rightarrow ((m - n) + (n - k)) = (m - k))$ i

- 53 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[k, n] \Rightarrow ((n-k) + (m-n)) = (m-k))$ i
- 54 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((m+k) - (n+k)) = (m-n))$ i
- 55 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((m+k) - (k+n)) = (m-n))$ i
- 56 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((k+m) - (n+k)) = (m-n))$ i
- 57 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((k+m) - (k+n)) = (m-n))$ i
- 58 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((m+k) - (m-n)) = (k+n))$ i
- 59 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \omega[k] \Rightarrow ((k+m) - (m-n)) = (k+n))$ i
- 60 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[m, k] \Rightarrow ((k-n) - (k-m)) = (m-n))$ i
- 61 $\vdash \forall n \forall m \forall k (\leq[k, n] \ \& \ \leq[n, m] \Rightarrow ((m-k) - (n-k)) = (m-n))$ i
- 62 $\vdash \forall n \forall m \forall k (\leq[m, (k+n)] \ \& \ \leq[n, m] \Rightarrow (k - (m-n)) = ((k+n) - m))$ i
- 63 $\vdash \forall n \forall m \forall k (\leq[k, (m-n)] \Rightarrow ((m-n) - k) = (m - (n+k)))$ i
- 64 $\vdash \forall a \forall b \forall c \forall d (\leq[d, c] \ \& \ \omega[a] \ \& \ \omega[b]$
 $\Rightarrow ((a+b)+(c-d)) = (((a+b)+c) - d))$ i
- 65 $\vdash \forall a \forall b \forall c \forall d (\leq[d, c] \ \& \ \omega[a] \ \& \ \omega[b]$
 $\Rightarrow ((c-d)+(a+b)) = (((c+a)+b) - d))$ i
- 66 $\vdash \forall a \forall b \forall c \forall d (\leq[d, c] \ \& \ \leq[d, a] \ \& \ \omega[b]$
 $\Rightarrow ((a+b)+(c-d)) = ((a-d)+(b+c)))$ i
- 67 $\vdash \forall a \forall b \forall c \forall d (\leq[d, c] \ \& \ \leq[d, b] \ \& \ \omega[a]$
 $\Rightarrow ((a+b)+(c-d)) = ((b-d)+(a+c)))$ i
- 68 $\vdash \forall a \forall b \forall c \forall d (\leq[b, a] \ \& \ \leq[d, c]$
 $\Rightarrow ((a-b)+(c-d)) = ((a+c)-(b+d)))$ i
- 69 $\vdash \forall a \forall b \forall c \forall d (\leq[b, a] \ \& \ \leq[d, c] \ \& \ \leq[d, a] \ \& \ \leq[b, c]$
 $\Rightarrow ((a-b)+(c-d)) = ((a-d)+(c-b)))$ i
- 70 $\vdash \forall a \forall b \forall c \forall d (\leq[(c-d), (a+b)]$
 $\Rightarrow ((a+b)-(c-d)) = (((a+b)+d)-c))$ i
- 71 $\vdash \forall a \forall b \forall c \forall d (\leq[(c+d), (a-b)]$
 $\Rightarrow ((a-b)-(c+d)) = (a-((b+c)+d)))$ i
- 72 $\vdash \forall n \forall m \forall k ((m-n) = (k-n) \Rightarrow m = k)$ i
- 73 $\vdash \forall n \forall m \forall k ((m-n) = (m-k) \Rightarrow n = k)$ i
- 74 $\vdash \forall n \forall m (\leq[n, m] \Rightarrow \leq[(m-n), m])$ i
- 75 $\vdash \forall n \forall m \forall k (\leq[k, (m-n)] \Rightarrow \leq[k, m])$ i
- 76 $\vdash \forall n \forall m \forall k (\leq[m, k] \ \& \ \omega[n] \Rightarrow \leq[(k-m), (k+n)])$ i
- 77 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[m, k] \Rightarrow \leq[(m-n), (k-n)])$ i
- 78 $\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[m, k] \Rightarrow \leq[(k-m), (k-n)])$ i
- 79 $\vdash \forall n \forall m \forall k (\leq[(m-n), (k-n)] \Rightarrow \leq[m, k])$ i
- 80 $\vdash \forall n \forall m \forall k (\leq[k, (m-n)] \Rightarrow \leq[(n+k), m])$ i
- 81 $\vdash \forall n \forall m \forall k (\leq[k, (m-n)] \Rightarrow \leq[(k+n), m])$ i
- 82 $\vdash \forall n \forall m \forall k (\leq[m, (k+n)] \ \& \ \leq[n, m] \Rightarrow \leq[(m-n), k])$ i
- 83 $\vdash \forall n \forall m \forall k (\leq[m, (n+k)] \ \& \ \leq[n, m] \Rightarrow \leq[(m-n), k])$ i
- 84 $\vdash \forall n \forall m \forall k (\leq[(n+k), m] \Rightarrow \leq[k, (m-n)])$ i
- 85 $\vdash \forall n \forall m \forall k (\leq[(k+n), m] \Rightarrow \leq[k, (m-n)])$ i
- 86 $\vdash \forall n \forall m \forall k (\leq[(m-n), k] \Rightarrow \leq[m, (n+k)])$ i
- 87 $\vdash \forall n \forall m \forall k (\leq[(m-n), k] \Rightarrow \leq[m, (k+n)])$ i
- 88 $\vdash \forall n \forall m \forall k (\leq[(m-k), (m-n)] \Rightarrow \leq[n, k])$ i
- 89 $\vdash \forall n \forall m \forall k \forall j (\leq[n, j] \ \& \ \leq[j, k] \ \& \ \leq[k, m] \Rightarrow \leq[(k-j), (m-n)])$ i
- 90 $\vdash \forall n \forall m \forall k (\leq[(m-n), k] \Rightarrow (k - (m-n)) = ((k+n) - m))$ i

- 91 $\vdash \forall a \forall b \forall c \forall d (\leq[(b+c)+d], a] \Rightarrow ((a-b)-(c+d)) = (a-((b+c)+d)))$ i
- 92 $\vdash \forall a \forall b \forall c \forall d (\leq[(c-d), (a-b)] \Rightarrow ((a-b)-(c-d)) = ((a+d)-(b+c)))$ i
- 93 $\vdash \forall a \forall b \forall c \forall d (\leq[b, d] \& \leq[d, c] \& \leq[c, a] \Rightarrow ((a-b)-(c-d)) = ((a-c)+(d-b)))$ i
- 94 $\vdash \forall n \forall m (\leq[n, m] \& \neg n = 0 \Rightarrow <[(m-n), m])$ i
- 95 $\vdash \forall n \forall m \forall k (<[n, m] \& \leq[m, k] \Rightarrow <[(k-m), (k-n)])$ i
- 96 $\vdash \forall n \forall m \forall k (\leq[n, m] \& <[m, k] \Rightarrow <[(m-n), (k-n)])$ i
- 97 $\vdash \forall n \forall m \forall k (<[(m-n), (k-n)] \Rightarrow <[m, k])$ i
- 98 $\vdash \forall n \forall m \forall k (<[(m-k), (m-n)] \Rightarrow <[n, k])$ i
- 99 $\vdash \forall n \forall m (<[n, m] \Rightarrow \leq[1, (m-n)])$ i

!CHAPTER 7 MULTIPLICATION;

- 1 $\vdash \forall n \forall m \forall k \forall P \forall R (\omega[n] \& \omega[k] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \Rightarrow (\otimes[n, m, k] \Leftrightarrow \mathfrak{N}[k, ((R \uparrow P)^I)]))$ i
- 2 $\vdash \forall n \forall m \forall k \forall P \forall R (\omega[n] \& \omega[k] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& \otimes[n, m, k] \Rightarrow \mathfrak{N}[k, ((R \uparrow P)^I)])$ i
- 3 $\vdash \forall n \forall m \forall k \forall P \forall R (\omega[n] \& \omega[k] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& \mathfrak{N}[k, ((R \uparrow P)^I)] \Rightarrow \otimes[n, m, k])$ i
- 4 $\vdash \forall n \forall m \forall k \forall P \forall R (\omega[n] \& \omega[k] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& (R^D) \subseteq P \& \otimes[n, m, k] \Rightarrow \mathfrak{N}[k, (R^I)])$ i
- 5 $\vdash \forall n \forall m \forall k \forall P \forall R (\omega[n] \& \omega[k] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& (R^D) \subseteq P \& \mathfrak{N}[k, (R^I)] \Rightarrow \otimes[n, m, k])$ i
- 6 $\vdash \forall n \forall m \forall k (\omega[n] \& \omega[m] \& \omega[k] \& \otimes[n, m, k] \Rightarrow \exists P \exists R (\mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& \mathfrak{N}[k, (R^I)]))$ i
- 7 $\vdash \forall n \forall m (\omega[n] \& \omega[m] \Rightarrow \exists a (\omega[a] \& \otimes[n, m, a]))$ i
- 8 $\vdash \forall n \forall m (\omega[n] \& \omega[m] \Rightarrow \forall a \forall b ((\omega[a] \& \otimes[n, m, a]) \& (\omega[b] \& \otimes[n, m, b]) \Rightarrow a = b))$ i
- 9 $\mathbb{T} \mathbf{x} ; (\mathbf{n} \times \mathbf{m}) ; \omega[\mathbf{n}] \& \omega[\mathbf{m}] ; (\omega[\mathbf{a}] \& \otimes[\mathbf{n}, \mathbf{m}, \mathbf{a}])$ i
- 10 $\vdash \forall n \forall m (\omega[n] \& \omega[m] \Rightarrow \omega[(n \times m)])$ i
- 11 $\vdash (\omega[n] \& \omega[m] \Rightarrow \otimes[n, m, (n \times m)])$ i
- 12 $\vdash \forall n \forall m \forall k ((n \times m) = k \Rightarrow \omega[k])$ i
- 13 $\vdash \forall n \forall m \forall k (k = (n \times m) \Rightarrow \omega[k])$ i
- 14 $\vdash \forall n \forall m \forall k (\omega[n] \& \omega[m] \& \omega[k] \& \otimes[n, m, k] \Rightarrow k = (n \times m))$ i
- 15 $\vdash \forall n \forall m \forall k (\omega[n] \& \omega[m] \& \omega[k] \& \otimes[n, m, k] \Rightarrow (n \times m) = k)$ i
- 16 $\vdash \forall n \forall m \forall R \forall P (\omega[n] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& (R^D) \subseteq P \Rightarrow \mathfrak{N}[(n \times m), (R^I)])$ i
- 17 $\vdash \forall n \forall m \forall k \forall R \forall P (\omega[n] \& \omega[k] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& (R^D) \subseteq P \& \mathfrak{N}[k, (R^I)] \Rightarrow k = (n \times m))$ i
- 18 $\vdash \forall n \forall m \forall k \forall R \forall P (\omega[n] \& \omega[k] \& \mathfrak{N}[n, P] \& R \cup_m P \& \mathbf{1} R \& (R^D) \subseteq P \& \mathfrak{N}[k, (R^I)]$

- $\Rightarrow (n \times m) = k$) i
- 19 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m]$
 $\Rightarrow \exists P \exists R (\mathfrak{N}_{[n,P]} \ \& \ R \cup_m P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \&$
 $\mathfrak{N}_{[(n \times m), (R^I)]})$ i
- !CHAPTER 8 BASIC LAWS OF MULTIPLICATION;**
- 1 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (n \times k) = (m \times k))$ i
- 2 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[k] \ \& \ n = m \Rightarrow (k \times n) = (k \times m))$ i
- 3 $\vdash \forall n (\omega[n] \Rightarrow (0 \times n) = 0)$ i
- 4 $\vdash \forall n (\omega[n] \Rightarrow (n \times 0) = 0)$ i
- 5 $\vdash \forall n \forall m (\neg (n \times m) = 0 \ \& \ \omega[m] \Rightarrow \neg n = 0)$ i
- 6 $\vdash \forall n \forall m (\neg (n \times m) = 0 \ \& \ \omega[n] \Rightarrow \neg m = 0)$ i
- 7 $\vdash \forall n \forall m (\neg (n \times m) = 0 \ \& \ \omega[n] \ \& \ \omega[m] \Rightarrow \neg n = 0 \ \& \ \neg m = 0)$ i
- 8 $\vdash \forall n \forall m \forall k ((n \times m) = k \ \& \ \neg k = 0 \Rightarrow \neg n = 0)$ i
- 9 $\vdash \forall n \forall m \forall k ((n \times m) = k \ \& \ \neg k = 0 \Rightarrow \neg m = 0)$ i
- 10 $\vdash \forall n \forall m \forall k ((n \times m) = k \ \& \ \neg k = 0 \Rightarrow \neg n = 0 \ \& \ \neg m = 0)$ i
- 11 $\vdash (0 \times 0) = 0$ i
- 12 $\vdash (0 \times 1) = 0$ i
- 13 $\vdash (1 \times 0) = 0$ i
- 14 $\vdash \forall n (\omega[n] \Rightarrow (1 \times n) = n)$ i
- 15 $\vdash (1 \times 1) = 1$ i
- 16 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\Rightarrow ((n + m) \times k) = ((n \times k) + (m \times k)))$ i
- 17 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow ((n + 1) \times m) = ((n \times m) + m))$ i
- 18 $\vdash \forall n (\omega[n] \Rightarrow (n \times 1) = n)$ i
- 19 $\vdash (2 \times 2) = 4$ i
- 20 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\Rightarrow (n \times (m + k)) = ((n \times m) + (n \times k)))$ i
- 21 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow (n \times (m + 1)) = ((n \times m) + n))$ i
- 22 $\vdash \forall n \forall m \forall k \forall j (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j]$
 $\Rightarrow ((n + m) \times (k + j))$
 $= (((n \times k) + (n \times j))$
 $+ ((m \times k) + (m \times j))))$ i
- 23 $\vdash \forall n \forall m \forall k \forall j (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \omega[j]$
 $\Rightarrow ((n + m) \times (k + j))$
 $= ((n \times k)$
 $+ ((n \times j) + ((m \times k) + (m \times j)))))$ i
- 24 $\vdash \forall n \forall m \forall i \forall j (\omega[n] \ \& \ \omega[m] \ \& \ \omega[i] \ \& \ \omega[j]$
 $\Rightarrow \exists k ((n + m) \times (i + j)) = ((n \times i) + k))$ i
- 25 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow (n \times m) = (m \times n))$ i
- 26 $\vdash \forall n \forall m \forall k ((n \times m) = k \Rightarrow (m \times n) = k)$ i
- 27 $\vdash \forall n \forall m \forall k (k = (n \times m) \Rightarrow k = (m \times n))$ i
- 28 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\Rightarrow ((n \times m) \times k) = (n \times (m \times k)))$ i
- 29 $\vdash \forall n \forall m \forall k \forall j (((n \times m) \times k) = j \Rightarrow (n \times (m \times k)) = j)$ i
- 30 $\vdash \forall n \forall m \forall k \forall j (j = ((n \times m) \times k) \Rightarrow j = (n \times (m \times k)))$ i

- 31 $\vdash \forall n \forall m \forall k \forall j ((n \times (m \times k)) = j \Rightarrow ((n \times m) \times k) = j)$ i
- 32 $\vdash \forall n \forall m \forall k \forall j (j = (n \times (m \times k)) \Rightarrow j = ((n \times m) \times k))$ i
- 33 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\Rightarrow ((n \times m) \times k) = ((n \times k) \times m))$ i
- 34 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$
 $\Rightarrow (k \times (n \times m)) = (n \times (k \times m)))$ i
- 35 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ \omega[k]$
 $\Rightarrow ((m - n) \times k) = ((m \times k) - (n \times k)))$ i
- 36 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ \omega[k]$
 $\Rightarrow (k \times (m - n)) = ((k \times m) - (k \times n)))$ i
- 37 $\vdash \forall n \forall m ((n \times m) = 0 \ \& \ \neg n = 0 \Rightarrow m = 0)$ i
- 38 $\vdash \forall n \forall m ((n \times m) = 0 \ \& \ \neg m = 0 \Rightarrow n = 0)$ i
- 39 $\vdash \forall n \forall m ((n \times m) = 0 \Rightarrow n = 0 \vee m = 0)$ i
- 40 $\vdash \forall n \forall m (\neg n = 0 \ \& \ \neg m = 0 \Rightarrow \neg (n \times m) = 0)$ i
- 41 $\vdash \forall n \forall m \forall k ((n \times k) = (m \times k) \ \& \ \neg k = 0 \ \& \ \leq[m,n] \Rightarrow n = m)$ i
- 42 $\vdash \forall n \forall m \forall k ((n \times k) = (m \times k) \ \& \ \neg k = 0 \Rightarrow n = m)$ i
- 43 $\vdash \forall n \forall m \forall k ((k \times n) = (m \times k) \ \& \ \neg k = 0 \Rightarrow n = m)$ i
- 44 $\vdash \forall n \forall m \forall k ((n \times k) = (k \times m) \ \& \ \neg k = 0 \Rightarrow n = m)$ i
- 45 $\vdash \forall n \forall m \forall k ((k \times n) = (k \times m) \ \& \ \neg k = 0 \Rightarrow n = m)$ i
- 46 $\vdash \forall n \forall m ((n \times m) = n \ \& \ \neg n = 0 \Rightarrow m = 1)$ i
- 47 $\vdash \forall n \forall m ((m \times n) = n \ \& \ \neg n = 0 \Rightarrow m = 1)$ i
- 48 $\vdash \forall n \forall m \forall k \forall j (\leq[n,m] \ \& \ \leq[k,j] \Rightarrow \leq[(n \times k), (m \times j)])$ i
- 49 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ \omega[k] \Rightarrow \leq[(n \times k), (m \times k)])$ i
- 50 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ \omega[k] \Rightarrow \leq[(k \times n), (k \times m)])$ i
- 51 $\vdash \forall n \forall m \forall k (<[n,m] \ \& \ \omega[k] \ \& \ \neg k = 0 \Rightarrow <[(n \times k), (m \times k)])$ i
- 52 $\vdash \forall n \forall m \forall k (<[n,m] \ \& \ \omega[k] \ \& \ \neg k = 0 \Rightarrow <[(k \times n), (k \times m)])$ i
- 53 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg m = 0 \Rightarrow \leq[n, (n \times m)])$ i
- 54 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg m = 0 \Rightarrow \leq[n, (m \times n)])$ i
- 55 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ \omega[k] \ \& \ \neg k = 0 \Rightarrow \leq[n, (m \times k)])$ i
- 56 $\vdash \forall n \forall m \forall k (\leq[n,m] \ \& \ \omega[k] \ \& \ \neg k = 0 \Rightarrow \leq[n, (k \times m)])$ i
- 57 $\vdash \forall n \forall m \forall k (<[n,m] \ \& \ \omega[k] \ \& \ \neg k = 0 \Rightarrow <[n, (k \times m)])$ i
- 58 $\vdash \forall n \forall m \forall a \forall b (\omega[n] \ \& \ \omega[m] \ \& \ \omega[a] \ \& \ \omega[b] \ \& \ \neg n = 0 \ \& \ \neg m = 0$
 $\Rightarrow \exists k (\leq[a, (k \times n)] \ \& \ \leq[b, (k \times m)]))$ i
- 59 $\vdash \forall n \forall m \forall k (\leq[(n \times k), (m \times k)] \ \& \ \neg k = 0 \Rightarrow \leq[n,m])$ i
- 60 $\vdash \forall n \forall m \forall k (\leq[(k \times n), (k \times m)] \ \& \ \neg k = 0 \Rightarrow \leq[n,m])$ i
- 61 $\vdash \forall n \forall m \forall k (<[(n \times k), (m \times k)] \Rightarrow <[n,m])$ i
- 62 $\vdash \forall n \forall m \forall k (<[(k \times n), (k \times m)] \Rightarrow <[n,m])$ i
- 63 $\vdash \forall n \forall m ((n \times m) = 1 \Rightarrow n = 1)$ i
- 64 $\vdash \forall n \forall m ((n \times m) = 1 \Rightarrow m = 1)$ i
- 65 $\vdash \forall n \forall m ((n \times m) = 1 \Rightarrow n = 1 \ \& \ m = 1)$ i
- 66 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg m = 0$
 $\Rightarrow \exists q \exists r (n = ((q \times m) + r) \ \& \ <[r,m]))$ i
- 67 $\vdash \forall n \forall m \forall q \forall r \forall x \forall y (n = ((q \times m) + r) \ \& \ <[r,m]$
 $\ \& \ n = ((y \times m) + x) \ \& \ <[x,m]$
 $\Rightarrow \omega[q] \ \& \ \omega[y] \ \& \ \neg <[q,y])$ i
- 68 $\vdash \forall n \forall m \forall q \forall r \forall x \forall y (n = ((q \times m) + r) \ \& \ <[r,m]$

- $\& n = ((y \times m) + x) \& \leq [x, m]$
 $\Rightarrow y = q \& x = r$ i
- 69 $\vdash \forall n \forall x \forall y (\exists c \exists d n = ((cx) - (dy)) \& \neg y = 0$
 $\Rightarrow \exists c \exists d n = ((cx) - (dx)))$ i
- 70 $\vdash \forall n \forall m \forall t \forall q \forall r (n = ((qt) + r) \& \exists c \exists d t = ((cx) - (dx))$
 $\Rightarrow \exists c \exists d r = ((cn) - (dm)))$ i

!CHAPTER 9 INFINITE INTERVALS;

- 1 $\mathbb{D} \ \infty ; (b \infty) ; ; \{a : \leq [b, a]\}$ i
- 2 $\vdash \forall b \forall x ((b \infty)[x] \Leftrightarrow \leq [b, x])$ i
- 3 $\vdash \forall b \forall x ((b \infty)[x] \Rightarrow \leq [b, x])$ i
- 4 $\vdash \forall b \forall x (\leq [b, x] \Rightarrow (b \infty)[x])$ i
- 5 $\vdash \forall b \forall x (\leq [x, b] \Rightarrow \neg (b \infty)[x])$ i
- 6 $\vdash \forall b (\omega[b] \Rightarrow \neg (b \infty) \equiv \phi)$ i
- 7 $\vdash \forall b (\neg (b \infty) \equiv \phi \Rightarrow \omega[b])$ i
- 8 $\vdash \forall b \forall x ((b \infty)[x] \Rightarrow \omega[x])$ i
- 9 $\vdash \forall b (b \infty) \subseteq \omega$ i
- 10 $\vdash (0 \infty) \equiv \omega$ i

!CHAPTER 10 DYADIC ADDITION, SUBTRACTION, AND MULTIPLICATION;

- 1 $\mathbb{D} \ \oplus ; (\oplus c) ; ; ((\oplus \diamond c) \lceil \omega \rfloor \omega)$ i
- 2 $\vdash \forall c \forall x \forall y (\omega[c] \& (\oplus c)[x, y] \Leftrightarrow (x+c) = y)$ i
- 3 $\vdash \forall c \forall x \forall y (\omega[c] \& (\oplus c)[x, y] \Rightarrow (x+c) = y)$ i
- 4 $\vdash \forall c \forall x \forall y ((x+c) = y \Rightarrow (\oplus c)[x, y])$ i
- 5 $\vdash \forall c \forall x (\omega[x] \& \omega[c] \Rightarrow (\oplus c)[x, (x+c)])$ i
- 6 $\vdash \forall c \forall x \forall y (\omega[c] \& (\oplus c)[x, y] \Rightarrow \omega[x] \& \omega[y])$ i
- 7 $\vdash (\oplus 0) \equiv (\mathbb{I}\omega)$ i
- 8 $\vdash (\oplus 1) \equiv (\sigma \lceil \omega)$ i
- 9 $\vdash \forall c (\omega[c] \Rightarrow ((\oplus c)^D) \equiv \omega)$ i
- 10 $\vdash \forall c (\omega[c] \Rightarrow \mathbf{f} (\oplus c))$ i
- 11 $\vdash \forall c (\omega[c] \Rightarrow (\oplus c) \mathbf{F} \omega)$ i
- 12 $\vdash \forall c \forall x (\omega[x] \& \omega[c] \Rightarrow ((\oplus c) \acute{x}) = (x+c))$ i
- 13 $\vdash \forall c (\omega[c] \Rightarrow ((\oplus c)^I) \equiv (c \infty))$ i
- 14 $\vdash \forall c (\omega[c] \Rightarrow \mathbf{1} (\oplus c))$ i
- 15 $\vdash \forall c (\omega[c] \Rightarrow (\oplus c) \mathbf{1} (c \infty))$ i
- 16 $\vdash \forall c (\omega[c] \Rightarrow \omega \sim (c \infty))$ i
- 17 $\mathbb{D} \ \ominus ; (\ominus c) ; ; ((\ominus \diamond c) \lceil (c \infty) \rfloor \omega)$ i
- 18 $\vdash \forall c \forall x \forall y ((\ominus c)[x, y] \Leftrightarrow (x-c) = y)$ i
- 19 $\vdash \forall c \forall x \forall y ((\ominus c)[x, y] \Rightarrow (x-c) = y)$ i
- 20 $\vdash \forall c \forall x \forall y ((x-c) = y \Rightarrow (\ominus c)[x, y])$ i
- 21 $\vdash \forall c \forall x (\leq [c, x] \Rightarrow (\ominus c)[x, (x-c)])$ i
- 22 $\vdash \forall c \forall x \forall y ((\ominus c)[x, y] \Rightarrow \omega[c] \& \omega[x] \& \omega[y])$ i
- 23 $\vdash \forall c (\omega[c] \Rightarrow ((\ominus c)^*) \equiv (\oplus c))$ i
- 24 $\vdash \forall c (\omega[c] \Rightarrow ((\oplus c)^*) \equiv (\ominus c))$ i
- 25 $\vdash (\ominus 0) \equiv (\mathbb{I}\omega)$ i
- 26 $\vdash (\ominus 1) \equiv ((\sigma \lceil \omega)^*)$ i
- 27 $\vdash \forall c (\omega[c] \Rightarrow \mathbf{f} (\ominus c))$ i
- 28 $\vdash \forall c \forall x (\leq [c, x] \Rightarrow ((\ominus c) \acute{x}) = (x-c))$ i

- 29 $\vdash \forall c (\omega[c] \Rightarrow ((\Theta c)^D) \equiv (c \infty))$ i
30 $\vdash \forall c (\omega[c] \Rightarrow \mathbf{1} (\Theta c))$ i
31 $\vdash \forall G \forall n \forall i (\mathbf{f} G \ \& \ \omega[n] \ \& \ \omega[i] \ \& \ (G^D)[i]$
 $\Rightarrow (((\Theta n) \circ G)^{(n+i)}) = (G^i))$ i
32 $\mathbb{D} \ \otimes ; (\otimes c) ; ; (((\otimes \hat{c}) \lceil \omega \rceil \omega)$ i
33 $\vdash \forall c \forall x \forall y (\omega[c] \ \& \ (\otimes c)[x,y] \Leftrightarrow (cXx) = y)$ i
34 $\vdash \forall c \forall x \forall y (\omega[c] \ \& \ (\otimes c)[x,y] \Rightarrow (cXx) = y)$ i
35 $\vdash \forall c \forall x \forall y ((cXx) = y \Rightarrow (\otimes c)[x,y])$ i
36 $\vdash \forall c \forall x (\omega[c] \ \& \ \omega[x] \Rightarrow (\otimes c)[x, (cXx)])$ i
37 $\vdash \forall c \forall x \forall y (\omega[c] \ \& \ (\otimes c)[x,y] \Rightarrow \omega[x] \ \& \ \omega[y])$ i
38 $\vdash (\otimes 1) \equiv (\mathbb{I}\omega)$ i
39 $\vdash \forall c (\omega[c] \Rightarrow ((\otimes c)^D) \equiv \omega)$ i
40 $\vdash \forall c (\omega[c] \Rightarrow \mathbf{f} (\otimes c))$ i
41 $\vdash \forall c (\omega[c] \Rightarrow (\otimes c) \mathbf{F} \omega)$ i
42 $\vdash \forall c \forall x (\omega[c] \ \& \ \omega[x] \Rightarrow ((\otimes c)^x) = (cXx))$ i
43 $\vdash \forall R \forall c \forall x (\omega[c] \ \& \ \omega[(R^x)] \Rightarrow ((R \circ (\otimes c))^x) = (cX(R^x))) ;$

!CHAPTER 11 THE FUNDAMENTAL THEOREMS OF ELEMENTARY ARITHMETIC;

! 1. THE FIRST FUNDAMENTAL THEOREM OF ARITHMETIC (THE MINOR).
Addition is repeated succession. i

- 1 $\vdash \forall n \forall R \forall a (\omega[n] \ \& \ \omega[a] \ \& \ (R^0) = a$
 $\ \& \ \forall i (\langle [i,n] \Rightarrow (R^i) = ((R^i)'))$
 $\Rightarrow (R^n) = (a+n))$ i

! 2. THE SECOND FUNDAMENTAL THEOREM OF ARITHMETIC (THE MAJOR).
Multiplication is repeated addition. i

- 2 $\vdash \forall n \forall R \forall a (\omega[n] \ \& \ \omega[a] \ \& \ (R^0) = 0$
 $\ \& \ \forall i (\langle [i,n] \Rightarrow (R^{i+1}) = ((R^i)+a))$
 $\Rightarrow (R^n) = (n \times a))$ i

SECTION VI: THE GREATEST COMMON DIVISOR

!CHAPTER 1 DIVISION;

- 1 $\mathbb{S} \mid ; \ n \mid m ; \exists x (n \times x) = m$ i
2 $\vdash \forall n \forall m (n \mid m \Rightarrow \omega[n] \ \& \ \omega[m])$ i
3 $\vdash \forall n \forall m (n \mid m \Rightarrow \omega[n])$ i
4 $\vdash \forall n \forall m (n \mid m \Rightarrow \omega[m])$ i
5 $\vdash \forall n \forall m (n \mid m \Rightarrow \exists x (x \times n) = m)$ i
6 $\vdash \forall n \forall m \forall x ((x \times n) = m \Rightarrow n \mid m)$ i
7 $\vdash \forall n \forall m \forall x (m = (x \times n) \Rightarrow n \mid m)$ i
8 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow n \mid (n \times m))$ i
9 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow n \mid (m \times n))$ i
10 $\vdash \forall n (\omega[n] \Rightarrow n \mid n)$ i
11 $\vdash \forall n (\omega[n] \Rightarrow n \mid 0)$ i
12 $\vdash \forall n (0 \mid n \Rightarrow n = 0)$ i
13 $\vdash \forall n \forall m (n \mid m \ \& \ \neg m = 0 \Rightarrow \neg n = 0)$ i
14 $\vdash \forall n (\omega[n] \Rightarrow 1 \mid n)$ i
15 $\vdash \forall n (n \mid 1 \Rightarrow n = 1)$ i
16 $\vdash 0 \mid 0$ i
17 $\vdash 1 \mid 0$ i
18 $\vdash 2 \mid 0$ i

- 19 $\vdash 1 \mid 1$ i
20 $\vdash 1 \mid 2$ i
21 $\vdash 2 \mid 2$ i
22 $\vdash \forall n \forall m (n \mid m \vee \neg n \mid m)$ i
23 $\vdash \forall n \forall m \forall k (n \mid m \& m \mid n \Rightarrow n = m)$ i
24 $\vdash \forall n \forall m \forall k (n \mid m \& m \mid k \Rightarrow n \mid k)$ i
25 $\vdash \forall n \forall m \forall k (n \mid m \& n \mid k \Rightarrow n \mid (m + k))$ i
26 $\vdash \forall n \forall m \forall k (n \mid m \& n \mid k \& \leq[k, m] \Rightarrow n \mid (m - k))$ i
27 $\vdash \forall n \forall m \forall k (n \mid m \& \omega[k] \Rightarrow n \mid (m \times k))$ i
28 $\vdash \forall n \forall m \forall k (n \mid m \& \omega[k] \Rightarrow n \mid (k \times m))$ i
29 $\vdash \forall n \forall a \forall b \forall x \forall y (n \mid x \& n \mid y \& \omega[a] \& \omega[b] \Rightarrow n \mid ((a \times x) + (b \times y)))$ i
30 $\vdash \forall n \forall a \forall b \forall x \forall y (n \mid x \& n \mid y \& \leq[(b \times y), (a \times x)] \Rightarrow n \mid ((a \times x) - (b \times y)))$ i
31 $\vdash \forall n \forall m \forall k (n \mid (m + k) \& n \mid m \Rightarrow n \mid k)$ i
32 $\vdash \forall n \forall m \forall k (n \mid (m + k) \& n \mid k \Rightarrow n \mid m)$ i
33 $\vdash \forall n \forall m \forall k (n \mid (m - k) \& n \mid k \Rightarrow n \mid m)$ i
34 $\vdash \forall n \forall m \forall k (n \mid (m - k) \& n \mid m \Rightarrow n \mid k)$ i
35 $\vdash \forall a \forall b \forall c \forall n (a = (b + c) \& n \mid b \& n \mid c \Rightarrow n \mid a)$ i
36 $\vdash \forall a \forall b \forall c \forall n (a = (b + c) \& n \mid a \& n \mid b \Rightarrow n \mid c)$ i
37 $\vdash \forall a \forall b \forall c \forall n (a = (b + c) \& n \mid a \& n \mid c \Rightarrow n \mid b)$ i
38 $\vdash \forall a \forall b \forall c \forall n ((b + c) = a \& n \mid b \& n \mid c \Rightarrow n \mid a)$ i
39 $\vdash \forall a \forall b \forall c \forall n ((b + c) = a \& n \mid a \& n \mid b \Rightarrow n \mid c)$ i
40 $\vdash \forall a \forall b \forall c \forall n ((b + c) = a \& n \mid a \& n \mid c \Rightarrow n \mid b)$ i
41 $\vdash \forall a \forall b \forall c \forall n (a = (b - c) \& n \mid b \& n \mid c \Rightarrow n \mid a)$ i
42 $\vdash \forall a \forall b \forall c \forall n (a = (b - c) \& n \mid a \& n \mid b \Rightarrow n \mid c)$ i
43 $\vdash \forall a \forall b \forall c \forall n (a = (b - c) \& n \mid a \& n \mid c \Rightarrow n \mid b)$ i
44 $\vdash \forall a \forall b \forall c \forall n ((b - c) = a \& n \mid b \& n \mid c \Rightarrow n \mid a)$ i
45 $\vdash \forall a \forall b \forall c \forall n ((b - c) = a \& n \mid a \& n \mid b \Rightarrow n \mid c)$ i
46 $\vdash \forall a \forall b \forall c \forall n ((b - c) = a \& n \mid a \& n \mid c \Rightarrow n \mid b)$ i
47 $\vdash \forall n \forall m (n \mid m \& \neg m = 0 \Rightarrow \leq[n, m])$ i
48 $\vdash \forall n (n \mid 2 \Rightarrow n = 1 \vee n = 2)$ i
49 $\vdash \forall n (n \mid 2 \Leftrightarrow n = 1 \vee n = 2)$ i
50 $\vdash \forall n \forall m (n \mid m \& <[m, n] \Rightarrow m = 0)$ i
51 $\vdash \forall n \forall m \forall q \forall r (n = ((q \times m) + r) \& r = 0 \Rightarrow m \mid n)$ i
52 $\vdash \forall n \forall m \forall q \forall r (n = ((q \times m) - r) \& r = 0 \Rightarrow m \mid n)$ i
53 $\vdash \forall n \forall m \forall q \forall r (n = ((q \times m) + r) \& \neg m \mid n \Rightarrow \neg r = 0)$ i
54 $\vdash \forall n \forall m \forall q \forall r (n = ((q \times m) - r) \& \neg m \mid n \Rightarrow \neg r = 0)$ i
55 $\vdash \forall n \forall m \forall q \forall r (n = ((q \times m) + r) \& <[r, m] \& m \mid n \Rightarrow r = 0)$ i
56 $\vdash \forall P (f P \& P \subseteq \omega \& \neg P[0] \Rightarrow \exists x (\omega[x] \& \neg x = 0 \& \forall y (P[y] \Rightarrow y \mid x)))$ i
57 $\vdash \forall P (f P \& P \subseteq \omega \& \neg P[0] \& \neg P[1] \Rightarrow \exists x (\omega[x] \& \neg x = 1 \& \forall y (P[y] \Rightarrow \neg y \mid x)))$ i

!CHAPTER 2 FINITE INTERVALS;

- 1 $\mathbb{D} _ ; (b _ c) ; ; \{a : \leq[b, a] \& \leq[a, c]\}$ i
2 $\vdash \forall b \forall c \forall x ((b _ c)[x] \Leftrightarrow \leq[b, x] \& \leq[x, c])$ i

- 3 $\vdash \forall b \forall c \forall x ((b _ c)[x] \Rightarrow \leq[b,x] \ \& \ \leq[x,c])$ i
- 4 $\vdash \forall b \forall c \forall x (\leq[b,x] \ \& \ \leq[x,c] \Rightarrow (b _ c)[x])$ i
- 5 $\vdash \forall b \forall c \forall x ((b _ c)[x] \Rightarrow \leq[b,x])$ i
- 6 $\vdash \forall b \forall c \forall x ((b _ c)[x] \Rightarrow \leq[x,c])$ i
- 7 $\vdash \forall b \forall c (\leq[b,c] \Rightarrow (b _ c)[b])$ i
- 8 $\vdash \forall b \forall c (\leq[b,c] \Rightarrow (b _ c)[c])$ i
- 9 $\vdash \forall b \forall c \forall x (<[x,b] \Rightarrow \neg (b _ c)[x])$ i
- 10 $\vdash \forall b \forall c \forall x (<[c,x] \Rightarrow \neg (b _ c)[x])$ i
- 11 $\vdash \forall b \forall c \neg ((b+1) _ c)[b]$ i
- 12 $\vdash \forall b \forall c \neg (b _ c)[(c+1)]$ i
- 13 $\vdash \forall b \forall c (\neg (b _ c) \equiv \phi \Rightarrow \leq[b,c])$ i
- 14 $\vdash \forall b \forall c (\leq[b,c] \Rightarrow \neg (b _ c) \equiv \phi)$ i
- 15 $\vdash \forall b \forall c (\neg (b _ c) \equiv \phi \Leftrightarrow \leq[b,c])$ i
- 16 $\vdash \forall b \forall c (\neg \leq[b,c] \Rightarrow (b _ c) \equiv \phi)$ i
- 17 $\vdash \forall b \forall c (<[c,b] \Rightarrow (b _ c) \equiv \phi)$ i
- 18 $\vdash (1 _ 0) \equiv \phi$ i
- 19 $\vdash \forall b \forall c (\neg (b _ c) \equiv \phi \Rightarrow \omega[b] \ \& \ \omega[c])$ i
- 20 $\vdash \forall b \forall c (\neg \omega[b] \Rightarrow (b _ c) \equiv \phi)$ i
- 21 $\vdash \forall b \forall c (\neg \omega[c] \Rightarrow (b _ c) \equiv \phi)$ i
- 22 $\vdash \forall b \forall c \forall x ((b _ c)[x] \Rightarrow \omega[x])$ i
- 23 $\vdash \forall b \forall c (b _ c) \subseteq \omega$ i
- 24 $\vdash \forall a \forall b \forall c \forall d (\leq[c,a] \ \& \ \leq[b,d] \Rightarrow (a _ b) \subseteq (c _ d))$ i
- 25 $\vdash \forall a \forall b \forall c (\leq[b,c] \Rightarrow (a _ b) \subseteq (a _ c))$ i
- 26 $\vdash \forall a \forall b \forall c (\leq[c,a] \Rightarrow (a _ b) \subseteq (c _ b))$ i
- 27 $\vdash \forall a \forall b \forall c (\leq[a,(c+1)] \ \& \ \leq[c,b]$
 $\Rightarrow ((a _ c) \cup ((c+1) _ b)) \subseteq (a _ b))$ i
- 28 $\vdash \forall a \forall b \forall c (\omega[c] \Rightarrow (a _ b) \subseteq ((a _ c) \cup ((c+1) _ b)))$ i
- 29 $\vdash \forall a \forall b \forall c (\leq[a,(c+1)] \ \& \ \leq[c,b]$
 $\Rightarrow ((a _ c) \cup ((c+1) _ b)) \equiv (a _ b))$ i
- 30 $\vdash \forall b (\omega[b] \Rightarrow (b _ b) \equiv (b^\bullet))$ i
- 31 $\vdash (1 _ 1) \equiv (1^\bullet)$ i
- 32 $\vdash \forall b \forall c (\leq[b,c] \Rightarrow ((b+1) _ c) \cup (b^\bullet) \equiv (b _ c))$ i
- 33 $\vdash \forall b \forall c (\leq[b,(c+1)] \Rightarrow ((b _ c) \cup ((c+1)^\bullet)) \equiv (b _ (c+1)))$ i
- 34 $\vdash \forall b \forall c (\omega[b] \Rightarrow ((b+1) _ c) \equiv ((b _ c) \setminus (b^\bullet)))$ i
- 35 $\vdash \forall a \forall b \forall c \forall d (<[b,c] \Rightarrow ((a _ b) \cap (c _ d)) \equiv \phi)$ i
- 36 $\vdash \forall a \forall b \forall c (\leq[a,(c+1)] \ \& \ \leq[c,b]$
 $\Rightarrow ((a _ c) \cup ((c+1) _ b)) \equiv (a _ b)$
 $\ \& \ ((a _ c) \cap ((c+1) _ b)) \equiv \phi)$ i
- 37 $\vdash \forall A \forall B \forall a \forall b (\omega[a] \ \& \ \omega[b] \ \& \ A \equiv (1 _ a) \ \& \ B \equiv ((a+1) _ (a+b))$
 $\Rightarrow (A \cup B) \equiv (1 _ (a+b)) \ \& \ (A \cap B) \equiv \phi)$ i
- 38 $\vdash \forall n \forall P (P \subseteq (0 _ n) \ \& \ \neg P[0] \Rightarrow P \subseteq (1 _ n))$ i
- 39 $\vdash \forall P \forall b \forall c \forall x (P \subseteq (b _ c) \ \& \ P[x] \Rightarrow P \subseteq \omega \ \& \ \neg P \equiv \phi)$ i
- 40 $\vdash \forall n \forall P (\forall y (P[y] \Rightarrow \leq[y,n]) \Rightarrow P \subseteq (0 _ n))$ i
- 41 $\vdash \forall a \forall b \forall c (\leq[a,b] \Rightarrow (b _ c) \subseteq (a \ \infty))$ i
- 42 $\vdash \forall b \forall c (\omega[b] \Rightarrow ((b+1) _ c) \subseteq (b \ \infty))$ i
- 43 $\vdash \forall n \forall m \forall k \forall x (\omega[k] \ \& \ (n _ m)[x] \Leftrightarrow ((n+k) _ (m+k))[(x+k)])$ i

- 44 $\vdash \forall n \forall m \forall k \forall x (\omega[k] \ \& \ (n _ m)[x] \Rightarrow ((n+k) _ (m+k))[(x+k)])$;
45 $\vdash \forall n \forall m \forall k \forall x (((n+k) _ (m+k))[(x+k)] \Rightarrow (n _ m)[x])$;
46 $\vdash \forall n \forall m \forall k \forall x ((n _ m)[(x-k)] \Rightarrow ((n+k) _ (m+k))[x])$;
47 $\vdash \forall n \forall m \forall k \forall x (((n+k) _ (m+k))[x] \Rightarrow (n _ m)[(x-k)])$;
48 $\vdash \forall n \forall m \forall k \forall x ((n _ m)[(x-k)] \Leftrightarrow ((n+k) _ (m+k))[x])$;
49 $\vdash \forall n \forall m \forall k \forall x (\leq[k,n] \ \& \ (n _ m)[x] \Leftrightarrow ((n-k) _ (m-k))[(x-k)])$;
50 $\vdash \forall n \forall m \forall k \forall x (\leq[k,n] \ \& \ (n _ m)[x] \Rightarrow ((n-k) _ (m-k))[(x-k)])$;
51 $\vdash \forall n \forall m \forall k \forall x (((n-k) _ (m-k))[(x-k)] \Rightarrow (n _ m)[x])$;
52 $\vdash \forall n \forall m \forall k \forall x (\leq[k,n] \ \& \ (n _ m)[(x+k)] \Rightarrow ((n-k) _ (m-k))[x])$;
53 $\vdash \forall n \forall m \forall k \forall x (((n-k) _ (m-k))[x] \Rightarrow \leq[k,n] \ \& \ (n _ m)[(x+k)])$;
54 $\vdash \forall n \forall m \forall k \forall x (\leq[k,n] \ \& \ (n _ m)[(x+k)] \Leftrightarrow ((n-k) _ (m-k))[x])$;
55 $\vdash \forall n \forall m \forall k \forall x (((n-k) _ (m-k))[x] \Rightarrow (n _ m)[(x+k)])$;
56 $\vdash \forall n \forall m \forall k (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \Rightarrow (n _ m) \sim ((n+k) _ (m+k)))$;
57 $\vdash \forall n \forall m \forall F ((F^D) \equiv ((n+1) _ m) \ \& \ \leq[n,m]$
 $\Rightarrow ((\oplus n) \circ F)^D \equiv (1 _ (m-n)))$;
58 $\vdash \forall n (\omega[n] \Rightarrow \mathfrak{N}[n, (1 _ n)])$;
59 $\vdash \forall n (\omega[n] \Rightarrow \mathfrak{N}[(n+1), (0 _ n)])$;
60 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow \mathfrak{N}[(m-n)+1, (n _ m)])$;
61 $\vdash \forall b \forall c \ f (b _ c)$;
62 $\vdash \forall P \forall b \forall c (P \subseteq (b _ c) \Rightarrow f \ P)$;
63 $\vdash \forall P \forall b \forall c \forall x (P \subseteq (b _ c) \ \& \ P[x] \Rightarrow P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f \ P)$;
64 $\vdash \forall P (\exists x \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x]) \Rightarrow f (\omega \cap P))$;
- !CHAPTER 3 THE LEAST NATURAL NUMBER;**
- 1 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi$
 $\Rightarrow \exists a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[a,y])))$;
2 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi$
 $\Rightarrow \forall a \forall b ((\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[a,y]))$
 $\ \& \ (\omega[b] \ \& \ P[b] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[b,y]))$
 $\Rightarrow a = b))$;
3 $\mathbb{T} \ \mu ; (\mu P) ; \neg (\omega \cap P) \equiv \phi ;$
 $(\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[a,y]))$;
4 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi$
 $\Rightarrow \omega[(\mu P)] \ \& \ P[(\mu P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[(\mu P),y]))$;
5 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \Rightarrow \omega[(\mu P)])$;
6 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \Rightarrow P[(\mu P)])$;
7 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \Rightarrow \omega[(\mu P)] \ \& \ P[(\mu P)])$;
8 $\vdash \forall P \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[(\mu P),y])$;
9 $\vdash \forall P \forall y (<[y,(\mu P)] \Rightarrow \neg P[y])$;
10 $\vdash \forall P \forall a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[a,y])$
 $\Rightarrow (\mu P) = a)$;
11 $\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ P \subseteq Q \Rightarrow \leq[(\mu Q),(\mu P)])$;

- 12 $\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ P \equiv Q \Rightarrow (\mu P) = (\mu Q))$ i
13 $\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ Q \equiv P \Rightarrow (\mu P) = (\mu Q))$ i
14 $\vdash \forall n \forall P \forall Q ((\mu P) = n \ \& \ P \equiv Q \Rightarrow (\mu Q) = n)$ i
15 $\vdash \forall n \forall P \forall Q ((\mu P) = n \ \& \ Q \equiv P \Rightarrow (\mu Q) = n)$ i
16 $\vdash \forall P \forall Q (P \subseteq Q \ \& \ P[(\mu Q)] \Rightarrow (\mu P) = (\mu Q))$ i
17 $\vdash \forall n \forall P \forall Q (P \subseteq Q \ \& \ P[n] \ \& \ (\mu Q) = n \Rightarrow (\mu P) = n)$ i
18 $\vdash \forall P \forall Q (\leq[(\mu P), (\mu Q)] \Rightarrow (\mu(P \cup Q)) = (\mu P))$ i
19 $\vdash \forall P \forall Q (\leq[(\mu P), (\mu Q)] \Rightarrow (\mu(Q \cup P)) = (\mu P))$ i
20 $\vdash \forall P (P[0] \Rightarrow (\mu P) = 0)$ i
21 $\vdash \forall b \forall c (\leq[b, c] \Rightarrow (\mu(b _ c)) = b)$ i
22 $\vdash \forall P \forall b \forall c (P \subseteq (b _ c) \ \& \ P[b] \Rightarrow (\mu P) = b)$ i
23 $\vdash \forall a (\omega[a] \Rightarrow (\mu(a^*)) = a)$ i
24 $\vdash \forall P (\neg \exists m (\mu P) = m \Rightarrow \forall n (\omega[n] \Rightarrow \neg P[n]))$ i

! CHAPTER 4 THE GREATEST NATURAL NUMBER;

- 1 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \exists a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a])))$ i
2 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \forall a \forall b ((\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a])) \ \& \ (\omega[b] \ \& \ P[b] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, b])) \Rightarrow a = b))$ i
3 $\mathbb{T} \chi ; (\chi P) ; \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) ; (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]))$ i
4 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)]))$ i
5 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)])$ i
6 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow P[(\chi P)])$ i
7 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$ i
8 $\vdash \forall P \forall y (f (\omega \cap P) \ \& \ \omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$ i
9 $\vdash \forall P \forall y (<[(\chi P), y] \Rightarrow \neg P[y])$ i
10 $\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (P[y] \Rightarrow \leq[y, (\chi P)]))$ i
11 $\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)])$ i
12 $\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow P[(\chi P)])$ i
13 $\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$ i
14 $\vdash \forall P \forall a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]) \Rightarrow (\chi P) = a)$ i
15 $\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq[(\chi P), (\chi Q)])$ i
16 $\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q))$ i
17 $\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \equiv P \Rightarrow (\chi P) = (\chi Q))$ i
18 $\vdash \forall P \forall Q \forall n ((\chi P) = n \ \& \ P \equiv Q \Rightarrow (\chi Q) = n)$ i
19 $\vdash \forall P \forall Q \forall n ((\chi P) = n \ \& \ Q \equiv P \Rightarrow (\chi Q) = n)$ i
20 $\vdash \forall P \forall Q (P \subseteq Q \ \& \ P[(\chi Q)] \Rightarrow (\chi P) = (\chi Q))$ i
21 $\vdash \forall P \forall Q \forall n (P \subseteq Q \ \& \ P[n] \ \& \ (\chi Q) = n \Rightarrow (\chi P) = n)$ i
22 $\vdash \forall P \forall Q (\leq[(\chi P), (\chi Q)] \Rightarrow (\chi(P \cup Q)) = (\chi Q))$ i
23 $\vdash \forall P \forall Q (\leq[(\chi P), (\chi Q)] \Rightarrow (\chi(Q \cup P)) = (\chi Q))$ i

- 24 $\vdash \forall b \forall c (\leq [b, c] \Rightarrow (\chi(b _ c)) = c)$ i
25 $\vdash \forall P \forall b \forall c (P \subseteq (b _ c) \ \& \ P[c] \Rightarrow (\chi P) = c)$ i
26 $\vdash \forall a (\omega[a] \Rightarrow (\chi(a \bullet)) = a)$ i
27 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \leq [(\mu P), (\chi P)])$ i

!CHAPTER 5 THE GREATEST COMMON DIVISOR;

- 1 $\mathbb{D} \ \delta ; (\delta \ n \ m) ; ; \{ a : a | n \ \& \ a | m \}$ i
2 $\vdash \forall n \forall m \forall x ((\delta \ n \ m)[x] \Leftrightarrow x | n \ \& \ x | m)$ i
3 $\vdash \forall n \forall m \forall x ((\delta \ n \ m)[x] \Rightarrow x | n \ \& \ x | m)$ i
4 $\vdash \forall n \forall m \forall x (x | n \ \& \ x | m \Rightarrow (\delta \ n \ m)[x])$ i
5 $\vdash \forall n \forall m \forall x ((\delta \ n \ m)[x] \Rightarrow x | n)$ i
6 $\vdash \forall n \forall m \forall x ((\delta \ n \ m)[x] \Rightarrow x | m)$ i
7 $\vdash \forall n \forall m \forall x ((\delta \ n \ m)[x] \Rightarrow \omega[x])$ i
8 $\vdash \forall n \forall m (\delta \ n \ m) \subseteq \omega$ i
9 $\vdash \forall n \forall m (\delta \ n \ m) \subseteq (\delta \ m \ n)$ i
10 $\vdash \forall n \forall m (\delta \ n \ m) \equiv (\delta \ m \ n)$ i
11 $\vdash \forall n \forall m (\neg (\delta \ n \ m) \equiv \phi \Rightarrow \omega[n] \ \& \ \omega[m])$; i
12 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow (\delta \ n \ m)[1])$ i
13 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \Rightarrow \neg (\delta \ n \ m) \equiv \phi)$; i
14 $\vdash \forall n \forall m (\neg \omega[n] \Rightarrow (\delta \ n \ m) \equiv \phi)$ i
15 $\vdash \forall n \forall m (\neg \omega[n] \Rightarrow (\delta \ m \ n) \equiv \phi)$ i
16 $\vdash (\delta \ 0 \ 0) \equiv \omega$ i
17 $\vdash \forall n \forall m (\neg n = 0 \Rightarrow (\delta \ n \ m) \subseteq (1 _ n))$ i
18 $\vdash \forall n \forall m (\neg n = 0 \Rightarrow (\delta \ m \ n) \subseteq (1 _ n))$ i
19 $\vdash \forall n \forall m (\neg n = 0 \Rightarrow f (\delta \ n \ m))$ i
20 $\vdash \forall n \forall m (\neg n = 0 \Rightarrow f (\delta \ m \ n))$ i
21 $\vdash \forall n \forall m (\neg n = 0 \vee \neg m = 0 \Rightarrow f (\delta \ n \ m))$ i
22 $\vdash \forall n \forall m (n | m \Rightarrow (\delta \ n \ m)[n])$ i
23 $\vdash \forall n \forall m (n | m \Rightarrow (\delta \ m \ n)[n])$ i
24 $\vdash \forall n \forall m \forall k (n | k \Rightarrow (\delta \ n \ m) \subseteq (\delta \ k \ m))$ i
25 $\vdash \forall n \forall m \forall k (m | k \Rightarrow (\delta \ n \ m) \subseteq (\delta \ n \ k))$ i
26 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0)$
 $\Rightarrow (\delta \ n \ m) \subseteq \omega \ \& \ \neg (\delta \ n \ m) \equiv \phi \ \& \ f (\delta \ n \ m))$; i
27 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0)$
 $\Rightarrow \neg (\omega \cap (\delta \ n \ m)) \equiv \phi \ \& \ f (\omega \cap (\delta \ n \ m)))$; i
28 $\mathbb{D} \ \Delta ; (n \ \Delta \ m) ; \omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0) ;$
 $(\chi(\delta \ n \ m))$; i
29 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0)$
 $\Rightarrow \omega[(n \ \Delta \ m)] \ \& \ (\delta \ n \ m)[(n \ \Delta \ m)]$
 $\ \& \ \forall y ((\delta \ n \ m)[y] \Rightarrow \leq [y, (n \ \Delta \ m)]))$; i
30 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow \omega[(n \ \Delta \ m)])$; i
31 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0)$
 $\Rightarrow (n \ \Delta \ m) | n \ \& \ (n \ \Delta \ m) | m)$; i
32 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow (n \ \Delta \ m) | n)$; i
33 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow (n \ \Delta \ m) | m)$; i
34 $\vdash \forall n \forall m \forall k ((n \ \Delta \ m) = n \Rightarrow n | m)$ i
35 $\vdash \forall n \forall m \forall k ((n \ \Delta \ m) = m \Rightarrow m | n)$; i

- 36 $\vdash \forall n \forall m \forall a \forall b (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0) \ \& \ \omega[a] \ \& \ \omega[b] \Rightarrow (n \ \Delta \ m) \mid ((a \times n) + (b \times m))) ;$
- 37 $\vdash \forall n \forall m \forall a \forall b (\leq[(b \times m), (a \times n)] \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow (n \ \Delta \ m) \mid ((a \times n) - (b \times m))) ;$
- 38 $\vdash \forall n \forall m \forall t (\exists c \exists d \ t = ((c \times n) - (d \times m)) \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow (n \ \Delta \ m) \mid t)$ i
- 39 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg n = 0 \Rightarrow \leq[(n \ \Delta \ m), n])$ i
- 40 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg m = 0 \Rightarrow \leq[(n \ \Delta \ m), m])$ i
- 41 $\vdash \forall n \forall m \forall y (y \mid n \ \& \ y \mid m \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow \leq[y, (n \ \Delta \ m)])$ i
- 42 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow (m \ \Delta \ n) = (n \ \Delta \ m))$ i
- 43 $\vdash \forall n \forall m \forall k (k \mid n \ \& \ k \mid m \ \& \ \leq[(n \ \Delta \ m), k] \Rightarrow (n \ \Delta \ m) = k)$ i
- 44 $\vdash \forall n \forall m \forall k (k \mid n \ \& \ k \mid m \ \& \ (n \ \Delta \ m) \mid k \Rightarrow (n \ \Delta \ m) = k)$ i
- 45 $\vdash \forall n \forall m (n \mid m \ \& \ \neg n = 0 \Rightarrow (n \ \Delta \ m) = n)$ i
- 46 $\vdash \forall n \forall m (n \mid m \ \& \ \neg n = 0 \Rightarrow (m \ \Delta \ n) = n)$ i
- 47 $\vdash \forall n (\omega[n] \ \& \ \neg n = 0 \Rightarrow (n \ \Delta \ 0) = n)$ i
- 48 $\vdash \forall n (\omega[n] \ \& \ \neg n = 0 \Rightarrow (0 \ \Delta \ n) = n)$ i
- 49 $\vdash \forall n (\omega[n] \Rightarrow (n \ \Delta \ 1) = 1)$ i
- 50 $\vdash \forall n (\omega[n] \Rightarrow (1 \ \Delta \ n) = 1)$ i
- 51 $\vdash \forall n \forall m (\omega[n] \ \& \ \omega[m] \ \& \ \neg n = 0 \Rightarrow \exists c \exists d (n \ \Delta \ m) = ((c \times n) - (d \times m)))$ i
- 52 $\vdash \forall n \forall m \forall k (k \mid n \ \& \ k \mid m \ \& \ \neg n = 0 \Rightarrow k \mid (n \ \Delta \ m))$ i
- 53 $\vdash \forall n \forall m \forall k (k \mid n \ \& \ k \mid m \ \& \ (\neg n = 0 \vee \neg m = 0) \Rightarrow k \mid (n \ \Delta \ m)) ;$
- 54 $\vdash \forall n \forall m \forall k (n \mid (m \times k) \ \& \ (n \ \Delta \ m) = 1 \ \& \ \neg n = 0 \Rightarrow n \mid k)$ i
- 55 $\vdash \forall n \forall m \forall k (n \mid (m \times k) \ \& \ (n \ \Delta \ m) = 1 \Rightarrow n \mid k)$ i
- 56 $\vdash \forall a \forall b \forall q \forall r (\neg b = 0 \ \& \ a = ((q \times b) + r) \Rightarrow (a \ \Delta \ b) = (b \ \Delta \ r))$ i

SECTION VII: THE EUCLIDEAN ALGORITHM
! CHAPTER 1 FINITE SEQUENCES ;

- 1 $\S \theta ; \theta \ \mathbf{F} ; \exists n (\omega[n] \ \& \ \mathbf{F} \ \mathbf{IF} (1 _ n))$ i
- 2 $\vdash \forall \mathbf{F} (\theta \ \mathbf{F} \Rightarrow \mathbf{f} \ \mathbf{F})$ i
- 3 $\vdash \forall \mathbf{F} (\theta \ \mathbf{F} \Rightarrow \exists a (\omega[a] \ \& \ (\mathbf{F}^{\mathbf{D}}) \equiv (1 _ a)))$ i
- 4 $\vdash \forall \mathbf{F} (\theta \ \mathbf{F} \Rightarrow \exists n (\mathbf{F}^{\mathbf{D}}) \equiv (1 _ n))$ i
- 5 $\vdash \forall \mathbf{F} (\theta \ \mathbf{F} \Rightarrow (\mathbf{F}^{\mathbf{D}}) \subseteq \omega)$ i
- 6 $\vdash \forall \mathbf{F} (\theta \ \mathbf{F} \Rightarrow \exists n (\omega[n] \ \& \ \mathfrak{N}[n, (\mathbf{F}^{\mathbf{D}})]))$ i
- 7 $\vdash \forall \mathbf{F} (\theta \ \mathbf{F} \Rightarrow f (\mathbf{F}^{\mathbf{D}}))$ i
- 8 $\vdash \forall \mathbf{F} \forall \mathbf{G} (\theta \ \mathbf{F} \ \& \ \mathbf{F} \equiv \mathbf{G} \Rightarrow \theta \ \mathbf{G})$ i
- 9 $\vdash \forall \mathbf{F} \forall \mathbf{G} (\theta \ \mathbf{F} \ \& \ \mathbf{G} \equiv \mathbf{F} \Rightarrow \theta \ \mathbf{G})$ i
- 10 $\vdash \forall n \forall \mathbf{F} (\mathbf{F} \ \mathbf{IF} (1 _ n) \Rightarrow \theta \ \mathbf{F})$ i
- 11 $\vdash \forall n \forall \mathbf{F} ((\mathbf{F}^{\mathbf{D}}) \equiv (1 _ n) \ \& \ \mathbf{f} \ \mathbf{F} \Rightarrow \theta \ \mathbf{F})$ i
- 12 $\vdash \forall \mathbf{F} \forall \mathbf{G} \forall i (\theta \ \mathbf{F} \ \& \ \mathbf{F} \equiv \mathbf{G} \ \& \ (\mathbf{F}^{\mathbf{D}})[i] \Rightarrow (\mathbf{F}'i) = (\mathbf{G}'i))$ i
- 13 $\vdash \theta \ \Phi$ i
- 14 $\vdash \forall a \theta (1 \ \blacksquare \ a)$ i
- 15 $\vdash \forall \mathbf{F} \forall c (\theta \ \mathbf{F} \ \& \ \omega[c] \Rightarrow \mathbf{f} ((\Theta c) \circ \mathbf{F}))$ i
- 16 $\vdash \forall \mathbf{F} (\theta \ \mathbf{F} \Rightarrow \forall a \forall b ((\omega[a] \ \& \ (\mathbf{F}^{\mathbf{D}}) \equiv (1 _ a))$ i

- $\& (\omega[b] \& (F^D) \equiv (1 _ b))$
 $\Rightarrow a = b))$ i
- 17 $\mathbb{T} \lambda ; (\lambda F) ; \theta F ; (\omega[a] \& (F^D) \equiv (1 _ a))$ i
- 18 $\vdash \forall n \forall F (\theta F \& \omega[n] \& (F^D) \equiv (1 _ n) \Rightarrow (\lambda F) = n)$ i
- 19 $\vdash \forall n \forall F ((F^D) \equiv (1 _ n) \& \mathbf{f} F \& \omega[n] \Rightarrow (\lambda F) = n)$ i
- 20 $\vdash \forall F (\theta F \Rightarrow \omega[(\lambda F)] \& (F^D) \equiv (1 _ (\lambda F)))$ i
- 21 $\vdash \forall F (\theta F \Rightarrow \omega[(\lambda F)])$ i
- 22 $\vdash \forall n \forall F ((\lambda F) = n \Rightarrow \omega[n])$ i
- 23 $\vdash \forall F (\theta F \Rightarrow (F^D) \equiv (1 _ (\lambda F)))$ i
- 24 $\vdash \forall n \forall F ((\lambda F) = n \Rightarrow (F^D) \equiv (1 _ n))$ i
- 25 $\vdash \forall F \forall x (\theta F \& (F^D)[x] \Rightarrow (1 _ (\lambda F))[x])$ i
- 26 $\vdash \forall F \forall x \forall y (\theta F \& F[x,y] \Rightarrow (1 _ (\lambda F))[x]) ;$ i
- 27 $\vdash \forall F \forall x ((1 _ (\lambda F))[x] \Rightarrow \theta F)$ i
- 28 $\vdash \forall F \forall a \forall x (\theta F \& (F^D)[x] \Rightarrow \leq[1,x] \& \leq[x,(\lambda F)])$ i
- 29 $\vdash \forall F \forall a \forall i (\theta F \& (F'i) = a \Rightarrow \leq[1,i] \& \leq[i,(\lambda F)])$ i
- 30 $\vdash \forall F \forall a \forall i (\theta F \& (F'i) = a \Rightarrow \leq[i,(\lambda F)])$ i
- 31 $\vdash \forall F \forall x ((1 _ (\lambda F))[x] \Rightarrow (F^D)[x])$ i
- 32 $\vdash \forall F \forall G (\theta F \& \theta G \& (F^D) \equiv (G^D) \Rightarrow (\lambda F) = (\lambda G))$ i
- 33 $\vdash \forall F \forall G (\theta F \& \theta G \& (G^D) \equiv (F^D) \Rightarrow (\lambda F) = (\lambda G))$ i
- 34 $\vdash \forall F \forall G (\theta F \& F \equiv G \Rightarrow (\lambda F) = (\lambda G))$ i
- 35 $\vdash \forall F \forall G (\theta F \& G \equiv F \Rightarrow (\lambda F) = (\lambda G))$ i
- 36 $\vdash \forall n \forall F \forall G ((\lambda F) = n \& F \equiv G \Rightarrow (\lambda G) = n)$ i
- 37 $\vdash \forall n \forall F \forall G ((\lambda F) = n \& G \equiv F \Rightarrow (\lambda G) = n)$ i
- 38 $\vdash \forall F \forall n \forall m (\leq[m,n] \& (\lambda F) = n \Rightarrow (\lambda(F \lceil (1 _ m))) = m)$ i
- 39 $\vdash \forall F \forall n (\mathbf{f} F \& \omega[n] \& (1 _ n) \subseteq (F^D)$
 $\Rightarrow \exists G ((\lambda G) = n$
 $\& \forall i (\leq[1,i] \& \leq[i,n] \Rightarrow (G'i) = (F'i))))$ i
- 40 $\vdash \forall F \forall G (F \equiv G \& (1 _ (\lambda F))[i] \Rightarrow (F'i) = (G'i))$ i
- 41 $\vdash \forall F \forall G ((\lambda F) = (\lambda G) \& \forall i ((1 _ (\lambda F))[i] \Rightarrow (F'i) = (G'i))$
 $\Rightarrow F \equiv G)$ i
- 42 $\vdash \forall F \forall G ((\lambda F) = (\lambda G)$
 $\& \forall i (\leq[1,i] \& \leq[i,(\lambda F)] \Rightarrow (F'i) = (G'i))$
 $\Rightarrow F \equiv G)$ i
- 43 $\vdash \forall F \forall G ((\lambda F) = 1 \& (\lambda G) = 1 \& (F'1) = (G'1) \Rightarrow F \equiv G)$ i
- 44 $\vdash \forall F \forall G ((\lambda F) = 2 \& (\lambda G) = 2 \& (F'1) = (G'1) \& (F'2) = (G'2)$
 $\Rightarrow F \equiv G)$ i
- 45 $\vdash \forall F \forall G ((\lambda F) = 3 \& (\lambda G) = 3 \& (F'1) = (G'1) \& (F'2) = (G'2)$
 $\& (F'3) = (G'3)$
 $\Rightarrow F \equiv G)$ i
- 46 $\vdash (\lambda \Phi) = 0$ i
- 47 $\vdash \forall F (F \equiv \Phi \Rightarrow (\lambda F) = 0)$ i
- 48 $\vdash \forall a (\lambda(1 \blacksquare a)) = 1$ i
- 49 $\vdash \forall F \forall a (F \equiv (1 \blacksquare a) \Rightarrow (\lambda F) = 1)$ i
- 50 $\vdash \forall F ((\lambda F) = 0 \Rightarrow (F^D) \equiv \phi)$ i
- 51 $\vdash \forall F ((\lambda F) = 0 \Rightarrow F \equiv \Phi)$ i
- 52 $\vdash \forall F ((\lambda F) = 1 \Rightarrow \exists b F \equiv (1 \blacksquare b))$ i

- 53 $\vdash \forall a \exists F ((\lambda F) = 1 \ \& \ (F'1) = a)$ i
- 54 $\vdash \forall n \forall m \forall F (\leq[m,n] \ \& \ (\lambda F) = n$
 $\Rightarrow (\lambda ((\oplus m) \circ (F \uparrow ((m+1) _ n)))) = (n-m))$ i
- 55 $\vdash \forall G \forall c (\theta G \ \& \ \omega[c]$
 $\Rightarrow (((\ominus c) \circ G)^D) \equiv ((c+1) _ (c+(\lambda G))))$ i
- 56 $\vdash \forall F \forall G (\theta F \ \& \ \theta G$
 $\Rightarrow ((F^D) \cup (((\ominus(\lambda F)) \circ G)^D)) \equiv (1 _ ((\lambda F)+(\lambda G)))$
 $\ \& \ ((F^D) \cap (((\ominus(\lambda F)) \circ G)^D)) \equiv \phi)$ i
- 57 $\vdash \forall F \forall G (\theta F \ \& \ \theta G$
 $\Rightarrow ((F^D) \cup (((\ominus(\lambda F)) \circ G)^D))$
 $\equiv (1 _ ((\lambda F)+(\lambda G))))$ i
- 58 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow ((F^D) \cap (((\ominus(\lambda F)) \circ G)^D)) \equiv \phi)$ i

!CHAPTER 2 CONCATENATION;

- 1 $\mathbb{D} \ \wedge \ ; \ (F^G) \ ; \ ; \ (F \sqcup ((\ominus(\lambda F)) \circ G))$ i
- 2 $\vdash \forall F \forall G \forall H (\theta F \ \& \ F \equiv G \Rightarrow (F^H) \equiv (G^H))$ i
- 3 $\vdash \forall F \forall G \forall H (\theta G \ \& \ F \equiv G \Rightarrow (F^H) \equiv (G^H))$ i
- 4 $\vdash \forall F \forall G \forall H (F \equiv G \Rightarrow (H^F) \equiv (H^G))$ i
- 5 $\vdash \forall F \forall G \forall H \forall I (\theta F \ \& \ F \equiv G \ \& \ I \equiv (F^H) \Rightarrow I \equiv (G^H))$ i
- 6 $\vdash \forall F \forall G \forall H \forall I (\theta F \ \& \ F \equiv G \ \& \ (F^H) \equiv I \Rightarrow (G^H) \equiv I)$ i
- 7 $\vdash \forall F \forall G \forall H \forall I (F \equiv G \ \& \ I \equiv (H^F) \Rightarrow I \equiv (H^G))$ i
- 8 $\vdash \forall F \forall G \forall H \forall I (F \equiv G \ \& \ (H^F) \equiv I \Rightarrow (H^G) \equiv I)$ i
- 9 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow ((F^G)^D) \equiv (1 _ ((\lambda F)+(\lambda G))))$ i
- 10 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \mathbf{f} (F^G))$ i
- 11 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow (\lambda(F^G)) = ((\lambda F)+(\lambda G)))$ i
- 12 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda F), (\lambda(F^G))]$ i
- 13 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \leq[(\lambda G), (\lambda(F^G))]$ i
- 14 $\vdash \forall F \forall G (\theta F \ \& \ \theta G \Rightarrow \theta (F^G))$ i
- 15 $\vdash \forall F \forall G \forall i (\theta G \ \& \ (1 _ (\lambda F))[i] \Rightarrow ((F^G)'i) = (F'i))$ i
- 16 $\vdash \forall F \forall G \forall i (\theta G \ \& \ \leq[1,i] \ \& \ \leq[i, (\lambda F)] \Rightarrow ((F^G)'i) = (F'i))$ i
- 17 $\vdash \forall F \forall G \forall n \forall i ((1 _ (\lambda G))[i] \ \& \ (\lambda F) = n$
 $\Rightarrow ((F^G)'(n+i)) = (G'i))$ i
- 18 $\vdash \forall F \forall G \forall i (\theta F \ \& \ (1 _ (\lambda G))[i] \Rightarrow ((F^G)'((\lambda F)+i)) = (G'i))$ i
- 19 $\vdash \forall F \forall G \forall i (\theta F \ \& \ \leq[1,i] \ \& \ \leq[i, (\lambda G)]$
 $\Rightarrow ((F^G)'((\lambda F)+i)) = (G'i))$ i
- 20 $\vdash \forall F \forall G \forall i (\theta G \ \& \ <[(\lambda F), i] \ \& \ \leq[i, (\lambda(F^G))]$
 $\Rightarrow ((F^G)'i) = (G'(i-(\lambda F))))$ i
- 21 $\vdash \forall F \forall G \forall H ((\lambda F) = ((\lambda G)+(\lambda H))$
 $\ \& \ \forall i (\leq[1,i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F'i) = (G'i))$
 $\ \& \ \forall i (<[(\lambda G), i] \ \& \ \leq[i, (\lambda F)]$
 $\Rightarrow (F'i) = (H'(i-(\lambda G))))$
 $\Rightarrow F \equiv (G^H))$ i
- 22 $\vdash \forall F \forall G \forall H ((\lambda F) = ((\lambda G)+(\lambda H))$
 $\ \& \ \forall i (\leq[1,i] \ \& \ \leq[i, (\lambda G)] \Rightarrow (F'i) = (G'i))$
 $\ \& \ \forall i (\leq[1,i] \ \& \ \leq[i, (\lambda H)] \Rightarrow (F'((\lambda G)+i)) = (H'i))$
 $\Rightarrow F \equiv (G^H))$ i
- 23 $\vdash \forall F \forall a (\theta F \Rightarrow (((1 \ \blacksquare \ a)^F)'1) = a)$ i
- 24 $\vdash \forall F \forall a \forall i (\leq[1,i] \ \& \ \leq[i, (\lambda F)]$ i

$$\begin{aligned}
& \& (T'(d+2)) = 0 \\
& \& \forall i (\leq[1,i] \& \leq[i,d] \Rightarrow \\
& \qquad (T'i) \\
& \qquad = (((S'i) \times (T'(i+1))) \\
& \qquad \qquad + (T'(i+2))) \\
& \qquad \& <[(T'(i+2)), (T'(i+1))] \\
& \qquad) \\
& \Rightarrow Q \equiv S \& R \equiv T))) \quad i
\end{aligned}$$

$$\begin{aligned}
2 \vdash \forall a \forall b (<[0,b] \& \leq[b,a] \\
\Rightarrow \exists Q \exists R \exists c ((\lambda Q) = c \\
& \& (\lambda R) = (c+2) \\
& \& (R'1) = a \\
& \& (R'2) = b \\
& \& (R'(c+1)) = (a \Delta b) \\
& \& (R'(c+2)) = 0 \\
& \& \forall i (\leq[1,i] \& \leq[i,c] \\
& \qquad \Rightarrow (R'i) = (((Q'i) \times (R'(i+1))) \\
& \qquad \qquad + (R'(i+2))) \\
& \qquad \& <[(R'(i+2)), (R'(i+1))]) \\
& \& \forall S \forall T \forall d \\
& \quad ((\lambda S) = d \\
& \quad \& (\lambda T) = (d+2) \\
& \quad \& (T'1) = a \\
& \quad \& (T'2) = b \\
& \quad \& (T'(d+2)) = 0 \\
& \quad \& \forall i (\leq[1,i] \& \leq[i,d] \\
& \qquad \Rightarrow (T'i) \\
& \qquad = (((S'i) \times (T'(i+1))) \\
& \qquad \qquad + (T'(i+2))) \\
& \qquad \& <[(T'(i+2)), (T'(i+1))]) \\
& \Rightarrow Q \equiv S \& R \equiv T))) \quad i
\end{aligned}$$

$$\begin{aligned}
3 \vdash \forall a \forall b (<[0,b] \& \leq[b,a] \\
\Rightarrow \exists Q \exists R \exists c ((\lambda Q) = c \\
& \& (\lambda R) = (c+2) \\
& \& (R'1) = a \\
& \& (R'2) = b \\
& \& (R'(c+1)) = (a \Delta b) \\
& \& (R'(c+2)) = 0 \\
& \& \forall i (\leq[1,i] \& \leq[i,c] \\
& \qquad \Rightarrow (R'i) = (((Q'i) \times (R'(i+1))) \\
& \qquad \qquad + (R'(i+2))) \\
& \qquad \& <[(R'(i+2)), (R'(i+1))]))) \quad i
\end{aligned}$$

$$\begin{aligned}
4 \vdash \forall a \forall b \forall Q \forall R \forall S \forall T \forall c \forall d \\
(<[0,b] \& \leq[b,a] \\
& \& (\lambda Q) = c \& (\lambda R) = (c+2) \\
& \& (R'1) = a \& (R'2) = b \& (R'(c+2)) = 0 \\
& \& \forall i (\leq[1,i] \& \leq[i,c] \\
& \qquad \Rightarrow (R'i) = (((Q'i) \times (R'(i+1))) + (R'(i+2))) \\
& \qquad \& <[(R'(i+2)), (R'(i+1))]) \\
& \& (\lambda S) = d \& (\lambda T) = (d+2) \\
& \& (T'1) = a \& (T'2) = b \& (T'(d+2)) = 0 \\
& \& \forall i (\leq[1,i] \& \leq[i,d] \\
& \qquad \Rightarrow (T'i) = (((S'i) \times (T'(i+1))) + (S'(i+2))) \\
& \qquad \& <[(S'(i+2)), (S'(i+1))]) \\
& \Rightarrow Q \equiv S \& R \equiv T) \quad i
\end{aligned}$$

- 5 $\vdash \forall a \forall b \forall Q \forall R \forall c$
 ($\lt[0,b] \ \& \ \leq[b,a]$
 $\& \ (\lambda Q) = c \ \& \ (\lambda R) = (c+2)$
 $\& \ (R'1) = a \ \& \ (R'2) = b \ \& \ (R'(c+2)) = 0$
 $\& \ \forall i \ (\leq[1,i] \ \& \ \leq[i,c]$
 $\Rightarrow (R'i) = (((Q'i) \times (R'(i+1))) + (R'(i+2)))$
 $\& \ \lt[(R'(i+2)), (R'(i+1))]$)
 $\Rightarrow (R'(c+1)) = (a \ \Delta \ b))$ i
- 6 $\vdash \forall a \forall b \forall Q \forall R \forall c$
 ($\lt[0,b] \ \& \ \leq[b,a]$
 $\& \ (R'1) = a \ \& \ (R'2) = b \ \& \ (R'(c+1)) = 0$
 $\& \ \forall i \ (\leq[1,i] \ \& \ \leq[i,c]$
 $\Rightarrow (R'i) = (((Q'i) \times (R'(i+1))) + (R'(i+2)))$
 $\& \ \lt[(R'(i+2)), (R'(i+1))]$)
 $\Rightarrow (R'(c+1)) = (a \ \Delta \ b))$ i
- 7 $\vdash \forall n \forall m \forall k \ (\lt[0,m] \ \& \ \leq[m,n] \ \& \ \omega[k] \ \& \ \neg k = 0$
 $\Rightarrow (k \times (n \ \Delta \ m)) = ((k \times n) \ \Delta \ (k \times m)))$ i
- 8 $\vdash \forall n \forall m \forall k \ ((\neg n = 0 \vee \neg m = 0) \ \& \ \leq[m,n] \ \& \ \omega[k] \ \& \ \neg k = 0$
 $\Rightarrow (k \times (n \ \Delta \ m)) = ((k \times n) \ \Delta \ (k \times m)))$ i
- 9 $\vdash \forall n \forall m \forall k \ (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ (\neg n = 0 \vee \neg m = 0) \ \& \ \neg k = 0$
 $\Rightarrow (k \times (n \ \Delta \ m)) = ((k \times n) \ \Delta \ (k \times m)))$ i
- !CHAPTER 4 PRIME NUMBERS AND THEIR INFINITUDE;**
- 1 $\mathbb{D} \ \pi ; \ \omega ; \ \{ a : \omega[a] \ \& \ \neg a = 1$
 $\& \ \forall n \ (n|a \Rightarrow n = 1 \vee n = a) \}$ i
- 2 $\vdash \forall x \ (\pi[x] \Leftrightarrow \omega[x] \ \& \ \neg x = 1 \ \& \ \forall n \ (n|x \Rightarrow n = 1 \vee n = x))$ i
- 3 $\vdash \forall x \ (\pi[x] \Rightarrow \omega[x] \ \& \ \neg x = 1 \ \& \ \forall n \ (n|x \Rightarrow n = 1 \vee n = x))$ i
- 4 $\vdash \forall x \ (\omega[x] \ \& \ \neg x = 1 \ \& \ \forall n \ (n|x \Rightarrow n = 1 \vee n = x) \Rightarrow \pi[x])$ i
- 5 $\vdash \forall x \ (\pi[x] \Rightarrow \omega[x])$ i
- 6 $\vdash \pi \subseteq \omega$ i
- 7 $\vdash \forall x \ (\pi[x] \Rightarrow \neg x = 1)$ i
- 8 $\vdash \forall x \forall n \ (\pi[x] \ \& \ n|x \Rightarrow n = 1 \vee n = x)$ i
- 9 $\vdash \forall x \forall n \ (n|x \ \& \ \neg n = 1 \ \& \ \neg n = x \Rightarrow \neg \pi[x])$ i
- 10 $\vdash \forall x \ (\omega[x] \ \& \ \neg \pi[x] \ \& \ \neg x = 1$
 $\Rightarrow \exists m \ (\neg m = 1 \ \& \ \neg m = x \ \& \ m|x))$;
- 11 $\vdash \neg \pi[0]$ i
- 12 $\vdash \neg \pi[1]$ i
- 13 $\vdash \pi[2]$ i
- 14 $\vdash \forall x \ (\pi[x] \Rightarrow \neg x = 0)$ i
- 15 $\vdash \forall x \forall n \ (\pi[x] \ \& \ \omega[n] \Rightarrow \omega[x] \ \& \ \omega[n] \ \& \ (\neg x = 0 \vee \neg n = 0))$ i
- 16 $\vdash \forall x \forall n \ (\pi[x] \ \& \ \omega[n] \Rightarrow (x \ \Delta \ n) = 1 \vee (x \ \Delta \ n) = x)$ i
- 17 $\vdash \forall x \forall n \ (\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \ \Delta \ n) = 1 \Rightarrow (x \ \Delta \ n) = x)$ i
- 18 $\vdash \forall x \forall n \ (\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \ \Delta \ n) = 1 \Rightarrow x|n)$ i
- 19 $\vdash \forall x \forall n \forall m \ (\pi[x] \ \& \ x|(n \times m) \Rightarrow x|n \vee x|m)$ i
- 20 $\vdash \forall n \ (\omega[n] \ \& \ \neg n = 1 \Rightarrow \exists m \ (\pi[m] \ \& \ m|n))$ i
- 21 $\vdash \iota \ \pi$ i

