

! CHAPTER 9

DOUBLET, TRIPLET, AND QUADRUPLT PREDICATES;

! This chapter introduces predicates which are satisfied by several things.  $(a\ b\ †)$  is satisfied by the things  $a$  and  $b$ ,  $(a\ b\ c\ †)$  by the things  $a$ ,  $b$ , and  $c$ , and  $(a\ b\ c\ d\ †)$  by the things  $a$ ,  $b$ ,  $c$ , and  $d$ .

The predicates will be applied only when the things are distinct, and so the propositions proved herein are evidently a preparation for arithmetic with the numbers 2, 3, and 4. The number of propositions is kept to a bare minimum. For instance, the equivalence of  $(a\ b\ †)$  with  $(a^{\bullet})$  when  $b$  is identical to  $a$  is not proven, as it is not needed. i

! 1.  $†$  represents a doublet predicate (which is satisfied by two things). i

$\mathbb{D}\ † ; (a\ b\ †) ; ; \{a : a = a \vee a = b\}$  i

! 2. i

$\vdash \forall a \forall b \forall x ( (a\ b\ †)[x] \Leftrightarrow x = a \vee x = b )$  i

$a, b$  , ! 1 (Prem) i

$\forall x ( \{a : a = a \vee x = b\}[x] \Leftrightarrow x = a \vee x = b )$  , ! 2 (Pred) i

$\forall x ( (a\ b\ †)[x] \Leftrightarrow x = a \vee x = b )$  , ! 3 ( $\mathbb{D}$ I: P1,2) i

$\forall a \forall b \forall x ( (a\ b\ †)[x] \Leftrightarrow x = a \vee x = b )$  ! 4 ( $\forall$ I: 1,3) i

$\square$

! 3. i

$\vdash \forall a \forall b (a\ b\ †) \equiv ((a^{\bullet}) \cup (b^{\bullet}))$  i

$a, b$  , ! 1 (Prem) i

$x$  , ! 2 (Prem) i

$( (a\ b\ †)[x] \Leftrightarrow x = a \vee x = b )$  , ! 3 ( $\forall$ E: P2) i

$(a\ b\ †)[x] \Leftrightarrow x = a \vee x = b$  , ! 4 ( $(\ )$ E: 3) i

$(a\ b\ †)[x]$  , ! 5 (Prem) i

$(a\ b\ †)[x] \Rightarrow x = a \vee x = b$  , ! 6 ( $\Leftrightarrow$ E: 4) i

$x = a \vee x = b$  , ! 7 ( $\Rightarrow$ E: 5,6) i

$x = a$  , ! 8 (Prem) i

$((a^{\bullet}) \cup (b^{\bullet}))[a]$  , ! 9 ( $\forall$ E: C8.33) i

$((a^{\bullet}) \cup (b^{\bullet}))[x]$  , ! 10 ( $=$ E: 8,9) i

$\mathbf{x} = \mathbf{a} \Rightarrow ((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}]$	,!	11 ( $\Rightarrow$ I: 8,10)	i
$\mathbf{x} = \mathbf{b}$	,!	12 (Prem)	i
$((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{b}]$	,!	13 ( $\forall$ E: C8.32)	i
$((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}]$	,!	14 ( $=$ E: 12,13)	i
$\mathbf{x} = \mathbf{b} \Rightarrow ((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}]$	,!	15 ( $\Rightarrow$ I: 12,14)	i
$((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}]$	,!	16 ( $\forall$ E: 7,11,15)	i
$(\mathbf{a} \mathbf{b} \dagger)[\mathbf{x}] \Rightarrow ((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}]$	,!	17 ( $\Rightarrow$ I: 5,16)	i
$((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}]$	,!	18 (Prem)	i
$( ((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}] \Leftrightarrow (\mathbf{a}^\bullet)[\mathbf{x}] \vee \mathbf{x} = \mathbf{b} )$	,!	19 ( $\forall$ E: C8.28)	i
$((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}] \Leftrightarrow (\mathbf{a}^\bullet)[\mathbf{x}] \vee \mathbf{x} = \mathbf{b}$	,!	20 ( $(())$ E: 19)	i
$((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}] \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}] \vee \mathbf{x} = \mathbf{b}$	,!	21 ( $\Leftrightarrow$ E: 20)	i
$(\mathbf{a}^\bullet)[\mathbf{x}] \vee \mathbf{x} = \mathbf{b}$	,!	22 ( $\Rightarrow$ E: 18,21)	i
$(\mathbf{a}^\bullet)[\mathbf{x}]$	,!	23 (Prem)	i
$( (\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a} )$	,!	24 ( $\forall$ E: C8.3)	i
$(\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a}$	,!	25 ( $(())$ E: 24)	i
$\mathbf{x} = \mathbf{a}$	,!	26 ( $\Rightarrow$ E: 23,25)	i
$\mathbf{x} = \mathbf{a} \vee \mathbf{x} = \mathbf{b}$	,!	27 ( $\forall$ I: 26)	i
$(\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a} \vee \mathbf{x} = \mathbf{b}$	,!	28 ( $\Rightarrow$ I: 23,27)	i
$\mathbf{x} = \mathbf{b}$	,!	29 (Prem)	i
$\mathbf{x} = \mathbf{a} \vee \mathbf{x} = \mathbf{b}$	,!	30 ( $\forall$ I: 29)	i
$\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = \mathbf{a} \vee \mathbf{x} = \mathbf{b}$	,!	31 ( $\Rightarrow$ I: 29,30)	i
$\mathbf{x} = \mathbf{a} \vee \mathbf{x} = \mathbf{b}$	,!	32 ( $\forall$ E: 22,28,31)	i
$\mathbf{x} = \mathbf{a} \vee \mathbf{x} = \mathbf{b} \Rightarrow (\mathbf{a} \mathbf{b} \dagger)[\mathbf{x}]$	,!	33 ( $\Leftrightarrow$ E: 4)	i
$(\mathbf{a} \mathbf{b} \dagger)[\mathbf{x}]$	,!	34 ( $\Rightarrow$ E: 32,33)	i
$((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}] \Rightarrow (\mathbf{a} \mathbf{b} \dagger)[\mathbf{x}]$	,!	35 ( $\Rightarrow$ I: 18,34)	i
$(\mathbf{a} \mathbf{b} \dagger)[\mathbf{x}] \Leftrightarrow ((\mathbf{a}^\bullet) \cup (\mathbf{b}^\bullet))[\mathbf{x}]$	,!	36 ( $\Leftrightarrow$ I: 17,35)	i

$( (a \ b \ †)[x] \Leftrightarrow ((a^\bullet) \cup (b^\bullet))[x] )$  ,! 37 ((I: 36) i  
 $\forall x ( (a \ b \ †)[x] \Leftrightarrow ((a^\bullet) \cup (b^\bullet))[x] )$  ,! 38 ( $\forall$ I: 2,37) i  
 $(a \ b \ †) \equiv ((a^\bullet) \cup (b^\bullet))$  ,! 39 ( $\mathfrak{S}$ I: C1.7) i  
 $\forall a \forall b (a \ b \ †) \equiv ((a^\bullet) \cup (b^\bullet))$  ! 40 ( $\forall$ I: 1,39) i

□

! 4. i

$\vdash \forall a \forall b \forall x ( \neg x = a \ \& \ \neg x = b \Rightarrow \neg (a \ b \ †)[x] )$  i  
**a, b, x** ,! 1 (Prem) i  
 $\neg x = a \ \& \ \neg x = b$  ,! 2 (Prem) i  
 $( \neg x = a \ \& \ \neg x = b \Rightarrow \neg (x = a \vee x = b) )$  ,! 3 ( $\forall$ E: I3.14) i  
 $\neg x = a \ \& \ \neg x = b \Rightarrow \neg (x = a \vee x = b)$  ,! 4 ((E: 3) i  
 $\neg (x = a \vee x = b)$  ,! 5 ( $\Rightarrow$ E: 2,4) i  
 $(a \ b \ †)[x]$  ,! 6 (Prem) i  
 $( (a \ b \ †)[x] \Leftrightarrow x = a \vee x = b )$  ,! 7 ( $\forall$ E: P2) i  
 $(a \ b \ †)[x] \Leftrightarrow x = a \vee x = b$  ,! 8 ((E: 7) i  
 $(a \ b \ †)[x] \Rightarrow x = a \vee x = b$  ,! 9 ( $\Leftrightarrow$ E: 8) i  
 $x = a \vee x = b$  ,! 10 ( $\Rightarrow$ E: 6,9) i  
 $(x = a \vee x = b)$  ,! 11 ((I: 10) i  
 $\mathfrak{F}$  ,! 12 ( $\mathfrak{F}$ I: 5,11) i  
 $(a \ b \ †)[x] \Rightarrow \mathfrak{F}$  ,! 13 ( $\Rightarrow$ I: 6,12) i  
 $\neg (a \ b \ †)[x]$  ,! 14 ( $\neg$ I: 13) i  
 $\neg x = a \ \& \ \neg x = b \Rightarrow \neg (a \ b \ †)[x]$  ,! 15 ( $\Rightarrow$ I: 2,14) i  
 $( \neg x = a \ \& \ \neg x = b \Rightarrow \neg (a \ b \ †)[x] )$  ,! 16 ((I: 15) i  
 $\forall a \forall b \forall x ( \neg x = a \ \& \ \neg x = b \Rightarrow \neg (a \ b \ †)[x] )$  ! 17 ( $\forall$ I: 1,16) i

□

! 5.  $\forall$  represents a triplet predicate (which is satisfied by three things). i

$\mathbb{D} \quad \forall ; (a \ b \ c \ \vee) ; ; \{a : a = a \vee a = b \vee a = c\} \quad i$   
**! 6.**  $i$   
 $\vdash \forall a \forall b \forall c \forall x ( (a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c ) \quad i$   
 $a, b, c \quad ,! \ 1 \text{ (Prem)} \quad i$   
 $\forall x ( \{a : a = a \vee a = b \vee a = c\}[x] \Leftrightarrow x = a \vee x = b \vee x = c ) \quad ,! \ 2 \text{ (Pred)} \quad i$   
 $\forall x ( (a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c ) \quad ,! \ 3 \text{ (DI: P5,2)} \quad i$   
 $\forall a \forall b \forall c \forall x ( (a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c ) \quad ! \ 4 \text{ (\forall I: 1,3)} \quad i$   
 $\square$   
**! 7.**  $i$   
 $\vdash \forall a \forall b \forall c (a \ b \ c \ \vee) \equiv ((a \ b \ \dagger) \cup (c^\bullet)) \quad i$   
 $a, b, c \quad ,! \ 1 \text{ (Prem)} \quad i$   
 $x \quad ,! \ 2 \text{ (Prem)} \quad i$   
 $( (a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c ) \quad ,! \ 3 \text{ (\forall E: P6)} \quad i$   
 $(a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c \quad ,! \ 4 \text{ (()E: 3)} \quad i$   
 $( ((a \ b \ \dagger) \cup (c^\bullet))[x] \Leftrightarrow (a \ b \ \dagger)[x] \vee x = c ) \quad ,! \ 5 \text{ (\forall E: C8.28)} \quad i$   
 $((a \ b \ \dagger) \cup (c^\bullet))[x] \Leftrightarrow (a \ b \ \dagger)[x] \vee x = c \quad ,! \ 6 \text{ (()E: 5)} \quad i$   
 $( (a \ b \ \dagger)[x] \Leftrightarrow x = a \vee x = b ) \quad ,! \ 7 \text{ (\forall E: P2)} \quad i$   
 $(a \ b \ \dagger)[x] \Leftrightarrow x = a \vee x = b \quad ,! \ 8 \text{ (()E: 7)} \quad i$   
 $(a \ b \ c \ \vee)[x] \quad ,! \ 9 \text{ (Prem)} \quad i$   
 $(a \ b \ c \ \vee)[x] \Rightarrow x = a \vee x = b \vee x = c \quad ,! \ 10 \text{ (\Leftrightarrow E: 4)} \quad i$   
 $x = a \vee x = b \vee x = c \quad ,! \ 11 \text{ (\Rightarrow E: 9,10)} \quad i$   
 $x = a \vee x = b \quad ,! \ 12 \text{ (Prem)} \quad i$

$x = a \vee x = b \Rightarrow (a \ b \ \dagger)[x]$  ,! 13 ( $\Leftrightarrow$ E: 8) i  
 $(a \ b \ \dagger)[x]$  ,! 14 ( $\Rightarrow$ E: 12,13) i  
 $(a \ b \ \dagger)[x] \vee x = c$  ,! 15 ( $\vee$ I: 14) i  
 $x = a \vee x = b \Rightarrow (a \ b \ \dagger)[x] \vee x = c$  ,! 16 ( $\Rightarrow$ I: 12,15) i  
 $x = c$  ,! 17 (Prem) i  
 $(a \ b \ \dagger)[x] \vee x = c$  ,! 18 ( $\vee$ I: 17) i  
 $x = c \Rightarrow (a \ b \ \dagger)[x] \vee x = c$  ,! 19 ( $\Rightarrow$ I: 17,18) i  
 $(a \ b \ \dagger)[x] \vee x = c$  ,! 20 ( $\vee$ E: 11,12,19) i  
 $(a \ b \ \dagger)[x] \vee x = c \Rightarrow ((a \ b \ \dagger) \cup (c^\bullet))[x]$  ,! 21 ( $\Leftrightarrow$ E: 6) i  
 $((a \ b \ \dagger) \cup (c^\bullet))[x]$  ,! 22 ( $\Rightarrow$ E: 21) i  
 $(a \ b \ c \ \forall)[x] \Rightarrow ((a \ b \ \dagger) \cup (c^\bullet))[x]$  ,! 23 ( $\Rightarrow$ I: 20,22) i  
 $((a \ b \ \dagger) \cup (c^\bullet))[x]$  ,! 24 (Prem) i  
 $((a \ b \ \dagger) \cup (c^\bullet))[x] \Rightarrow (a \ b \ \dagger)[x] \vee x = c$  ,! 25 ( $\Leftrightarrow$ E: 6) i  
 $(a \ b \ \dagger)[x] \vee x = c$  ,! 26 ( $\Rightarrow$ E: 24,25) i  
 $(a \ b \ \dagger)[x]$  ,! 27 (Prem) i  
 $(a \ b \ \dagger)[x] \Rightarrow x = a \vee x = b$  ,! 28 ( $\Leftrightarrow$ E: 8) i  
 $x = a \vee x = b$  ,! 29 ( $\Rightarrow$ E: 27,28) i  
 $x = a \vee x = b \vee x = c$  ,! 30 ( $\vee$ I: 29) i  
 $(a \ b \ \dagger)[x] \Rightarrow x = a \vee x = b \vee x = c$  ,! 31 ( $\Rightarrow$ I: 27,30) i  
 $x = c$  ,! 32 (Prem) i  
 $x = a \vee x = b \vee x = c$  ,! 33 ( $\vee$ I: 32) i  
 $x = c \Rightarrow x = a \vee x = b \vee x = c$  ,! 34 ( $\Rightarrow$ I: 32,33) i  
 $x = a \vee x = b \vee x = c$  ,! 35 ( $\vee$ E: 26,31,34) i  
 $x = a \vee x = b \vee x = c \Rightarrow (a \ b \ c \ \forall)[x]$  ,! 36 ( $\Leftrightarrow$ E: 4) i

$(a \ b \ c \ \vee)[x]$	, ! 37 ( $\Rightarrow$ E: 35,36)	i
$((a \ b \ \dagger) \cup (c^\bullet))[x] \Rightarrow (a \ b \ c \ \vee)[x]$	, ! 38 ( $\Rightarrow$ I: 24)	i
$(a \ b \ c \ \vee)[x] \Leftrightarrow ((a \ b \ \dagger) \cup (c^\bullet))[x]$	, ! 39 ( $\Leftrightarrow$ I: 23,38)	i
$( (a \ b \ c \ \vee)[x] \Leftrightarrow ((a \ b \ \dagger) \cup (c^\bullet))[x] )$	, ! 40 ( $(\ )$ I: 39)	i
$\forall x ( (a \ b \ c \ \vee)[x] \Leftrightarrow ((a \ b \ \dagger) \cup (c^\bullet))[x] )$	, ! 41 ( $\forall$ I: 2,40)	i
$(a \ b \ c \ \vee) \equiv ((a \ b \ \dagger) \cup (c^\bullet))$	, ! 42 ( $\S$ I: C1.7,41)	i
$\forall a \forall b \forall c (a \ b \ c \ \vee) \equiv ((a \ b \ \dagger) \cup (c^\bullet))$	! 43 ( $\forall$ I: 1,42)	i
$\square$		
! 8. <span style="float: right;">i</span>		
$\vdash \forall a \forall b \forall c \forall x ( \neg x = a \ \& \ \neg x = b \ \& \ \neg x = c$		
$\Rightarrow \neg (a \ b \ c \ \vee)[x] )$		i
$a, b, c, x$	, ! 1 (Prem)	i
$\neg x = a \ \& \ \neg x = b \ \& \ \neg x = c$	, ! 2 (Prem)	i
$(a \ b \ c \ \vee)[x]$	, ! 3 (Prem)	i
$( (a \ b \ c \ \vee)[x] \Leftrightarrow x = a \ \vee \ x = b \ \vee \ x = c )$	, ! 4 ( $\forall$ E: P6)	i
$(a \ b \ c \ \vee)[x] \Leftrightarrow x = a \ \vee \ x = b \ \vee \ x = c$	, ! 5 ( $(\ )$ E: 4)	i
$(a \ b \ c \ \vee)[x] \Rightarrow x = a \ \vee \ x = b \ \vee \ x = c$	, ! 6 ( $\Leftrightarrow$ E: 5)	i
$x = a \ \vee \ x = b \ \vee \ x = c$	, ! 7 ( $\Rightarrow$ E: 3,6)	i
$x = a \ \vee \ x = b$	, ! 8 (Prem)	i
$(x = a \ \vee \ x = b)$	, ! 9 ( $(\ )$ I: 8)	i
$\neg x = a \ \& \ \neg x = b$	, ! 10 ( $\&$ E: 2)	i
$( \neg x = a \ \& \ \neg x = b \Rightarrow \neg (x = a \ \vee \ x = b) )$	, ! 11 ( $\forall$ E: I3.14)	i
$\neg x = a \ \& \ \neg x = b \Rightarrow \neg (x = a \ \vee \ x = b)$	, ! 12 ( $(\ )$ E: 11)	i
$\neg (x = a \ \vee \ x = b)$	, ! 13 ( $\Rightarrow$ E: 10,12)	i
$\S$	, ! 14 ( $\S$ I: 9,13)	i

$x = a \vee x = b \Rightarrow \mathfrak{F}$	,! 15 ( $\Rightarrow$ I: 8,14)	i
$x = c$	,! 16 (Prem)	i
$\neg x = c$	,! 17 ( $\&$ E: 2)	i
$\mathfrak{F}$	,! 18 ( $\mathfrak{F}$ I: 16,17)	i
$x = c \Rightarrow \mathfrak{F}$	,! 19 ( $\Rightarrow$ I: 16,18)	i
$\mathfrak{F}$	,! 20 ( $\vee$ E: 7,15,19)	i
$(a \ b \ c \ \vee)[x] \Rightarrow \mathfrak{F}$	,! 21 ( $\Rightarrow$ I: 3,20)	i
$\neg (a \ b \ c \ \vee)[x]$	,! 22 ( $\neg$ I: 21)	i
$\neg x = a \ \& \ \neg x = b \ \& \ \neg x = c \Rightarrow \neg (a \ b \ c \ \vee)[x]$	,! 23 ( $\Rightarrow$ I: 2,22)	i
$(\neg x = a \ \& \ \neg x = b \ \& \ \neg x = c \Rightarrow \neg (a \ b \ c \ \vee)[x])$	,! 24 ( $(\ )$ I: 23)	i
$\forall a \forall b \forall c \forall x (\neg x = a \ \& \ \neg x = b \ \& \ \neg x = c \Rightarrow \neg (a \ b \ c \ \vee)[x])$	! 25 ( $\forall$ I: 1,24)	i

□

! 9.  $\forall$  represents a quadruplet predicate (which is satisfied by four things). i

$\mathbb{D} \ \forall ; (a \ b \ c \ d \ \forall) ; ; \{a : a = a \vee a = b \vee a = c \vee a = d\}$  i

! 10. i

$\vdash \forall a \forall b \forall c \forall d \forall x ((a \ b \ c \ d \ \forall)[x] \Leftrightarrow x = a \vee x = b \vee x = c \vee x = d)$  i

$a, b, c, d$  ,! 1 (Prem) i

$\forall x (\{a : a = a \vee a = b \vee a = c \vee a = d\}[x] \Leftrightarrow x = a \vee x = b \vee x = c \vee x = d)$  ,! 2 (Pred) i

$\forall x ((a \ b \ c \ d \ \forall)[x] \Leftrightarrow x = a \vee x = b \vee x = c \vee x = d)$  ,! 3 ( $\mathbb{D}$ I: P9,2) i

$\forall a \forall b \forall c \forall d \forall x ((a \ b \ c \ d \ \forall)[x] \Leftrightarrow x = a \vee x = b \vee x = c \vee x = d)$  ! 4 ( $\forall$ I: 1,3) i

□

! 11. i

$\vdash \forall a \forall b \forall c \forall d (a \ b \ c \ d \ \forall) \equiv ((a \ b \ c \ \vee) \cup (d \bullet))$  i

$a, b, c, d$	,! 1 (Prem)	i
$x$	,! 2 (Prem)	i
$( (a \ b \ c \ d \ \forall)[x] \Leftrightarrow x = a \vee x = b \vee x = c \vee x = d )$	,! 3 ( $\forall E$ : P10)	i
$(a \ b \ c \ d \ \forall)[x] \Leftrightarrow x = a \vee x = b \vee x = c \vee x = d$	,! 4 ( $(\ )E$ : 3)	i
$( ((a \ b \ c \ \vee) \cup (d^\bullet))[x] \Leftrightarrow (a \ b \ c \ \vee)[x] \vee x = d )$	,! 5 ( $\forall E$ : C8.28)	i
$((a \ b \ c \ \vee) \cup (d^\bullet))[x] \Leftrightarrow (a \ b \ c \ \vee)[x] \vee x = d$	,! 6 ( $(\ )E$ : 5)	i
$( (a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c )$	,! 7 ( $\forall E$ : 6)	i
$(a \ b \ c \ \vee)[x] \Leftrightarrow x = a \vee x = b \vee x = c$	,! 8 ( $(\ )E$ : 7)	i
$(a \ b \ c \ d \ \forall)[x]$	,! 9 (Prem)	i
$(a \ b \ c \ d \ \forall)[x] \Rightarrow x = a \vee x = b \vee x = c \vee x = d$	,! 10 ( $\Leftrightarrow E$ : 4)	i
$x = a \vee x = b \vee x = c \vee x = d$	,! 11 ( $\Rightarrow E$ : 9,10)	i
$x = a \vee x = b \vee x = c$	,! 12 (Prem)	i
$x = a \vee x = b \vee x = c \Rightarrow (a \ b \ c \ \vee)[x]$	,! 13 ( $\Leftrightarrow E$ : 8)	i
$(a \ b \ c \ \vee)[x]$	,! 14 ( $\Rightarrow E$ : 12,13)	i
$(a \ b \ c \ \vee)[x] \vee x = d$	,! 15 ( $\vee I$ : 14)	i
$x = a \vee x = b \vee x = c \Rightarrow (a \ b \ c \ \vee)[x] \vee x = d$	,! 16 ( $\Rightarrow I$ : 12,15)	i
$x = d$	,! 17 (Prem)	i
$(a \ b \ c \ \vee)[x] \vee x = d$	,! 18 ( $\vee I$ : 17)	i
$x = d \Rightarrow (a \ b \ c \ \vee)[x] \vee x = d$	,! 19 ( $\Rightarrow I$ : 17,18)	i
$(a \ b \ c \ \vee)[x] \vee x = d$	,! 20 ( $\vee E$ : 11,16,19)	i
$(a \ b \ c \ \vee)[x] \vee x = d \Rightarrow ((a \ b \ c \ \vee) \cup (d^\bullet))[x]$	,! 21 ( $\Leftrightarrow E$ : 6)	i

$((a \ b \ c \ \vee) \cup (d^\bullet))[x]$  ,! 22 ( $\Rightarrow$ E: 20,21) ;  
 $(a \ b \ c \ d \ \forall)[x] \Rightarrow ((a \ b \ c \ \vee) \cup (d^\bullet))[x]$  ,! 23 ( $\Rightarrow$ I: 9,22) ;  
 $((a \ b \ c \ \vee) \cup (d^\bullet))[x]$  ,! 24 (Prem) ;  
 $((a \ b \ c \ \vee) \cup (d^\bullet))[x] \Rightarrow (a \ b \ c \ \vee)[x] \vee x = d$  ,! 25 ( $\Leftrightarrow$ E: 6) ;  
 $(a \ b \ c \ \vee)[x] \vee x = d$  ,! 26 ( $\Rightarrow$ E: 24,25) ;  
 $(a \ b \ c \ \vee)[x]$  ,! 27 (Prem) ;  
 $(a \ b \ c \ \vee)[x] \Rightarrow x = a \vee x = b \vee x = c$  ,! 28 ( $\Leftrightarrow$ E: 8) ;  
 $x = a \vee x = b \vee x = c$  ,! 29 ( $\Rightarrow$ E: 28) ;  
 $x = a \vee x = b \vee x = c \vee x = d$  ,! 30 ( $\vee$ I: 29) ;  
 $(a \ b \ c \ \vee)[x] \Rightarrow x = a \vee x = b \vee x = c \vee x = d$  ,! 31 ( $\Rightarrow$ I: 27,30) ;  
 $x = d$  ,! 32 (Prem) ;  
 $x = a \vee x = b \vee x = c \vee x = d$  ,! 33 ( $\vee$ I: 32) ;  
 $x = d \Rightarrow x = a \vee x = b \vee x = c \vee x = d$  ,! 34 ( $\Rightarrow$ I: 32,33) ;  
 $x = a \vee x = b \vee x = c \vee x = d$  ,! 35 ( $\vee$ E: 26,31,34) ;  
 $x = a \vee x = b \vee x = c \vee x = d \Rightarrow (a \ b \ c \ d \ \forall)[x]$  ,! 36 ( $\Leftrightarrow$ E: 4) ;  
 $(a \ b \ c \ d \ \forall)[x]$  ,! 37 ( $\Rightarrow$ E: 35,36) ;  
 $((a \ b \ c \ \vee) \cup (d^\bullet))[x] \Rightarrow (a \ b \ c \ d \ \forall)[x]$  ,! 38 ( $\Rightarrow$ I: 24,37) ;  
 $(a \ b \ c \ d \ \forall)[x] \Leftrightarrow ((a \ b \ c \ \vee) \cup (d^\bullet))[x]$  ,! 39 ( $\Leftrightarrow$ I: 23,38) ;  
 $((a \ b \ c \ d \ \forall)[x] \Leftrightarrow ((a \ b \ c \ \vee) \cup (d^\bullet))[x])$  ,! 40 ( $(\ )$ I: 39) ;  
 $\forall x((a \ b \ c \ d \ \forall)[x] \Leftrightarrow ((a \ b \ c \ \vee) \cup (d^\bullet))[x])$  ,! 41 ( $\forall$ I: 40) ;  
 $(a \ b \ c \ d \ \forall) \equiv ((a \ b \ c \ \vee) \cup (d^\bullet))$  ,! 42 ( $\equiv$ I: C1.7,41) ;

$$\forall a \forall b \forall c \forall d (a \wedge b \wedge c \wedge d \rightarrow \Psi) \equiv ((a \wedge b \wedge c \wedge \Psi) \cup (d \bullet))$$

! 43 ( $\forall I$ : 1,42) ;

□