

! CHAPTER 11

IDENTITY PREDICATES;

! This chapter introduces a predicate of equality, written \mathbb{E} . While the '=' sign is a primitive term, and is implicitly a two-place predicate, it cannot be manipulated--taking its restriction, for example--like other predicates, because it does not have their form. \mathbb{E} , which holds from x to y precisely when y equals x , will be like other predicates, and so it will be possible to manipulate it. i

! 1. \mathbb{E} represents an equality predicate. i

$\mathbb{D} \ \mathbb{E} \ ; \ \mathbb{E} \ ; \ ; \ \{a,b : a = b\}$ i

! 2. i

$\vdash \forall x \forall y (\mathbb{E}[x,y] \Leftrightarrow x = y)$ i

$\forall x \forall y (\{a,b : a = b\}[x,y] \Leftrightarrow x = y)$, ! 1 (Pred) i

$\forall x \forall y (\mathbb{E}[x,y] \Leftrightarrow x = y)$! 2 (\mathbb{D} I: P1,1) i

□

! 3. i

$\vdash \forall x \forall y (\mathbb{E}[x,y] \Rightarrow x = y)$ i

$\mathbf{x,y}$, ! 1 (Prem) i

$(\mathbb{E}[\mathbf{x,y}] \Leftrightarrow \mathbf{x} = \mathbf{y})$, ! 2 (\forall E: P2) i

$\mathbb{E}[\mathbf{x,y}] \Leftrightarrow \mathbf{x} = \mathbf{y}$, ! 3 ($(())$ E: 2) i

$\mathbb{E}[\mathbf{x,y}] \Rightarrow \mathbf{x} = \mathbf{y}$, ! 4 (\Leftrightarrow E: 3) i

$(\mathbb{E}[\mathbf{x,y}] \Rightarrow \mathbf{x} = \mathbf{y})$, ! 5 ($(())$ I: 4) i

$\forall x \forall y (\mathbb{E}[x,y] \Rightarrow x = y)$! 6 (\forall I: 1,5) i

□

! 4. i

$\vdash \forall x \forall y (x = y \Rightarrow \mathbb{E}[x,y])$ i

$\mathbf{x,y}$, ! 1 (Prem) i

$(\mathbb{E}[\mathbf{x,y}] \Leftrightarrow \mathbf{x} = \mathbf{y})$, ! 2 (\forall E: P2) i

$\mathbb{E}[\mathbf{x,y}] \Leftrightarrow \mathbf{x} = \mathbf{y}$, ! 3 ($(())$ E: 2) i

$\mathbf{x} = \mathbf{y} \Rightarrow \mathbb{E}[\mathbf{x,y}]$, ! 4 (\Leftrightarrow E: 3) i

$(\mathbf{x} = \mathbf{y} \Rightarrow \mathbb{E}[\mathbf{x,y}])$, ! 5 ($(())$ I: 4) i

$\forall x \forall y (x = y \Rightarrow \mathbb{E}[x,y])$! 6 ($\forall I$: 1,5) i

□

! 5. i

$\vdash \forall x \mathbb{E}[x,x]$ i

x ,! 1 (Prem) i

$x = x$,! 2 (=I) i

$(x = x \Rightarrow \mathbb{E}[x,x])$,! 3 ($\forall E$: P4) i

$x = x \Rightarrow \mathbb{E}[x,x]$,! 4 ($(\Rightarrow)E$: 3) i

$\mathbb{E}[x,x]$,! 5 ($\Rightarrow E$: 2,4) i

$\forall x \mathbb{E}[x,x]$! 6 ($\forall I$: 1,5) i

□

! 6. i

$\vdash \forall x \forall y (\mathbb{E}[x,y] \Rightarrow \mathbb{E}[y,x])$ i

x, y ,! 1 (Prem) i

$\mathbb{E}[x,y]$,! 2 (Prem) i

$(\mathbb{E}[x,y] \Rightarrow x = y)$,! 3 ($\forall E$: P3) i

$\mathbb{E}[x,y] \Rightarrow x = y$,! 4 ($(\Rightarrow)E$: 3) i

$x = y$,! 5 ($\Rightarrow E$: 2,4) i

$\mathbb{E}[x,x]$,! 6 (=E: 2,5) i

$\mathbb{E}[y,x]$,! 7 (=E: 5,6) i

$\mathbb{E}[x,y] \Rightarrow \mathbb{E}[y,x]$,! 8 ($\Rightarrow I$: 2,7) i

$(\mathbb{E}[x,y] \Rightarrow \mathbb{E}[y,x])$,! 9 ($(\Rightarrow)I$: 8) i

$\forall x \forall y (\mathbb{E}[x,y] \Rightarrow \mathbb{E}[y,x])$! 10 ($\forall I$: 1,9) i

□

! 7. i

$\vdash \mathbb{E} \equiv (\mathbb{E}^*)$ i

$(\forall x \forall y (\mathbb{E}[x,y] \Rightarrow \mathbb{E}[y,x]) \Rightarrow \mathbb{E} \equiv (\mathbb{E}^*))$,! 1 ($\forall E$: C3.14) i

$\forall x \forall y (\mathbb{E}[x,y] \Rightarrow \mathbb{E}[y,x]) \Rightarrow \mathbb{E} \equiv (\mathbb{E}^*)$,! 2 ($(\Rightarrow)E$: 1) i

$\mathbb{E} \equiv (\mathbb{E}^*)$! 3 ($\Rightarrow\mathbb{E}$: P6,2)	i
\square		
! 8.		i
$\vdash (\mathbb{E}^*) \equiv \mathbb{E}$		i
$(\mathbb{E} \equiv (\mathbb{E}^*) \Rightarrow (\mathbb{E}^*) \equiv \mathbb{E})$,! 1 ($\forall\mathbb{E}$: C1.8)	i
$\mathbb{E} \equiv (\mathbb{E}^*) \Rightarrow (\mathbb{E}^*) \equiv \mathbb{E}$,! 2 ($(\)\mathbb{E}$: 1)	i
$(\mathbb{E}^*) \equiv \mathbb{E}$! 3 ($\Rightarrow\mathbb{E}$: P7,2)	i
\square		
! 9.		i
$\vdash (\mathbb{E}^D) \equiv \mathbb{U}$		i
\mathbf{x}	,! 1 (Prem)	i
$\mathbb{E}[\mathbf{x}, \mathbf{x}]$,! 2 ($\forall\mathbb{E}$: P5)	i
$\exists y \mathbb{E}[\mathbf{x}, y]$,! 3 ($\exists\mathbb{I}$: 2)	i
$\forall x \exists y \mathbb{E}[x, y]$,! 4 ($\forall\mathbb{I}$: 1,3)	i
$(\forall x \exists y \mathbb{E}[x, y] \Rightarrow (\mathbb{E}^D) \equiv \mathbb{U})$,! 5 ($\forall\mathbb{E}$: C5.25)	i
$\forall x \exists y \mathbb{E}[x, y] \Rightarrow (\mathbb{E}^D) \equiv \mathbb{U}$,! 6 ($(\)\mathbb{E}$: 5)	i
$(\mathbb{E}^D) \equiv \mathbb{U}$! 7 ($\Rightarrow\mathbb{E}$: 4,6)	i
\square		
! 10.		i
$\vdash \mathbf{f} \mathbb{E}$		i
$\mathbf{x}, \mathbf{y}, \mathbf{z}$,! 1 (Prem)	i
$\mathbb{E}[\mathbf{x}, \mathbf{y}] \ \& \ \mathbb{E}[\mathbf{x}, \mathbf{z}]$,! 2 (Prem)	i
$\mathbb{E}[\mathbf{x}, \mathbf{y}]$,! 3 ($\&\mathbb{E}$: 2)	i
$(\mathbb{E}[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{y})$,! 4 ($\forall\mathbb{E}$: P3)	i
$\mathbb{E}[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{y}$,! 5 ($(\)\mathbb{E}$: 4)	i
$\mathbf{x} = \mathbf{y}$,! 6 ($\Rightarrow\mathbb{E}$: 3,5)	i
$\mathbb{E}[\mathbf{x}, \mathbf{z}]$,! 7 ($\&\mathbb{E}$: 2)	i
$\mathbb{E}[\mathbf{y}, \mathbf{z}]$,! 8 ($=\mathbb{E}$: 6,7)	i

$(\mathbb{E}[y, z] \Rightarrow y = z)$,!	9	($\forall E$: P3)	i
$\mathbb{E}[y, z] \Rightarrow y = z$,!	10	($()E$: 9)	i
$y = z$,!	11	($\Rightarrow E$: 8,10)	i
$\mathbb{E}[x, y] \ \& \ \mathbb{E}[x, z] \Rightarrow y = z$,!	12	($\Rightarrow I$: 2,11)	i
$(\mathbb{E}[x, y] \ \& \ \mathbb{E}[x, z] \Rightarrow y = z)$,!	13	($()I$: 12)	i
$\forall x \forall y \forall z (\mathbb{E}[x, y] \ \& \ \mathbb{E}[x, z] \Rightarrow y = z)$,!	14	($\forall I$: 1,13)	i
f \mathbb{E}	!	15	($\$I$: C8.1,14)	i
\square				
! 11.				i
$\vdash \mathbb{E} \text{ F } \mathbb{U}$				i
$(\mathbb{E}^D) \equiv \mathbb{U} \ \& \ \mathbf{f} \ \mathbb{E}$,!	1	($\&I$: P9,P10)	i
$\mathbb{E} \text{ F } \mathbb{U}$!	2	($\$I$: C8.10,1)	i
\square				
! 12.				i
$\vdash \forall A (\mathbb{E} \lceil A) \equiv (\mathbb{E} \lfloor A)$				i
A	,!	1	(Prem)	i
$(\forall x \forall y (\mathbb{E}[x, y] \Rightarrow (A[x] \Leftrightarrow A[y]))) \Rightarrow (\mathbb{E} \lceil A) \equiv (\mathbb{E} \lfloor A)$,!	2	($\forall E$: C7.53)	i
$\forall x \forall y (\mathbb{E}[x, y] \Rightarrow (A[x] \Leftrightarrow A[y])) \Rightarrow (\mathbb{E} \lceil A) \equiv (\mathbb{E} \lfloor A)$,!	3	($()E$: 2)	i
x, y	,!	4	(Prem)	i
$\mathbb{E}[x, y]$,!	5	(Prem)	i
$(\mathbb{E}[x, y] \Rightarrow x = y)$,!	6	($\forall E$: P3)	i
$\mathbb{E}[x, y] \Rightarrow x = y$,!	7	($()E$: 6)	i
x = y	,!	8	($\Rightarrow E$: 5,7)	i
A[x]	,!	9	(Prem)	i
A[y]	,!	10	($=E$: 8,9)	i
$A[x] \Rightarrow A[y]$,!	11	($\Rightarrow I$: 9,10)	i
A[y]	,!	12	(Prem)	i
A[x]	,!	13	($=E$: 8,12)	i

$\mathbf{A}[y] \Rightarrow \mathbf{A}[x]$,! 14 (\Rightarrow I: 12,13)	i
$\mathbf{A}[x] \Leftrightarrow \mathbf{A}[y]$,! 15 (\Leftrightarrow I: 11,14)	i
$(\mathbf{A}[x] \Leftrightarrow \mathbf{A}[y])$,! 16 ($(\)$ I: 15)	i
$\mathbb{E}[x,y] \Rightarrow (\mathbf{A}[x] \Leftrightarrow \mathbf{A}[y])$,! 17 (\Rightarrow I: 5,16)	i
$(\mathbb{E}[x,y] \Rightarrow (\mathbf{A}[x] \Leftrightarrow \mathbf{A}[y]))$,! 18 ($(\)$ I: 17)	i
$\forall x \forall y (\mathbb{E}[x,y] \Rightarrow (\mathbf{A}[x] \Leftrightarrow \mathbf{A}[y]))$,! 19 (\forall I: 4,18)	i
$(\mathbb{E} \vdash \mathbf{A}) \equiv (\mathbb{E} \lfloor \mathbf{A})$,! 20 (\Rightarrow E: 3,19)	i
$\forall \mathbf{A} (\mathbb{E} \vdash \mathbf{A}) \equiv (\mathbb{E} \lfloor \mathbf{A})$! 21 (\forall I: 1,20)	i
\square		
! 13.		i
$\vdash \forall \mathbf{A} ((\mathbb{E} \vdash \mathbf{A})^*) \equiv (\mathbb{E} \vdash \mathbf{A})$		i
\mathbf{A}	,! 1 (Prem)	i
$(\mathbb{E} \vdash \mathbf{A}) \equiv (\mathbb{E} \lfloor \mathbf{A})$,! 2 (\forall E: P12)	i
$((\mathbb{E} \vdash \mathbf{A}) \equiv (\mathbb{E} \lfloor \mathbf{A}) \Rightarrow ((\mathbb{E} \vdash \mathbf{A})^*) \equiv ((\mathbb{E} \lfloor \mathbf{A})^*))$,! 3 (\forall E: C3.20)	i
$(\mathbb{E} \vdash \mathbf{A}) \equiv (\mathbb{E} \lfloor \mathbf{A}) \Rightarrow ((\mathbb{E} \vdash \mathbf{A})^*) \equiv ((\mathbb{E} \lfloor \mathbf{A})^*)$,! 4 ($(\)$ E: 3)	i
$((\mathbb{E} \vdash \mathbf{A})^*) \equiv ((\mathbb{E} \lfloor \mathbf{A})^*)$,! 5 (\Rightarrow E: 2,4)	i
$((\mathbb{E} \lfloor \mathbf{A})^*) \equiv ((\mathbb{E}^*) \vdash \mathbf{A})$,! 6 (\forall E: C7.52)	i
$((\mathbb{E} \vdash \mathbf{A})^*) \equiv ((\mathbb{E} \lfloor \mathbf{A})^*) \ \& \ ((\mathbb{E} \lfloor \mathbf{A})^*) \equiv ((\mathbb{E}^*) \vdash \mathbf{A})$,! 7 ($\&$ I: 5,6)	i
$((\mathbb{E}^*) \equiv \mathbb{E} \Rightarrow ((\mathbb{E}^*) \vdash \mathbf{A}) \equiv (\mathbb{E} \vdash \mathbf{A}))$,! 8 (\forall E: C7.12)	i
$(\mathbb{E}^*) \equiv \mathbb{E} \Rightarrow ((\mathbb{E}^*) \vdash \mathbf{A}) \equiv (\mathbb{E} \vdash \mathbf{A})$,! 9 ($(\)$ E: 8)	i
$((\mathbb{E}^*) \vdash \mathbf{A}) \equiv (\mathbb{E} \vdash \mathbf{A})$,! 10 (\Rightarrow E: P8,9)	i
$((\mathbb{E} \vdash \mathbf{A})^*) \equiv ((\mathbb{E} \lfloor \mathbf{A})^*) \ \& \ ((\mathbb{E} \lfloor \mathbf{A})^*) \equiv ((\mathbb{E}^*) \vdash \mathbf{A})$ $\& \ ((\mathbb{E}^*) \vdash \mathbf{A}) \equiv (\mathbb{E} \vdash \mathbf{A})$,! 11 ($\&$ I: 7,10)	i
$(((\mathbb{E} \vdash \mathbf{A})^*) \equiv ((\mathbb{E} \lfloor \mathbf{A})^*) \ \& \ ((\mathbb{E} \lfloor \mathbf{A})^*) \equiv ((\mathbb{E}^*) \vdash \mathbf{A})$ $\& \ ((\mathbb{E}^*) \vdash \mathbf{A}) \equiv (\mathbb{E} \vdash \mathbf{A})$ $\Rightarrow ((\mathbb{E} \vdash \mathbf{A})^*) \equiv (\mathbb{E} \vdash \mathbf{A}))$		

,! 12 ($\forall E$: C1.19) ;

$((\mathbb{E} \lceil \mathbf{A})^*) \equiv ((\mathbb{E} \lfloor \mathbf{A})^*) \ \& \ ((\mathbb{E} \lfloor \mathbf{A})^*) \equiv ((\mathbb{E}^*) \lceil \mathbf{A})$
 $\& \ ((\mathbb{E}^*) \lceil \mathbf{A}) \equiv (\mathbb{E} \lceil \mathbf{A})$
 $\Rightarrow ((\mathbb{E} \lceil \mathbf{A})^*) \equiv (\mathbb{E} \lceil \mathbf{A})$

,! 13 ($(\)E$: 12) ;

$((\mathbb{E} \lceil \mathbf{A})^*) \equiv (\mathbb{E} \lceil \mathbf{A})$

,! 14 ($\Rightarrow E$: 11,13) ;

$\forall \mathbf{A} ((\mathbb{E} \lceil \mathbf{A})^*) \equiv (\mathbb{E} \lceil \mathbf{A})$

! 15 ($\forall I$: 1,14) ;

\square

! 14.

;

$\vdash \forall \mathbf{R} (\mathbb{E} \circ \mathbf{R}) \equiv \mathbf{R}$

;

\mathbf{R}

,! 1 (Prem) ;

\mathbf{x}, \mathbf{y}

,! 2 (Prem) ;

$((\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \exists \mathbf{z} (\mathbb{E}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{R}[\mathbf{z}, \mathbf{y}]))$

,! 3 ($\forall E$: C10.2) ;

$(\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \exists \mathbf{z} (\mathbb{E}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{R}[\mathbf{z}, \mathbf{y}])$

,! 4 ($(\)E$: 3) ;

$(\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}]$

,! 5 (Prem) ;

$(\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}] \Rightarrow \exists \mathbf{z} (\mathbb{E}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{R}[\mathbf{z}, \mathbf{y}])$

,! 6 ($\Leftrightarrow E$: 4) ;

$\exists \mathbf{z} (\mathbb{E}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{R}[\mathbf{z}, \mathbf{y}])$

,! 7 ($\Rightarrow E$: 5,6) ;

$(\mathbb{E}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{R}[\mathbf{z}, \mathbf{y}])$

,! 8 ($\exists E$: 7) ;

$\mathbb{E}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{R}[\mathbf{z}, \mathbf{y}]$

,! 9 ($(\)E$: 8) ;

$\mathbb{E}[\mathbf{x}, \mathbf{z}]$

,! 10 ($\&E$: 9) ;

$\mathbf{R}[\mathbf{z}, \mathbf{y}]$

,! 11 ($\&E$: 9) ;

$(\mathbb{E}[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{x} = \mathbf{z})$

,! 12 ($\forall E$: P3) ;

$\mathbb{E}[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{x} = \mathbf{z}$

,! 13 ($(\)E$: 12) ;

$\mathbf{x} = \mathbf{z}$

,! 14 ($\Rightarrow E$: 10,13) ;

$\mathbf{R}[\mathbf{x}, \mathbf{y}]$

,! 15 ($=E$: 11,14) ;

$(\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbf{R}[\mathbf{x}, \mathbf{y}]$

,! 16 ($\Rightarrow I$: 5,15) ;

$\mathbf{R}[\mathbf{x}, \mathbf{y}]$

,! 17 (Prem) ;

$\mathbb{E}[\mathbf{x}, \mathbf{x}]$

,! 18 ($\forall E$: P5) ;

$\mathbb{E}[\mathbf{x}, \mathbf{x}] \ \& \ \mathbf{R}[\mathbf{x}, \mathbf{y}]$

,! 19 ($\&I$: 17,18) ;

$(\mathbb{E}[\mathbf{x}, \mathbf{x}] \ \& \ \mathbf{R}[\mathbf{x}, \mathbf{y}])$,! 20 ((I: 19)	i
$\exists z(\mathbb{E}[\mathbf{x}, z] \ \& \ \mathbf{R}[z, \mathbf{y}])$,! 21 (\exists I: 20)	i
$\exists z(\mathbb{E}[\mathbf{x}, z] \ \& \ \mathbf{R}[z, \mathbf{y}]) \Rightarrow (\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}]$,! 22 (\Leftrightarrow E: 4)	i
$(\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}]$,! 23 (\Rightarrow E: 21,22)	i
$\mathbf{R}[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}]$,! 24 (\Rightarrow I: 17,23)	i
$(\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x}, \mathbf{y}]$,! 25 (\Leftrightarrow I: 16,24)	i
$((\mathbb{E} \circ \mathbf{R})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbf{R}[\mathbf{x}, \mathbf{y}])$,! 26 ((I: 25)	i
$\forall x \forall y ((\mathbb{E} \circ \mathbf{R})[x, y] \Leftrightarrow \mathbf{R}[x, y])$,! 27 (\forall I: 2,26)	i
$(\mathbb{E} \circ \mathbf{R}) \equiv \mathbf{R}$,! 28 (\S I: C1.5)	i
$\forall \mathbf{R} (\mathbb{E} \circ \mathbf{R}) \equiv \mathbf{R}$! 29 (\forall I: 1,28)	i
\square		
! 15.		i
$\vdash \forall \mathbf{R} (\mathbf{R} \circ \mathbb{E}) \equiv \mathbf{R}$		i
\mathbf{R}	,! 1 (Prem)	i
$((\mathbf{R} \circ \mathbb{E})^*) \equiv ((\mathbb{E}^*) \circ (\mathbf{R}^*))$,! 2 (\forall E: C10.11)	i
$((\mathbb{E}^*) \equiv \mathbb{E} \Rightarrow ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \equiv (\mathbb{E} \circ (\mathbf{R}^*)))$,! 3 (\forall E: C10.9)	i
$(\mathbb{E}^*) \equiv \mathbb{E} \Rightarrow ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \equiv (\mathbb{E} \circ (\mathbf{R}^*))$,! 4 ((E: 3)	i
$((\mathbb{E}^*) \circ (\mathbf{R}^*)) \equiv (\mathbb{E} \circ (\mathbf{R}^*))$,! 5 (\Rightarrow E: P7,4)	i
$((\mathbf{R} \circ \mathbb{E})^*) \equiv ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \ \& \ ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \equiv (\mathbb{E} \circ (\mathbf{R}^*))$,! 6 ($\&$ I: 2,5)	i
$(\mathbb{E} \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*)$,! 7 (\forall E: P14)	i
$((\mathbf{R} \circ \mathbb{E})^*) \equiv ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \ \& \ ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \equiv (\mathbb{E} \circ (\mathbf{R}^*))$ $\ \& \ (\mathbb{E} \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*)$,! 8 ($\&$ I: 6,7)	i
$(((\mathbf{R} \circ \mathbb{E})^*) \equiv ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \ \& \ ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \equiv (\mathbb{E} \circ (\mathbf{R}^*))$ $\ \& \ (\mathbb{E} \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*)$ $\Rightarrow ((\mathbf{R} \circ \mathbb{E})^*) \equiv (\mathbf{R}^*))$,! 9 (\forall E: C1.19)	i
$((\mathbf{R} \circ \mathbb{E})^*) \equiv ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \ \& \ ((\mathbb{E}^*) \circ (\mathbf{R}^*)) \equiv (\mathbb{E} \circ (\mathbf{R}^*))$		

$\& (\mathbb{E} \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*)$		
$\Rightarrow ((\mathbf{R} \circ \mathbb{E})^*) \equiv (\mathbf{R}^*)$, ! 10 ((E: 9) i
$((\mathbf{R} \circ \mathbb{E})^*) \equiv (\mathbf{R}^*)$, ! 11 (\Rightarrow E: 8,10) i
$(((\mathbf{R} \circ \mathbb{E})^*) \equiv (\mathbf{R}^*) \Rightarrow (\mathbf{R} \circ \mathbb{E}) \equiv ((\mathbf{R}^*)^*))$, ! 12 (\forall E: C3.24) i
$((\mathbf{R} \circ \mathbb{E})^*) \equiv (\mathbf{R}^*) \Rightarrow (\mathbf{R} \circ \mathbb{E}) \equiv ((\mathbf{R}^*)^*)$, ! 13 ((E: 12) i
$(\mathbf{R} \circ \mathbb{E}) \equiv ((\mathbf{R}^*)^*)$, ! 14 (\Rightarrow E: 11,13) i
$((\mathbf{R}^*)^*) \equiv \mathbf{R}$, ! 15 (\forall E: C3.17) i
$(\mathbf{R} \circ \mathbb{E}) \equiv ((\mathbf{R}^*)^*) \& ((\mathbf{R}^*)^*) \equiv \mathbf{R}$, ! 16 (&I: 14,15) i
$((\mathbf{R} \circ \mathbb{E}) \equiv ((\mathbf{R}^*)^*) \& ((\mathbf{R}^*)^*) \equiv \mathbf{R} \Rightarrow (\mathbf{R} \circ \mathbb{E}) \equiv \mathbf{R})$, ! 17 (\forall E: C1.15) i
$(\mathbf{R} \circ \mathbb{E}) \equiv ((\mathbf{R}^*)^*) \& ((\mathbf{R}^*)^*) \equiv \mathbf{R} \Rightarrow (\mathbf{R} \circ \mathbb{E}) \equiv \mathbf{R}$, ! 18 ((E: 17) i
$(\mathbf{R} \circ \mathbb{E}) \equiv \mathbf{R}$, ! 19 (\Rightarrow E: 16,18) i
$\forall \mathbf{R} (\mathbf{R} \circ \mathbb{E}) \equiv \mathbf{R}$! 20 (\forall I: 1,19) i

□

! 16. ($\mathbb{I}\mathbf{A}$) represents an equality predicate restricted to the domain \mathbf{A} . i

$\mathbb{D} \ \mathbb{I} \ ; \ (\mathbb{I}\mathbf{A}) \ ; \ ; \ (\mathbb{E} \ \lceil \ \mathbf{A})$ i

! 17. i

$\vdash \forall \mathbf{A} \forall \mathbf{x} \forall \mathbf{y} ((\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbf{x} = \mathbf{y} \& \mathbf{A}[\mathbf{x}])$ i

$\mathbf{A}, \mathbf{x}, \mathbf{y}$, ! 1 (Prem) i

$((\mathbb{E} \ \lceil \ \mathbf{A})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbb{E}[\mathbf{x}, \mathbf{y}] \& \mathbf{A}[\mathbf{x}])$, ! 2 (\forall E: C7.2) i

$(\mathbb{E} \ \lceil \ \mathbf{A})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbb{E}[\mathbf{x}, \mathbf{y}] \& \mathbf{A}[\mathbf{x}]$, ! 3 ((E: 2) i

$(\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbb{E}[\mathbf{x}, \mathbf{y}] \& \mathbf{A}[\mathbf{x}]$, ! 4 (\mathbb{D} I: P16,3) i

$(\mathbb{E}[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbf{x} = \mathbf{y})$, ! 5 (\forall E: P2) i

$\mathbb{E}[\mathbf{x}, \mathbf{y}] \Leftrightarrow \mathbf{x} = \mathbf{y}$, ! 6 ((E: 5) i

$(\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}]$, ! 7 (Prem) i

$(\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbb{E}[\mathbf{x}, \mathbf{y}] \& \mathbf{A}[\mathbf{x}]$, ! 8 (\Leftrightarrow E: 4) i

$\mathbb{E}[x, y] \ \& \ \mathbf{A}[x]$, ! 9 (\Rightarrow E: 7,8)	i
$\mathbb{E}[x, y]$, ! 10 ($\&$ E: 9)	i
$\mathbf{A}[x]$, ! 11 ($\&$ E: 9)	i
$\mathbb{E}[x, y] \Rightarrow x = y$, ! 12 (\Leftrightarrow E: 6)	i
$x = y$, ! 13 (\Rightarrow E: 10,12)	i
$x = y \ \& \ \mathbf{A}[x]$, ! 14 ($\&$ I: 11,13)	i
$(\mathbb{I}\mathbf{A})[x, y] \Rightarrow x = y \ \& \ \mathbf{A}[x]$, ! 15 (\Rightarrow I: 7,14)	i
$x = y \ \& \ \mathbf{A}[x]$, ! 16 (Prem)	i
$x = y$, ! 17 ($\&$ E: 16)	i
$\mathbf{A}[x]$, ! 18 ($\&$ E: 16)	i
$x = y \Rightarrow \mathbb{E}[x, y]$, ! 19 (\Leftrightarrow E: 6)	i
$\mathbb{E}[x, y]$, ! 20 (\Rightarrow E: 17,19)	i
$\mathbb{E}[x, y] \ \& \ \mathbf{A}[x]$, ! 21 ($\&$ I: 18,20)	i
$\mathbb{E}[x, y] \ \& \ \mathbf{A}[x] \Rightarrow (\mathbb{I}\mathbf{A})[x, y]$, ! 22 (\Leftrightarrow E: 4)	i
$(\mathbb{I}\mathbf{A})[x, y]$, ! 23 (\Rightarrow E: 21,22)	i
$x = y \ \& \ \mathbf{A}[x] \Rightarrow (\mathbb{I}\mathbf{A})[x, y]$, ! 24 (\Rightarrow I: 16,23)	i
$(\mathbb{I}\mathbf{A})[x, y] \Leftrightarrow x = y \ \& \ \mathbf{A}[x]$, ! 25 (\Leftrightarrow I: 15,24)	i
$((\mathbb{I}\mathbf{A})[x, y] \Leftrightarrow x = y \ \& \ \mathbf{A}[x])$, ! 26 ($(())$ I: 25)	i
$\forall x \forall y ((\mathbb{I}\mathbf{A})[x, y] \Leftrightarrow x = y \ \& \ \mathbf{A}[x])$! 27 (\forall I: 1,26)	i
\square		
! 18.		i
$\vdash \forall x \forall y ((\mathbb{I}\mathbf{A})[x, y] \Rightarrow x = y \ \& \ \mathbf{A}[x])$		i
x, y	, ! 1 (Prem)	i
$((\mathbb{I}\mathbf{A})[x, y] \Leftrightarrow x = y \ \& \ \mathbf{A}[x])$, ! 2 (\forall E: P17)	i
$(\mathbb{I}\mathbf{A})[x, y] \Leftrightarrow x = y \ \& \ \mathbf{A}[x]$, ! 3 ($(())$ E: 2)	i
$(\mathbb{I}\mathbf{A})[x, y] \Rightarrow x = y \ \& \ \mathbf{A}[x]$, ! 4 (\Leftrightarrow E: 3)	i
$((\mathbb{I}\mathbf{A})[x, y] \Rightarrow x = y \ \& \ \mathbf{A}[x])$, ! 5 ($(())$ I: 4)	i
$\forall x \forall y ((\mathbb{I}\mathbf{A})[x, y] \Rightarrow x = y \ \& \ \mathbf{A}[x])$! 6 (\forall I: 1,5)	i

□

! 19.

$\vdash \forall x \forall y (x = y \ \& \ A[x] \Rightarrow (\mathbb{I}A)[x,y])$		i
$\mathbf{x, y}$,! 1 (Prem)	i
$((\mathbb{I}A)[x,y] \Leftrightarrow \mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}])$,! 2 ($\forall E$: P17)	i
$(\mathbb{I}A)[x,y] \Leftrightarrow \mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}]$,! 3 ($(\)E$: 2)	i
$\mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}] \Rightarrow (\mathbb{I}A)[x,y]$,! 4 ($\Leftrightarrow E$: 3)	i
$(\mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}] \Rightarrow (\mathbb{I}A)[x,y])$,! 5 ($(\)I$: 4)	i
$\forall x \forall y (x = y \ \& \ A[x] \Rightarrow (\mathbb{I}A)[x,y])$! 6 ($\forall I$: 1,5)	i

□

! 20.

$\vdash \forall A \forall x \forall y ((\mathbb{I}A)[x,y] \Rightarrow x = y)$		i
$\mathbf{A, x, y}$,! 1 (Prem)	i
$(\mathbb{I}A)[x,y]$,! 2 (Prem)	i
$((\mathbb{I}A)[x,y] \Rightarrow \mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}])$,! 3 ($\forall E$: P18)	i
$(\mathbb{I}A)[x,y] \Rightarrow \mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}]$,! 4 ($(\)E$: 3)	i
$\mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}]$,! 5 ($\Rightarrow E$: 2,4)	i
$\mathbf{x} = \mathbf{y}$,! 6 ($\&E$: 5)	i
$(\mathbb{I}A)[x,y] \Rightarrow \mathbf{x} = \mathbf{y}$,! 7 ($\Rightarrow I$: 2,6)	i
$((\mathbb{I}A)[x,y] \Rightarrow \mathbf{x} = \mathbf{y})$,! 8 ($(\)I$: 7)	i
$\forall A \forall x \forall y ((\mathbb{I}A)[x,y] \Rightarrow x = y)$! 9 ($\forall I$: 1,8)	i

□

! 21. The converse of the following proposition also holds.

$\vdash \forall A \forall B (A \subseteq B \Rightarrow (\mathbb{I}A) \subseteq (\mathbb{I}B))$		i
$\mathbf{A, B}$,! 1 (Prem)	i
$\mathbf{A} \subseteq \mathbf{B}$,! 2 (Prem)	i
$(\mathbf{A} \subseteq \mathbf{B} \Rightarrow (\mathbb{E} \lceil \mathbf{A}) \subseteq (\mathbb{E} \lceil \mathbf{B}))$,! 3 ($\forall E$: C7.10)	i
$\mathbf{A} \subseteq \mathbf{B} \Rightarrow (\mathbb{E} \lceil \mathbf{A}) \subseteq (\mathbb{E} \lceil \mathbf{B})$,! 4 ($(\)E$: 3)	i
$(\mathbb{E} \lceil \mathbf{A}) \subseteq (\mathbb{E} \lceil \mathbf{B})$,! 5 ($\Rightarrow E$: 2,4)	i

$(\mathbb{I}\mathbf{A}) \subseteq (\mathbb{E} \ulcorner \mathbf{B})$,! 6 ($\mathbb{D}\mathbb{I}$: P16,5)	i
$(\mathbb{I}\mathbf{A}) \subseteq (\mathbb{I}\mathbf{B})$,! 7 ($\mathbb{D}\mathbb{I}$: P16,6)	i
$\mathbf{A} \subseteq \mathbf{B} \Rightarrow (\mathbb{I}\mathbf{A}) \subseteq (\mathbb{I}\mathbf{B})$,! 8 ($\Rightarrow\mathbb{E}$: 2,7)	i
$(\mathbf{A} \subseteq \mathbf{B} \Rightarrow (\mathbb{I}\mathbf{A}) \subseteq (\mathbb{I}\mathbf{B}))$,! 9 ($(\)\mathbb{I}$: 8)	i
$\forall \mathbf{A} \forall \mathbf{B} (\mathbf{A} \subseteq \mathbf{B} \Rightarrow (\mathbb{I}\mathbf{A}) \subseteq (\mathbb{I}\mathbf{B}))$! 10 ($\forall\mathbb{I}$: 1,9)	i
\square		

! 22.

$\vdash \forall \mathbf{A} \forall \mathbf{B} (\mathbf{A} \equiv \mathbf{B} \Rightarrow (\mathbb{I}\mathbf{A}) \equiv (\mathbb{I}\mathbf{B}))$		i
\mathbf{A}, \mathbf{B}	,! 1 (Prem)	i
$\mathbf{A} \equiv \mathbf{B}$,! 2 (Prem)	i
$(\mathbf{A} \equiv \mathbf{B} \Rightarrow (\mathbb{E} \ulcorner \mathbf{A}) \equiv (\mathbb{E} \ulcorner \mathbf{B}))$,! 3 ($\forall\mathbb{E}$: C7.12)	i
$\mathbf{A} \equiv \mathbf{B} \Rightarrow (\mathbb{E} \ulcorner \mathbf{A}) \equiv (\mathbb{E} \ulcorner \mathbf{B})$,! 4 ($(\)\mathbb{E}$: 3)	i
$(\mathbb{E} \ulcorner \mathbf{A}) \equiv (\mathbb{E} \ulcorner \mathbf{B})$,! 5 ($\Rightarrow\mathbb{E}$: 2,4)	i
$(\mathbb{I}\mathbf{A}) \equiv (\mathbb{E} \ulcorner \mathbf{B})$,! 6 ($\mathbb{D}\mathbb{I}$: P16,5)	i
$(\mathbb{I}\mathbf{A}) \equiv (\mathbb{I}\mathbf{B})$,! 7 ($\mathbb{D}\mathbb{I}$: P16,6)	i
$\mathbf{A} \equiv \mathbf{B} \Rightarrow (\mathbb{I}\mathbf{A}) \equiv (\mathbb{I}\mathbf{B})$,! 8 ($\Rightarrow\mathbb{E}$: 2,7)	i
$(\mathbf{A} \equiv \mathbf{B} \Rightarrow (\mathbb{I}\mathbf{A}) \equiv (\mathbb{I}\mathbf{B}))$,! 9 ($(\)\mathbb{I}$: 8)	i
$\forall \mathbf{A} \forall \mathbf{B} (\mathbf{A} \equiv \mathbf{B} \Rightarrow (\mathbb{I}\mathbf{A}) \equiv (\mathbb{I}\mathbf{B}))$! 10 ($\forall\mathbb{I}$: 1,9)	i
\square		

! 23.

$\vdash \forall \mathbf{A} ((\mathbb{I}\mathbf{A})^*) \equiv (\mathbb{I}\mathbf{A})$		i
\mathbf{A}	,! 1 (Prem)	i
$((\mathbb{E} \ulcorner \mathbf{A})^*) \equiv (\mathbb{E} \ulcorner \mathbf{A})$,! 2 ($\forall\mathbb{E}$: P13)	i
$((\mathbb{I}\mathbf{A})^*) \equiv (\mathbb{I}\mathbf{A})$,! 3 ($\mathbb{D}\mathbb{I}$: P16,2)	i
$\forall \mathbf{A} ((\mathbb{I}\mathbf{A})^*) \equiv (\mathbb{I}\mathbf{A})$! 4 ($\forall\mathbb{I}$: 1,3)	i
\square		

! 24.

$\vdash \forall A ((\mathbb{I}A)^D) \equiv A$		i
\mathbf{A}	,! 1 (Prem)	i
$((\mathbb{E}^D) \equiv \mathbf{U} \Rightarrow ((\mathbb{E} \lceil \mathbf{A})^D) \equiv \mathbf{A})$,! 2 ($\forall E$: C7.36)	i
$(\mathbb{E}^D) \equiv \mathbf{U} \Rightarrow ((\mathbb{E} \lceil \mathbf{A})^D) \equiv \mathbf{A}$,! 3 ($(\)E$: 2)	i
$((\mathbb{E} \lceil \mathbf{A})^D) \equiv \mathbf{A}$,! 4 ($\Rightarrow E$: P9,3)	i
$((\mathbb{I}A)^D) \equiv \mathbf{A}$,! 5 ($\mathbb{D}I$: P16,4)	i
$\forall A ((\mathbb{I}A)^D) \equiv A$! 6 ($\forall I$: 1,5)	i

□

! 25.

$\vdash \forall A \mathbf{f} (\mathbb{I}A)$		i
\mathbf{A}	,! 1 (Prem)	i
$(\mathbf{f} \mathbb{E} \Rightarrow \mathbf{f} (\mathbb{E} \lceil \mathbf{A}))$,! 2 ($\forall E$: C8.4)	i
$\mathbf{f} \mathbb{E} \Rightarrow \mathbf{f} (\mathbb{E} \lceil \mathbf{A})$,! 3 ($(\)E$: 2)	i
$\mathbf{f} (\mathbb{E} \lceil \mathbf{A})$,! 4 ($\Rightarrow E$: P10,3)	i
$\mathbf{f} (\mathbb{I}A)$,! 5 ($\mathbb{D}I$: P16,4)	i
$\forall A \mathbf{f} (\mathbb{I}A)$! 6 ($\forall I$: 1,5)	i

□

! 26.

$\vdash \forall A (\mathbb{I}A) \mathbb{F} A$		i
\mathbf{A}	,! 1 (Prem)	i
$((\mathbb{I}A)^D) \equiv \mathbf{A}$,! 2 ($\forall E$: P24)	i
$\mathbf{f} (\mathbb{I}A)$,! 3 ($\forall E$: P24)	i
$((\mathbb{I}A)^D) \equiv \mathbf{A} \ \& \ \mathbf{f} (\mathbb{I}A)$,! 4 ($\&I$: 2,3)	i
$(\mathbb{I}A) \mathbb{F} A$,! 5 ($\mathbb{S}I$: C8.10,4)	i
$\forall A (\mathbb{I}A) \mathbb{F} A$! 6 ($\forall I$: 1,5)	i

□

! 27.

$\vdash \forall A (\mathbb{I}A) \mathbb{1} A$		i
---	--	---

A	,! 1 (Prem)	i
(IA) F A	,! 2 ($\forall E$: P26)	i
((IA) F A \Rightarrow ((IA)[*]) 1 A)	,! 3 ($\forall E$: C9.23)	i
(IA) F A \Rightarrow ((IA)[*]) 1 A	,! 4 (())E: 3)	i
((IA)[*]) 1 A	,! 5 ($\Rightarrow E$: 4)	i
((IA)[*]) \equiv (IA)	,! 6 ($\forall E$: P23)	i
((IA)[*]) 1 A & ((IA)[*]) \equiv (IA)	,! 7 (&I: 5,6)	i
(((IA)[*]) 1 A & ((IA)[*]) \equiv (IA) \Rightarrow (IA) 1 A)	,! 8 ($\forall E$ C9.27)	i
((IA)[*]) 1 A & ((IA)[*]) \equiv (IA) \Rightarrow (IA) 1 A	,! 9 (())E: 8)	i
(IA) 1 A	,! 10 ($\Rightarrow E$: 7,9)	i
$\forall A$ (IA) 1 A	! 11 ($\forall I$: 1,10)	i

□

! 28.

$\vdash \forall R \forall A ((R^D) \subseteq A \Rightarrow ((IA) \circ R) \equiv R)$		i
R, A	,! 1 (Prem)	i
$(R^D) \subseteq A$,! 2 (Prem)	i
$(E \circ R) \equiv R$,! 3 ($\forall E$: P14)	i
$((E \circ R) \lceil A) \equiv ((E \lceil A) \circ R)$,! 4 ($\forall E$: C10.20)	i
$((E \circ R) \lceil A) \equiv ((IA) \circ R)$,! 5 ($\mathbb{D}I$: P16,4)	i
$((E \circ R) \equiv R \Rightarrow ((E \circ R) \lceil A) \equiv (R \lceil A))$,! 6 ($\forall E$: C7.12)	i
$(E \circ R) \equiv R \Rightarrow ((E \circ R) \lceil A) \equiv (R \lceil A)$,! 7 (())E: 6)	i
$((E \circ R) \lceil A) \equiv (R \lceil A)$,! 8 ($\Rightarrow E$: 3,7)	i
$((E \circ R) \lceil A) \equiv ((IA) \circ R) \& ((E \circ R) \lceil A) \equiv (R \lceil A)$,! 9 (&I: 5,8)	i
$(((E \circ R) \lceil A) \equiv ((IA) \circ R) \& ((E \circ R) \lceil A) \equiv (R \lceil A) \Rightarrow ((IA) \circ R) \equiv (R \lceil A))$		

,! 10 ($\forall E$: C1.18) i

$$((E \circ R) \vdash A) \equiv ((\mathbb{I}A) \circ R) \ \& \ ((E \circ R) \vdash A) \equiv (R \vdash A)$$

$$\Rightarrow ((\mathbb{I}A) \circ R) \equiv (R \vdash A)$$

,! 11 ($()E$: 10) i

$$((\mathbb{I}A) \circ R) \equiv (R \vdash A)$$

,! 12 ($\Rightarrow E$: 9,11) i

$$((R^D) \subseteq A \Rightarrow (R \vdash A) \equiv R)$$

,! 13 ($\forall E$: C7.24) i

$$(R^D) \subseteq A \Rightarrow (R \vdash A) \equiv R$$

,! 14 ($()E$: 13) i

$$(R \vdash A) \equiv R$$

,! 15 ($\Rightarrow E$: 2,14) i

$$((\mathbb{I}A) \circ R) \equiv (R \vdash A) \ \& \ (R \vdash A) \equiv R$$

,! 16 ($\&I$: 12,15) i

$$((\mathbb{I}A) \circ R) \equiv (R \vdash A) \ \& \ (R \vdash A) \equiv R \Rightarrow ((\mathbb{I}A) \circ R) \equiv R$$

,! 17 ($\forall E$: C1.15) i

$$((\mathbb{I}A) \circ R) \equiv (R \vdash A) \ \& \ (R \vdash A) \equiv R \Rightarrow ((\mathbb{I}A) \circ R) \equiv R$$

,! 18 ($()E$: 17) i

$$((\mathbb{I}A) \circ R) \equiv R$$

,! 19 ($\Rightarrow E$: 16,18) i

$$(R^D) \subseteq A \Rightarrow ((\mathbb{I}A) \circ R) \equiv R$$

,! 20 ($\Rightarrow I$: 2,19) i

$$((R^D) \subseteq A \Rightarrow ((\mathbb{I}A) \circ R) \equiv R)$$

,! 21 ($()I$: 20) i

$$\forall R \forall A ((R^D) \subseteq A \Rightarrow ((\mathbb{I}A) \circ R) \equiv R)$$

! 22 ($\forall I$: 1,21) i

□

! 29.

$\vdash \forall R \forall S \forall A ((R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} R \Rightarrow (R \circ S) \equiv (\mathbb{I}A))$ i

R, S, A ,! 1 (Prem) i

$$(R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} R$$

,! 2 (Prem) i

$$(R^*) \equiv S$$

,! 3 ($\&E$: 2) i

$$(R^D) \equiv A$$

,! 4 ($\&E$: 2) i

1 R ,! 5 ($\&E$: 2) i

x, y ,! 6 (Prem) i

$$((R \circ S)[x, y] \Leftrightarrow \exists z (R[x, z] \ \& \ S[z, y]))$$

,! 7 ($\forall E$: C10.2) i

$$(R \circ S)[x, y] \Leftrightarrow \exists z (R[x, z] \ \& \ S[z, y])$$

,! 8 ($()E$: 7) i

$((\mathbb{I}A)[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{x} = \mathbf{y} \ \& \ A[\mathbf{x}])$,! 9 ($\forall E$: P17) ;
 $(\mathbb{I}A)[\mathbf{x},\mathbf{y}] \Leftrightarrow \mathbf{x} = \mathbf{y} \ \& \ A[\mathbf{x}]$,! 10 ($(\)E$: 9) ;
 $(\mathbf{R} \circ \mathbf{S})[\mathbf{x},\mathbf{y}]$,! 11 (Prem) ;
 $(\mathbf{R} \circ \mathbf{S})[\mathbf{x},\mathbf{y}] \Rightarrow \exists z(\mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{S}[z,\mathbf{y}])$
,! 12 ($\Leftrightarrow E$: 8) ;
 $\exists z(\mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{S}[z,\mathbf{y}])$,! 13 ($\Rightarrow E$: 11,12) ;
 $(\mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{S}[z,\mathbf{y}])$,! 14 ($\exists E$: 13) ;
 $\mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{S}[z,\mathbf{y}]$,! 15 ($(\)E$: 14) ;
 $\mathbf{R}[\mathbf{x},z]$,! 16 ($\&E$: 15) ;
 $\mathbf{S}[z,\mathbf{y}]$,! 17 ($\&E$: 15) ;
 $(\mathbf{R}^*) \equiv \mathbf{S} \ \& \ \mathbf{S}[z,\mathbf{y}]$,! 18 ($\&I$: 3,17) ;
 $((\mathbf{R}^*) \equiv \mathbf{S} \ \& \ \mathbf{S}[z,\mathbf{y}] \Rightarrow \mathbf{R}[\mathbf{y},z])$,! 19 ($\forall E$: C3.12) ;
 $(\mathbf{R}^*) \equiv \mathbf{S} \ \& \ \mathbf{S}[z,\mathbf{y}] \Rightarrow \mathbf{R}[\mathbf{y},z]$,! 20 ($(\)E$: 19) ;
 $\mathbf{R}[\mathbf{y},z]$,! 21 ($\Rightarrow E$: 18,20) ;
 $\mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{R}[\mathbf{y},z]$,! 22 ($\&I$: 16,21) ;
 $\mathbf{1} \ \mathbf{R} \ \& \ \mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{R}[\mathbf{y},z]$,! 23 ($\&I$: 5,22) ;
 $(\mathbf{1} \ \mathbf{R} \ \& \ \mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{R}[\mathbf{y},z] \Rightarrow \mathbf{x} = \mathbf{y})$
,! 24 ($\forall E$: C9.2) ;
 $\mathbf{1} \ \mathbf{R} \ \& \ \mathbf{R}[\mathbf{x},z] \ \& \ \mathbf{R}[\mathbf{y},z] \Rightarrow \mathbf{x} = \mathbf{y}$,! 25 ($(\)E$: 24) ;
 $\mathbf{x} = \mathbf{y}$,! 26 ($\Rightarrow E$: 23,25) ;
 $\mathbf{R}[\mathbf{x},z] \ \& \ (\mathbf{R}^D) \equiv \mathbf{A}$,! 27 ($\&I$: 4,16) ;
 $(\mathbf{R}[\mathbf{x},z] \ \& \ (\mathbf{R}^D) \equiv \mathbf{A} \Rightarrow \mathbf{A}[\mathbf{x}])$,! 28 ($\forall E$: C5.7) ;
 $\mathbf{R}[\mathbf{x},z] \ \& \ (\mathbf{R}^D) \equiv \mathbf{A} \Rightarrow \mathbf{A}[\mathbf{x}]$,! 29 ($(\)E$: 28) ;
 $\mathbf{A}[\mathbf{x}]$,! 30 ($\Rightarrow E$: 27,29) ;
 $\mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}]$,! 31 ($\&I$: 26,30) ;
 $\mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}] \Rightarrow (\mathbb{I}A)[\mathbf{x},\mathbf{y}]$,! 32 ($\Leftrightarrow E$: 10) ;
 $(\mathbb{I}A)[\mathbf{x},\mathbf{y}]$,! 33 ($\Rightarrow E$: 31,32) ;
 $(\mathbf{R} \circ \mathbf{S})[\mathbf{x},\mathbf{y}] \Rightarrow (\mathbb{I}A)[\mathbf{x},\mathbf{y}]$,! 34 ($\Rightarrow I$: 11,33) ;

$(\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}]$,!	35 (Prem)	i
$(\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}]$,!	36 (\Leftrightarrow E: 10)	i
$\mathbf{x} = \mathbf{y} \ \& \ \mathbf{A}[\mathbf{x}]$,!	37 (\Rightarrow E: 35,36)	i
$\mathbf{x} = \mathbf{y}$,!	38 ($\&$ E: 37)	i
$\mathbf{A}[\mathbf{x}]$,!	39 ($\&$ E: 37)	i
$\mathbf{A}[\mathbf{x}] \ \& \ (\mathbf{R}^{\mathbf{D}}) \equiv \mathbf{A}$,!	40 ($\&$ I: 4,39)	i
$(\mathbf{A}[\mathbf{x}] \ \& \ (\mathbf{R}^{\mathbf{D}}) \equiv \mathbf{A} \Rightarrow \exists \mathbf{y} \ \mathbf{R}[\mathbf{x}, \mathbf{y}])$,!	41 (\forall E: C5.9)	i
$\mathbf{A}[\mathbf{x}] \ \& \ (\mathbf{R}^{\mathbf{D}}) \equiv \mathbf{A} \Rightarrow \exists \mathbf{y} \ \mathbf{R}[\mathbf{x}, \mathbf{y}]$,!	42 ($(\)$ E: 41)	i
$\exists \mathbf{y} \ \mathbf{R}[\mathbf{x}, \mathbf{y}]$,!	43 (\Rightarrow E: 40,42)	i
$\mathbf{R}[\mathbf{x}, \mathbf{z}]$,!	44 (\exists E: 43)	i
$(\mathbf{R}^*) \equiv \mathbf{S} \ \& \ \mathbf{R}[\mathbf{x}, \mathbf{z}]$,!	45 ($\&$ I: 3,44)	i
$(\mathbf{R}^*) \equiv \mathbf{S} \ \& \ \mathbf{R}[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{S}[\mathbf{z}, \mathbf{x}])$,!	46 (\forall E: C3.10)	i
$(\mathbf{R}^*) \equiv \mathbf{S} \ \& \ \mathbf{R}[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{S}[\mathbf{z}, \mathbf{x}]$,!	47 ($(\)$ E: 46)	i
$\mathbf{S}[\mathbf{z}, \mathbf{x}]$,!	48 (\Rightarrow E: 45,47)	i
$\mathbf{S}[\mathbf{z}, \mathbf{y}]$,!	49 ($=$ E: 38,48)	i
$\mathbf{R}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{S}[\mathbf{z}, \mathbf{y}]$,!	50 ($\&$ I: 44,49)	i
$(\mathbf{R}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{S}[\mathbf{z}, \mathbf{y}])$,!	51 ($(\)$ I: 50)	i
$\exists \mathbf{z}(\mathbf{R}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{S}[\mathbf{z}, \mathbf{y}])$,!	52 (\exists I: 51)	i
$\exists \mathbf{z}(\mathbf{R}[\mathbf{x}, \mathbf{z}] \ \& \ \mathbf{S}[\mathbf{z}, \mathbf{y}]) \Rightarrow (\mathbf{R} \circ \mathbf{S})[\mathbf{x}, \mathbf{y}]$,!	53 (\Leftrightarrow E: 8)	i
$(\mathbf{R} \circ \mathbf{S})[\mathbf{x}, \mathbf{y}]$,!	54 (\Rightarrow E: 52,53)	i
$(\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{R} \circ \mathbf{S})[\mathbf{x}, \mathbf{y}]$,!	55 (\Rightarrow I: 35,54)	i
$(\mathbf{R} \circ \mathbf{S})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}]$,!	56 (\Leftrightarrow I: 34,55)	i
$(\mathbf{R} \circ \mathbf{S})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}])$,!	57 ($(\)$ I: 56)	i
$\forall \mathbf{x} \forall \mathbf{y} (\mathbf{R} \circ \mathbf{S})[\mathbf{x}, \mathbf{y}] \Leftrightarrow (\mathbb{I}\mathbf{A})[\mathbf{x}, \mathbf{y}])$,!	58 (\forall I: 6,57)	i
$(\mathbf{R} \circ \mathbf{S}) \equiv (\mathbb{I}\mathbf{A})$,!	59 (\S I: C1.5,58)	i
$(\mathbf{R}^*) \equiv \mathbf{S} \ \& \ (\mathbf{R}^{\mathbf{D}}) \equiv \mathbf{A} \ \& \ \mathbf{1} \ \mathbf{R} \Rightarrow (\mathbf{R} \circ \mathbf{S}) \equiv (\mathbb{I}\mathbf{A})$,!	60 (\Rightarrow I: 2,59)	i

$((R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A))$
, ! 61 ((I: 60) i

$\forall R \forall S \forall A ((R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A))$
! 62 (\forall I: 1,61) i

□

! 30. i

$\vdash \forall R \forall S \forall A (R \equiv (S^*) \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A))$ i

R, S, A , ! 1 (Prem) i

$R \equiv (S^*) \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R$, ! 2 (Prem) i

$R \equiv (S^*)$, ! 3 (&E: 2) i

$(R^D) \equiv A \ \& \ \mathbf{1} \ R$, ! 4 (&E: 2) i

$(R \equiv (S^*) \Rightarrow (R^*) \equiv S)$, ! 5 (\forall E: C3.26) i

$R \equiv (S^*) \Rightarrow (R^*) \equiv S$, ! 6 ((E: 5) i

$(R^*) \equiv S$, ! 7 (\Rightarrow E: 3,6) i

$(R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R$, ! 8 (&I: 4,7) i

$((R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A))$
, ! 9 (\forall E: P29) i

$(R^*) \equiv S \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A)$
, ! 10 ((E: 9) i

$(R \circ S) \equiv (\mathbb{I}A)$, ! 11 (\Rightarrow E: 8,10) i

$R \equiv (S^*) \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A)$
, ! 12 (\Rightarrow I: 2,11) i

$(R \equiv (S^*) \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A))$
, ! 13 ((I: 12) i

$\forall R \forall S \forall A (R \equiv (S^*) \ \& \ (R^D) \equiv A \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (\mathbb{I}A))$
! 14 (\forall I: 1,13) i

□

! 31. i

$\vdash \forall R \forall S \forall A ((R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ \mathbf{f} \ R \Rightarrow (S \circ R) \equiv (\mathbb{I}A))$ i

R, S, A , ! 1 (Prem) i

$(\mathbf{R}^*) \equiv \mathbf{S} \ \& \ (\mathbf{R}^I) \equiv \mathbf{A} \ \& \ \mathbf{f} \ \mathbf{R}$,! 2 (Prem)	i
$(\mathbf{R}^*) \equiv \mathbf{S}$,! 3 (&E: 2)	i
$(\mathbf{R}^I) \equiv \mathbf{A}$,! 4 (&E: 2)	i
$\mathbf{f} \ \mathbf{R}$,! 5 (&E: 2)	i
$((\mathbf{R}^*) \equiv \mathbf{S} \Rightarrow \mathbf{S} \equiv (\mathbf{R}^*))$,! 6 (\forall E: C1.8)	i
$(\mathbf{R}^*) \equiv \mathbf{S} \Rightarrow \mathbf{S} \equiv (\mathbf{R}^*)$,! 7 (()E: 6)	i
$\mathbf{S} \equiv (\mathbf{R}^*)$,! 8 (\Rightarrow E: 3,7)	i
$((\mathbf{R}^*) \equiv \mathbf{S} \Rightarrow ((\mathbf{R}^*)^D) \equiv (\mathbf{S}^D))$,! 9 (\forall E: C5.15)	i
$(\mathbf{R}^*) \equiv \mathbf{S} \Rightarrow ((\mathbf{R}^*)^D) \equiv (\mathbf{S}^D)$,! 10 (()E: 9)	i
$((\mathbf{R}^*)^D) \equiv (\mathbf{S}^D)$,! 11 (\Rightarrow E: 3,10)	i
$((\mathbf{R}^*)^D) \equiv (\mathbf{S}^D) \ \& \ (\mathbf{R}^I) \equiv \mathbf{A}$,! 12 (&I: 4,11)	i
$(\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D)$,! 13 (\forall E: C6.15)	i
$((\mathbf{R}^*)^D) \equiv (\mathbf{S}^D) \ \& \ (\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D) \ \& \ (\mathbf{R}^I) \equiv \mathbf{A}$,! 14 (&I: 12,13)	i
$(((\mathbf{R}^*)^D) \equiv (\mathbf{S}^D) \ \& \ (\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D) \ \& \ (\mathbf{R}^I) \equiv \mathbf{A} \Rightarrow (\mathbf{S}^D) \equiv \mathbf{A})$,! 15 (\forall E: III1.27)	i
$((\mathbf{R}^*)^D) \equiv (\mathbf{S}^D) \ \& \ (\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D) \ \& \ (\mathbf{R}^I) \equiv \mathbf{A} \Rightarrow (\mathbf{S}^D) \equiv \mathbf{A}$,! 16 (()E: 15)	i
$(\mathbf{S}^D) \equiv \mathbf{A}$,! 17 (\Rightarrow E: 14,16)	i
$\mathbf{S} \equiv (\mathbf{R}^*) \ \& \ (\mathbf{S}^D) \equiv \mathbf{A}$,! 18 (&I: 8,17)	i
$\mathbf{f} \ \mathbf{R} \ \& \ (\mathbf{R}^*) \equiv \mathbf{S}$,! 19 (&I: 3,5)	i
$(\mathbf{f} \ \mathbf{R} \ \& \ (\mathbf{R}^*) \equiv \mathbf{S} \Rightarrow \mathbf{1} \ \mathbf{S})$,! 20 (\forall E: C9.15)	i
$\mathbf{f} \ \mathbf{R} \ \& \ (\mathbf{R}^*) \equiv \mathbf{S} \Rightarrow \mathbf{1} \ \mathbf{S}$,! 21 (()E: 20)	i
$\mathbf{1} \ \mathbf{S}$,! 22 (\Rightarrow E: 19,21)	i
$\mathbf{S} \equiv (\mathbf{R}^*) \ \& \ (\mathbf{S}^D) \equiv \mathbf{A} \ \& \ \mathbf{1} \ \mathbf{S}$,! 23 (&I: 18,22)	i
$(\mathbf{S} \equiv (\mathbf{R}^*) \ \& \ (\mathbf{S}^D) \equiv \mathbf{A} \ \& \ \mathbf{1} \ \mathbf{S} \Rightarrow (\mathbf{S} \circ \mathbf{R}) \equiv (\mathbf{IA}))$,! 24 (\forall E: P30)	i
$\mathbf{S} \equiv (\mathbf{R}^*) \ \& \ (\mathbf{S}^D) \equiv \mathbf{A} \ \& \ \mathbf{1} \ \mathbf{S} \Rightarrow (\mathbf{S} \circ \mathbf{R}) \equiv (\mathbf{IA})$		

	,! 25 (())E: 24)	i
$(S \circ R) \equiv (IA)$,! 26 (\Rightarrow E: 23,25)	i
$(R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA)$		
	,! 27 (\Rightarrow I: 2,26)	i
$((R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA))$		
	,! 28 (())I: 27)	i
$\forall R \forall S \forall A ((R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA))$		
	! 29 (\forall I: 1,28)	i
\square		
! 32.		i
$\vdash \forall R \forall S \forall A (R \equiv (S^*) \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA))$		i
R, S, A	,! 1 (Prem)	i
$R \equiv (S^*) \ \& \ (R^I) \equiv A \ \& \ f R$,! 2 (Prem)	i
$R \equiv (S^*)$,! 3 (&E: 2)	i
$(R^I) \equiv A \ \& \ f R$,! 4 (&E: 2)	i
$(R \equiv (S^*) \Rightarrow (R^*) \equiv S)$,! 5 (\forall E: C3.26)	i
$R \equiv (S^*) \Rightarrow (R^*) \equiv S$,! 6 (())E: 5)	i
$(R^*) \equiv S$,! 7 (\Rightarrow E: 3,6)	i
$(R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ f R$,! 8 (&I: 4,7)	i
$((R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA))$		
	,! 9 (\forall E: P31)	i
$(R^*) \equiv S \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA)$		
	,! 10 (())E: 9)	i
$(S \circ R) \equiv (IA)$,! 11 (\Rightarrow E: 8,10)	i
$R \equiv (S^*) \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA)$		
	,! 12 (\Rightarrow I: 2,11)	i
$(R \equiv (S^*) \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA))$		
	,! 13 (())I: 12)	i
$\forall R \forall S \forall A (R \equiv (S^*) \ \& \ (R^I) \equiv A \ \& \ f R \Rightarrow (S \circ R) \equiv (IA))$		
	! 14 (\forall I: 1,13)	i

□

! 33.

$\vdash \forall R \forall S \forall T (R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R \Rightarrow ((R \circ S) \circ T) \equiv T)$

R, S, T ,! 1 (Prem) i

$R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R$,! 2 (Prem) i

$R \equiv (S^*)$,! 3 (&E: 2) i

$(T^D) \subseteq (R^D)$,! 4 (&E: 2) i

$\mathbf{1} \ R$,! 5 (&E: 2) i

$R \equiv (S^*) \ \& \ \mathbf{1} \ R$,! 6 (&I: 3,5) i

$(R^D) \equiv (R^D)$,! 7 (\forall E: III.9) i

$R \equiv (S^*) \ \& \ (R^D) \equiv (R^D) \ \& \ \mathbf{1} \ R$,! 8 (&I: 6,7) i

$(R \equiv (S^*) \ \& \ (R^D) \equiv (R^D) \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (I(R^D)))$
,! 9 (\forall E: P30) i

$R \equiv (S^*) \ \& \ (R^D) \equiv (R^D) \ \& \ \mathbf{1} \ R \Rightarrow (R \circ S) \equiv (I(R^D))$
,! 10 ((E: 9) i

$(R \circ S) \equiv (I(R^D))$,! 11 (\Rightarrow E: 8,10) i

$((R \circ S) \equiv (I(R^D)) \Rightarrow ((R \circ S) \circ T) \equiv ((I(R^D)) \circ T))$
,! 12 (\forall E: C10.9) i

$(R \circ S) \equiv (I(R^D)) \Rightarrow ((R \circ S) \circ T) \equiv ((I(R^D)) \circ T)$
,! 13 ((E: 12) i

$((R \circ S) \circ T) \equiv ((I(R^D)) \circ T)$,! 14 (\Rightarrow E: 11,13) i

$((T^D) \subseteq (R^D) \Rightarrow ((I(R^D)) \circ T) \equiv T)$,! 15 (\forall E: P28) i

$(T^D) \subseteq (R^D) \Rightarrow ((I(R^D)) \circ T) \equiv T$,! 16 ((E: 15) i

$((I(R^D)) \circ T) \equiv T$,! 17 (\Rightarrow E: 4,16) i

$((R \circ S) \circ T) \equiv ((I(R^D)) \circ T) \ \& \ ((I(R^D)) \circ T) \equiv T$
,! 18 (&I: 14,17) i

$(((R \circ S) \circ T) \equiv ((I(R^D)) \circ T) \ \& \ ((I(R^D)) \circ T) \equiv T$
 $\Rightarrow ((R \circ S) \circ T) \equiv T)$
,! 19 (\forall E: C1.15) i

$$((R \circ S) \circ T) \equiv ((\mathbb{I}(R^D)) \circ T) \ \& \ ((\mathbb{I}(R^D)) \circ T) \equiv T$$

$$\Rightarrow ((R \circ S) \circ T) \equiv T$$

,! 20 (()E: 19) i

$$((R \circ S) \circ T) \equiv T$$

,! 21 (\Rightarrow E: 18,20) i

$$R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R \Rightarrow ((R \circ S) \circ T) \equiv T$$

,! 22 (\Rightarrow I: 2,21) i

$$(R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R \Rightarrow ((R \circ S) \circ T) \equiv T)$$

,! 23 (()I: 22) i

$$\forall R \forall S \forall T (R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R \Rightarrow ((R \circ S) \circ T) \equiv T)$$

! 24 (\forall I: 1,23) i

□

! 34. i

$$\vdash \forall R \forall S \forall T (T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} \ T \Rightarrow (R \circ (S \circ T)) \equiv R)$$

i

$$R, S, T$$

,! 1 (Prem) i

$$T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} \ T$$

,! 2 (Prem) i

$$T \equiv (S^*)$$

,! 3 (&E: 2) i

$$(R^I) \subseteq (T^I)$$

,! 4 (&E: 2) i

$$\mathbf{f} \ T$$

,! 5 (&E: 2) i

$$((R \circ (S \circ T))^*) \equiv (((S \circ T)^*) \circ (R^*))$$

,! 6 (\forall E: C10.11) i

$$((S \circ T)^*) \equiv ((T^*) \circ (S^*))$$

,! 7 (\forall E: C10.11) i

$$(((S \circ T)^*) \equiv ((T^*) \circ (S^*)))$$

$$\Rightarrow (((S \circ T)^*) \circ (R^*)) \equiv (((T^*) \circ (S^*)) \circ (R^*))$$

,! 8 (\forall E: C10.9) i

$$((S \circ T)^*) \equiv ((T^*) \circ (S^*))$$

$$\Rightarrow (((S \circ T)^*) \circ (R^*)) \equiv (((T^*) \circ (S^*)) \circ (R^*))$$

,! 9 (()E: 8) i

$$(((S \circ T)^*) \circ (R^*)) \equiv (((T^*) \circ (S^*)) \circ (R^*))$$

,! 10 (\Rightarrow E: 7,9) i

$$((R \circ (S \circ T))^*) \equiv (((S \circ T)^*) \circ (R^*))$$

$$\ \& \ (((S \circ T)^*) \circ (R^*)) \equiv (((T^*) \circ (S^*)) \circ (R^*))$$

,! 11 (&I: 6,10) i

$(\mathbf{T} \equiv (\mathbf{S}^*) \Rightarrow (\mathbf{T}^*) \equiv ((\mathbf{S}^*)^*))$,! 12 ($\forall E$: C3.20) ;
 $\mathbf{T} \equiv (\mathbf{S}^*) \Rightarrow (\mathbf{T}^*) \equiv ((\mathbf{S}^*)^*)$,! 13 ($()E$: 12) ;
 $(\mathbf{T}^*) \equiv ((\mathbf{S}^*)^*)$,! 14 ($\Rightarrow E$: 3,13) ;
 $(\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D)$,! 15 ($\forall E$: C6.15) ;
 $(\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D) \ \& \ (\mathbf{R}^I) \subseteq (\mathbf{T}^I)$,! 16 ($\&I$: 4,15) ;
 $(\mathbf{T}^I) \equiv ((\mathbf{T}^*)^D)$,! 17 ($\forall E$: C6.15) ;
 $(\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D) \ \& \ (\mathbf{T}^I) \equiv ((\mathbf{T}^*)^D) \ \& \ (\mathbf{R}^I) \subseteq (\mathbf{T}^I)$
,! 18 ($\&I$: 16,17) ;
 $((\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D) \ \& \ (\mathbf{T}^I) \equiv ((\mathbf{T}^*)^D) \ \& \ (\mathbf{R}^I) \subseteq (\mathbf{T}^I))$
 $\Rightarrow ((\mathbf{R}^*)^D) \subseteq ((\mathbf{T}^*)^D))$
,! 19 ($\forall E$: II1.33) ;
 $(\mathbf{R}^I) \equiv ((\mathbf{R}^*)^D) \ \& \ (\mathbf{T}^I) \equiv ((\mathbf{T}^*)^D) \ \& \ (\mathbf{R}^I) \subseteq (\mathbf{T}^I)$
 $\Rightarrow ((\mathbf{R}^*)^D) \subseteq ((\mathbf{T}^*)^D)$
,! 20 ($()E$: 19) ;
 $((\mathbf{R}^*)^D) \subseteq ((\mathbf{T}^*)^D)$,! 21 ($\Rightarrow E$: 18,20) ;
 $(\mathbf{T}^*) \equiv ((\mathbf{S}^*)^*) \ \& \ ((\mathbf{R}^*)^D) \subseteq ((\mathbf{T}^*)^D)$,! 22 ($\&I$: 14,21) ;
 $(\mathbf{f} \mathbf{T} \Rightarrow \mathbf{1} (\mathbf{T}^*))$,! 23 ($\forall E$: C9.7) ;
 $\mathbf{f} \mathbf{T} \Rightarrow \mathbf{1} (\mathbf{T}^*)$,! 24 ($()E$: 23) ;
 $\mathbf{1} (\mathbf{T}^*)$,! 25 ($\Rightarrow E$: 5,24) ;
 $(\mathbf{T}^*) \equiv ((\mathbf{S}^*)^*) \ \& \ ((\mathbf{R}^*)^D) \subseteq ((\mathbf{T}^*)^D) \ \& \ \mathbf{1} (\mathbf{T}^*)$
,! 26 ($\&I$: 22,25) ;
 $((\mathbf{T}^*) \equiv ((\mathbf{S}^*)^*) \ \& \ ((\mathbf{R}^*)^D) \subseteq ((\mathbf{T}^*)^D) \ \& \ \mathbf{1} (\mathbf{T}^*))$
 $\Rightarrow (((\mathbf{T}^*) \circ (\mathbf{S}^*)) \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*))$
,! 27 ($\forall E$: P33) ;
 $(\mathbf{T}^*) \equiv ((\mathbf{S}^*)^*) \ \& \ ((\mathbf{R}^*)^D) \subseteq ((\mathbf{T}^*)^D) \ \& \ \mathbf{1} (\mathbf{T}^*)$
 $\Rightarrow (((\mathbf{T}^*) \circ (\mathbf{S}^*)) \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*)$
,! 28 ($()E$: 27) ;
 $(((\mathbf{T}^*) \circ (\mathbf{S}^*)) \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*)$,! 29 ($\Rightarrow E$: 26,28) ;
 $((\mathbf{R} \circ (\mathbf{S} \circ \mathbf{T}))^*) \equiv (((\mathbf{S} \circ \mathbf{T})^*) \circ (\mathbf{R}^*))$
 $\& \ (((\mathbf{S} \circ \mathbf{T})^*) \circ (\mathbf{R}^*)) \equiv (((\mathbf{T}^*) \circ (\mathbf{S}^*)) \circ (\mathbf{R}^*))$
 $\& \ (((\mathbf{T}^*) \circ (\mathbf{S}^*)) \circ (\mathbf{R}^*)) \equiv (\mathbf{R}^*)$
,! 30 ($\&I$: 11,29) ;

$$\begin{aligned}
& (((R \circ (S \circ T))^*) \equiv ((S \circ T)^* \circ (R^*))) \\
& \& \left(((S \circ T)^* \circ (R^*)) \equiv ((T^* \circ (S^*)) \circ (R^*)) \right) \\
& \& \left(((T^* \circ (S^*)) \circ (R^*)) \equiv (R^*) \right) \\
& \Rightarrow ((R \circ (S \circ T))^*) \equiv (R^*)
\end{aligned}$$

,! 31 ($\forall E$: C1.19) i

$$\begin{aligned}
& ((R \circ (S \circ T))^*) \equiv ((S \circ T)^* \circ (R^*)) \\
& \& \left(((S \circ T)^* \circ (R^*)) \equiv ((T^* \circ (S^*)) \circ (R^*)) \right) \\
& \& \left(((T^* \circ (S^*)) \circ (R^*)) \equiv (R^*) \right) \\
& \Rightarrow ((R \circ (S \circ T))^*) \equiv (R^*)
\end{aligned}$$

,! 32 ($()E$: 31) i

$$((R \circ (S \circ T))^*) \equiv (R^*) \quad ,! 33 (\Rightarrow E: 30,32) i$$

$$\left(((R \circ (S \circ T))^*) \equiv (R^*) \Rightarrow (R \circ (S \circ T)) \equiv R \right)$$

,! 34 ($\forall E$: C3.21) i

$$((R \circ (S \circ T))^*) \equiv (R^*) \Rightarrow (R \circ (S \circ T)) \equiv R$$

,! 35 ($()E$: 34) i

$$(R \circ (S \circ T)) \equiv R \quad ,! 36 (\Rightarrow E: 33,35) i$$

$$T \equiv (S^*) \& (R^I) \subseteq (T^I) \& \mathbf{f} T \Rightarrow (R \circ (S \circ T)) \equiv R$$

,! 37 ($\Rightarrow I$: 2,36) i

$$\left(T \equiv (S^*) \& (R^I) \subseteq (T^I) \& \mathbf{f} T \Rightarrow (R \circ (S \circ T)) \equiv R \right)$$

,! 38 ($()I$: 37) i

$$\forall R \forall S \forall T \left(T \equiv (S^*) \& (R^I) \subseteq (T^I) \& \mathbf{f} T \Rightarrow (R \circ (S \circ T)) \equiv R \right)$$

,! 39 ($\forall I$: 1,38) i

□

! 35. i

$$\vdash \forall R \forall S \forall T \left(R \equiv (S^*) \& (T^D) \subseteq (R^D) \& \mathbf{1} R \Rightarrow (R \circ (S \circ T)) \equiv T \right)$$

i

$$R, S, T \quad ,! 1 (\text{Prem}) \quad i$$

$$R \equiv (S^*) \& (T^D) \subseteq (R^D) \& \mathbf{1} R \quad ,! 2 (\text{Prem}) \quad i$$

$$\left(R \equiv (S^*) \& (T^D) \subseteq (R^D) \& \mathbf{1} R \Rightarrow ((R \circ S) \circ T) \equiv T \right)$$

,! 3 ($\forall E$: P33) i

$$R \equiv (S^*) \& (T^D) \subseteq (R^D) \& \mathbf{1} R \Rightarrow ((R \circ S) \circ T) \equiv T$$

,! 4 ($()E$: 3) i

$$((R \circ S) \circ T) \equiv T \quad ,! 5 (\Rightarrow E: 2,4) \quad i$$

$(R \circ (S \circ T)) \equiv ((R \circ S) \circ T)$,! 6 ($\forall E$: C10.10) ;
 $(R \circ (S \circ T)) \equiv ((R \circ S) \circ T) \ \& \ ((R \circ S) \circ T) \equiv T$,! 7 ($\&I$: 5,6) ;
 $((R \circ (S \circ T)) \equiv ((R \circ S) \circ T) \ \& \ ((R \circ S) \circ T) \equiv T$
 $\Rightarrow (R \circ (S \circ T)) \equiv T)$,! 8 ($\forall E$: C1.15) ;
 $(R \circ (S \circ T)) \equiv ((R \circ S) \circ T) \ \& \ ((R \circ S) \circ T) \equiv T$
 $\Rightarrow (R \circ (S \circ T)) \equiv T$,! 9 ($()E$: 8) ;
 $(R \circ (S \circ T)) \equiv T$,! 10 ($\Rightarrow E$: 7,9) ;
 $R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R \Rightarrow (R \circ (S \circ T)) \equiv T$,! 11 ($\Rightarrow I$: 2,10) ;
 $(R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R \Rightarrow (R \circ (S \circ T)) \equiv T)$,! 12 ($()I$: 11) ;
 $\forall R \forall S \forall T (R \equiv (S^*) \ \& \ (T^D) \subseteq (R^D) \ \& \ \mathbf{1} \ R \Rightarrow (R \circ (S \circ T)) \equiv T)$! 13 ($\forall I$: 1,12) ;
 \square
! 36. ;
 $\vdash \forall R \forall S \forall T (T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} \ T \Rightarrow ((R \circ S) \circ T) \equiv R)$;
 R, S, T ,! 1 (Prem) ;
 $T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} \ T$,! 2 (Prem) ;
 $(T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} \ T \Rightarrow (R \circ (S \circ T)) \equiv R)$,! 3 ($\forall E$: P34) ;
 $T \equiv (S^*) \ \& \ (R^I) \subseteq (T^I) \ \& \ \mathbf{f} \ T \Rightarrow (R \circ (S \circ T)) \equiv R$,! 4 ($()E$: 3) ;
 $(R \circ (S \circ T)) \equiv R$,! 5 ($\Rightarrow E$: 2,4) ;
 $(R \circ (S \circ T)) \equiv ((R \circ S) \circ T)$,! 6 ($\forall E$: C10.10) ;
 $(R \circ (S \circ T)) \equiv ((R \circ S) \circ T) \ \& \ (R \circ (S \circ T)) \equiv R$,! 7 ($\&I$: 5,6) ;
 $((R \circ (S \circ T)) \equiv ((R \circ S) \circ T) \ \& \ (R \circ (S \circ T)) \equiv R$
 $\Rightarrow ((R \circ S) \circ T) \equiv R)$

,! 8 ($\forall E$: C1.18) i

$$\begin{aligned} & (\mathbf{R} \circ (\mathbf{S} \circ \mathbf{T})) \equiv ((\mathbf{R} \circ \mathbf{S}) \circ \mathbf{T}) \ \& \ (\mathbf{R} \circ (\mathbf{S} \circ \mathbf{T})) \equiv \mathbf{R} \\ \Rightarrow & ((\mathbf{R} \circ \mathbf{S}) \circ \mathbf{T}) \equiv \mathbf{R} \end{aligned}$$

,! 9 ($()E$: 8) i

$$((\mathbf{R} \circ \mathbf{S}) \circ \mathbf{T}) \equiv \mathbf{R}$$

,! 10 ($\Rightarrow E$: 7,9) i

$$\mathbf{T} \equiv (\mathbf{S}^*) \ \& \ (\mathbf{R}^I) \subseteq (\mathbf{T}^I) \ \& \ \mathbf{f} \ \mathbf{T} \Rightarrow ((\mathbf{R} \circ \mathbf{S}) \circ \mathbf{T}) \equiv \mathbf{R}$$

,! 11 ($\Rightarrow I$: 2,10) i

$$(\mathbf{T} \equiv (\mathbf{S}^*) \ \& \ (\mathbf{R}^I) \subseteq (\mathbf{T}^I) \ \& \ \mathbf{f} \ \mathbf{T} \Rightarrow ((\mathbf{R} \circ \mathbf{S}) \circ \mathbf{T}) \equiv \mathbf{R})$$

,! 12 ($()I$: 11) i

$$\forall \mathbf{R} \forall \mathbf{S} \forall \mathbf{T} (\mathbf{T} \equiv (\mathbf{S}^*) \ \& \ (\mathbf{R}^I) \subseteq (\mathbf{T}^I) \ \& \ \mathbf{f} \ \mathbf{T} \Rightarrow ((\mathbf{R} \circ \mathbf{S}) \circ \mathbf{T}) \equiv \mathbf{R})$$

! 13 ($\forall I$: 1,12) i

□