

! CHAPTER 13

CORRESPONDENCES ;

! This chapter introduces the notion of a one-to-one correspondence between two predicates. $A \sim B$ if and only if there is a one-to-one functional relationship with domain A and image B . The notion is important because, as will be shown, it is related to equinumerosity (having the same number). ;

! 1. \sim represents correspondence. ;

$\$ \sim ; A \sim B ; \exists R (R \Vdash A \ \& \ R \Vdash B)$;

! 2. Equivalent predicates correspond. ;

$\vdash \forall A \forall B (A \equiv B \Rightarrow A \sim B)$;

A, B ,! 1 (Prem) ;

$A \equiv B$,! 2 (Prem) ;

$(\Vdash A) \Vdash A$,! 3 ($\forall E$: C11.26) ;

$(\Vdash A) \Vdash A$,! 4 ($\forall E$: C11.27) ;

$(\Vdash A) \Vdash A \ \& \ A \equiv B$,! 5 ($\&I$: 3,4) ;

$((\Vdash A) \Vdash A \ \& \ A \equiv B \Rightarrow (\Vdash A) \Vdash B)$,! 6 ($\forall E$: C9.28) ;

$(\Vdash A) \Vdash A \ \& \ A \equiv B \Rightarrow (\Vdash A) \Vdash B$,! 7 ($(\)E$: 6) ;

$(\Vdash A) \Vdash B$,! 8 ($\Rightarrow E$: 5,7) ;

$(\Vdash A) \Vdash A \ \& \ (\Vdash A) \Vdash B$,! 9 ($\&I$: 3,8) ;

$((\Vdash A) \Vdash A \ \& \ (\Vdash A) \Vdash B)$,! 10 ($(\)I$: 9) ;

$\exists R (R \Vdash A \ \& \ R \Vdash B)$,! 11 ($\exists I$: 10) ;

$A \sim B$,! 12 ($\$I$: P1,11) ;

$A \equiv B \Rightarrow A \sim B$,! 13 ($\Rightarrow I$: 2,12) ;

$(A \equiv B \Rightarrow A \sim B)$,! 14 ($(\)I$: 13) ;

$\forall A \forall B (A \equiv B \Rightarrow A \sim B)$! 15 ($\forall I$: 1,14) ;

\square

! 3. Reflexivity of Correspondence. ;

$\vdash \forall A A \sim A$;

A ,! 1 (Prem) ;

$A \equiv A$,! 2 ($\forall E$: II1.9) ;

$(A \equiv A \Rightarrow A \sim A)$,! 3 ($\forall E$: P2)	i
$A \equiv A \Rightarrow A \sim A$,! 4 ($()E$: 3)	i
$A \sim A$,! 5 ($\Rightarrow E$: 2,4)	i
$\forall A A \sim A$! 6 ($\forall I$: 1,5)	i
\square		
! 4. Symmetry of Correspondence.		i
$\vdash \forall A \forall B (A \sim B \Rightarrow B \sim A)$		i
A, B	,! 1 (Prem)	i
$A \sim B$,! 2 (Prem)	i
$\exists R (R \Vdash A \ \& \ R \Vdash B)$,! 3 ($\exists E$: P1,2)	i
$(R \Vdash A \ \& \ R \Vdash B)$,! 4 ($\exists E$: 3)	i
$R \Vdash A \ \& \ R \Vdash B$,! 5 ($()E$: 4)	i
$R \Vdash A$,! 6 ($\&E$: 5)	i
$(R \Vdash A \Rightarrow (R^*) \Vdash A)$,! 7 ($\forall E$: C9.23)	i
$R \Vdash A \Rightarrow (R^*) \Vdash A$,! 8 ($()E$: 7)	i
$(R^*) \Vdash A$,! 9 ($\Rightarrow E$: 6,8)	i
$\square \square \square R \Vdash B$,! 10 ($\&E$: 5)	i
$(R \Vdash B \Rightarrow (R^*) \Vdash B)$,! 11 ($\forall E$: C9.20)	i
$R \Vdash B \Rightarrow (R^*) \Vdash B$,! 12 ($()E$: 11)	i
$(R^*) \Vdash B$,! 13 ($\Rightarrow E$: 10,12)	i
$(R^*) \Vdash B \ \& \ (R^*) \Vdash A$,! 14 ($\&I$: 9,13)	i
$((R^*) \Vdash B \ \& \ (R^*) \Vdash A)$,! 15 ($()I$: 14)	i
$\exists R (R \Vdash B \ \& \ R \Vdash A)$,! 16 ($\exists I$: 15)	i
$B \sim A$,! 17 ($\exists I$: P1,16)	i
$A \sim B \Rightarrow B \sim A$,! 18 ($\Rightarrow I$: 2,17)	i
$(A \sim B \Rightarrow B \sim A)$,! 19 ($()I$: 18)	i
$\forall A \forall B (A \sim B \Rightarrow B \sim A)$! 20 ($\forall I$: 1,19)	i
\square		

! 5. Transitivity of Correspondence.

$\vdash \forall A \forall B \forall C (A \sim B \ \& \ B \sim C \Rightarrow A \sim C)$

A, B, C

,! 1 (Prem)

A ~ B & B ~ C

,! 2 (Prem)

A ~ B

,! 3 (&E: 2)

$\exists R (R \ \mathbb{F} \ A \ \& \ R \ \mathbb{1} \ B)$

,! 4 (\exists E: P1,3)

(R \ \mathbb{F} \ A \ \& \ R \ \mathbb{1} \ B)

,! 5 (\exists E: 4)

R \ \mathbb{F} \ A \ \& \ R \ \mathbb{1} \ B

,! 6 (()E: 5)

B ~ C

,! 7 (&E: 2)

$\exists S (R \ \mathbb{F} \ B \ \& \ R \ \mathbb{1} \ C)$

,! 8 (\exists E: P1,7)

(S \ \mathbb{F} \ B \ \& \ S \ \mathbb{1} \ C)

,! 9 (\exists E: 8)

S \ \mathbb{F} \ B \ \& \ S \ \mathbb{1} \ C

,! 10 (()E: 9)

R \ \mathbb{F} \ A \ \& \ R \ \mathbb{1} \ B \ \& \ S \ \mathbb{F} \ B \ \& \ S \ \mathbb{1} \ C

,! 11 (&I: 6,10)

**(R \ \mathbb{F} \ A \ \& \ R \ \mathbb{1} \ B \ \& \ S \ \mathbb{F} \ B \ \& \ S \ \mathbb{1} \ C
 $\Rightarrow (R \circ S) \ \mathbb{F} \ A \ \& \ (R \circ S) \ \mathbb{1} \ C)$**

,! 12 (\forall E: C10.26)

R \ \mathbb{F} \ A \ \& \ R \ \mathbb{1} \ B \ \& \ S \ \mathbb{F} \ B \ \& \ S \ \mathbb{1} \ C $\Rightarrow (R \circ S) \ \mathbb{F} \ A \ \& \ (R \circ S) \ \mathbb{1} \ C$

,! 13 (()E: 12)

(R \ \circ \ S) \ \mathbb{F} \ A \ \& \ (R \ \circ \ S) \ \mathbb{1} \ C

,! 14 (\Rightarrow E: 11,13)

((R \ \circ \ S) \ \mathbb{F} \ A \ \& \ (R \ \circ \ S) \ \mathbb{1} \ C)

,! 15 (()I: 14)

$\exists R (R \ \mathbb{F} \ A \ \& \ R \ \mathbb{1} \ C)$

,! 16 (\exists I: 15)

A ~ C

,! 17 (\exists I: P1,16)

A ~ B & B ~ C \Rightarrow A ~ C

,! 18 (\Rightarrow I: 2,17)

(A ~ B & B ~ C \Rightarrow A ~ C)

,! 19 (()I: 18)

$\forall A \forall B \forall C (A \sim B \ \& \ B \sim C \Rightarrow A \sim C)$

! 20 (\forall I: 1,19)

\square

! 6. Applications of Symmetry to Transitivity, n1.

$\vdash \forall A \forall B \forall C (A \sim B \ \& \ B \sim C \Rightarrow C \sim A)$

A, B, C

,! 1 (Prem)

$A \sim B \ \& \ B \sim C$,! 2 (Prem)	i
$(A \sim B \ \& \ B \sim C \Rightarrow A \sim C)$,! 3 ($\forall E$: P5)	i
$A \sim B \ \& \ B \sim C \Rightarrow A \sim C$,! 4 ($()E$: 3)	i
$A \sim C$,! 5 ($\Rightarrow E$: 2,4)	i
$(A \sim C \Rightarrow C \sim A)$,! 6 ($\forall E$: P4)	i
$A \sim C \Rightarrow C \sim A$,! 7 ($()E$: 6)	i
$C \sim A$,! 8 ($\Rightarrow E$: 5,7)	i
$A \sim B \ \& \ B \sim C \Rightarrow C \sim A$,! 9 ($\Rightarrow I$: 2,8)	i
$(A \sim B \ \& \ B \sim C \Rightarrow C \sim A)$,! 10 ($()I$: 9)	i
$\forall A \forall B \forall C (A \sim B \ \& \ B \sim C \Rightarrow C \sim A)$! 11 ($\forall I$: 1,10)	i

□

! 7. Applications of Symmetry to Transitivity, n2. i

$\vdash \forall A \forall B \forall C (A \sim B \ \& \ C \sim B \Rightarrow A \sim C)$		i
A, B, C	,! 1 (Prem)	i
$A \sim B \ \& \ C \sim B$,! 2 (Prem)	i
$A \sim B$,! 3 ($\&E$: 2)	i
$C \sim B$,! 4 ($\&E$: 2)	i
$(C \sim B \Rightarrow B \sim C)$,! 5 ($\forall E$: P4)	i
$C \sim B \Rightarrow B \sim C$,! 6 ($()E$: 5)	i
$B \sim C$,! 7 ($\Rightarrow E$: 4,6)	i
$A \sim B \ \& \ B \sim C$,! 8 ($\&I$: 3,7)	i
$(A \sim B \ \& \ B \sim C \Rightarrow A \sim C)$,! 9 ($\forall E$: P5)	i
$A \sim B \ \& \ B \sim C \Rightarrow A \sim C$,! 10 ($()E$: 9)	i
$A \sim C$,! 11 ($\Rightarrow E$: 8,10)	i
$A \sim B \ \& \ C \sim B \Rightarrow A \sim C$,! 12 ($\Rightarrow I$: 2,11)	i
$(A \sim B \ \& \ C \sim B \Rightarrow A \sim C)$,! 13 ($()I$: 12)	i
$\forall A \forall B \forall C (A \sim B \ \& \ C \sim B \Rightarrow A \sim C)$! 14 ($\forall I$: 1,13)	i

□

! 8. Applications of Symmetry to Transitivity, n3. i

$\vdash \forall A \forall B \forall C (A \sim B \ \& \ A \sim C \Rightarrow B \sim C)$ i

A, B, C ,! 1 (Prem) i

A ~ B & A ~ C ,! 2 (Prem) i

A ~ C ,! 3 (&E: 2) i

A ~ B ,! 4 (&E: 2) i

(A ~ B \Rightarrow B ~ A) ,! 5 (\forall E: P4) i

A ~ B \Rightarrow B ~ A ,! 6 (()E: 5) i

B ~ A ,! 7 (\Rightarrow E: 4,6) i

B ~ A & A ~ C ,! 8 (&I: 3,7) i

(B ~ A & A ~ C \Rightarrow B ~ C) ,! 9 (\forall E: P5) i

B ~ A & A ~ C \Rightarrow B ~ C ,! 10 (()E: 9) i

B ~ C ,! 11 (\Rightarrow E: 8,10) i

A ~ B & A ~ C \Rightarrow B ~ C ,! 12 (\Rightarrow I: 2,11) i

(A ~ B & A ~ C \Rightarrow B ~ C) ,! 13 (()I: 12) i

$\forall A \forall B \forall C (A \sim B \ \& \ A \sim C \Rightarrow B \sim C)$! 14 (\forall I: 1,13) i

□

! P9 to P14 apply P2 to transitivity. i

! 9. i

$\vdash \forall A \forall B \forall C (A \sim B \ \& \ A \equiv C \Rightarrow C \sim B)$ i

A, B, C ,! 1 (Prem) i

A ~ B & A \equiv C ,! 2 (Prem) i

A ~ B ,! 3 (&E: 2) i

A \equiv C ,! 4 (&E: 2) i

(A \equiv C \Rightarrow A ~ C) ,! 5 (\forall E: P2) i

A \equiv C \Rightarrow A ~ C ,! 6 (()E: 5) i

A ~ C ,! 7 (\Rightarrow E: 4,6) i

A ~ C & A ~ B ,! 8 (&I: 3,7) i

(A ~ C & A ~ B \Rightarrow C ~ B) ,! 9 (\forall E: P8) i

$A \sim C \ \& \ A \sim B \Rightarrow C \sim B$,! 10 (()E: 9)	i
$C \sim B$,! 11 (\Rightarrow E: 8,10)	i
$A \sim B \ \& \ A \equiv C \Rightarrow C \sim B$,! 12 (\Rightarrow I: 2,11)	i
($A \sim B \ \& \ A \equiv C \Rightarrow C \sim B$)	,! 13 (()I: 12)	i
$\forall A \forall B \forall C (A \sim B \ \& \ A \equiv C \Rightarrow C \sim B)$! 14 (\forall I: 1,13)	i
\square		

! 10. i

$\vdash \forall A \forall B \forall C (A \sim B \ \& \ C \equiv A \Rightarrow C \sim B)$		i
A, B, C	,! 1 (Prem)	i
$A \sim B \ \& \ C \equiv A$,! 2 (Prem)	i
$A \sim B$,! 3 ($\&$ E: 2)	i
$C \equiv A$,! 4 ($\&$ E: 2)	i
($C \equiv A \Rightarrow C \sim A$)	,! 5 (\forall E: P2)	i
$C \equiv A \Rightarrow C \sim A$,! 6 (()E: 5)	i
$C \sim A$,! 7 (\Rightarrow E: 4,6)	i
$C \sim A \ \& \ A \sim B$,! 8 ($\&$ I: 3,7)	i
($C \sim A \ \& \ A \sim B \Rightarrow C \sim B$)	,! 9 (\forall E: P5)	i
$C \sim A \ \& \ A \sim B \Rightarrow C \sim B$,! 10 (()E: 9)	i
$C \sim B$,! 11 (\Rightarrow E: 8,10)	i
$A \sim B \ \& \ C \equiv A \Rightarrow C \sim B$,! 12 (\Rightarrow I: 2,11)	i
($A \sim B \ \& \ C \equiv A \Rightarrow C \sim B$)	,! 13 (()I: 12)	i
$\forall A \forall B \forall C (A \sim B \ \& \ C \equiv A \Rightarrow C \sim B)$! 14 (\forall I: 1,13)	i
\square		

! 11. i

$\vdash \forall A \forall B \forall C (A \sim B \ \& \ B \equiv C \Rightarrow A \sim C)$		i
A, B, C	,! 1 (Prem)	i
$A \sim B \ \& \ B \equiv C$,! 2 (Prem)	i
$A \sim B$,! 3 ($\&$ E: 2)	i

$B \equiv C$,! 4 (&E: 2)	i
$(B \equiv C \Rightarrow B \sim C)$,! 5 (\forall E: P2)	i
$B \equiv C \Rightarrow B \sim C$,! 6 (()E: 5)	i
$B \sim C$,! 7 (\Rightarrow E: 4,6)	i
$A \sim B \ \& \ B \sim C$,! 8 (&I: 3,7)	i
$(A \sim B \ \& \ B \sim C \Rightarrow A \sim C)$,! 9 (\forall E: P5)	i
$A \sim B \ \& \ B \sim C \Rightarrow A \sim C$,! 10 (()E: 9)	i
$A \sim C$,! 11 (\Rightarrow E: 8,10)	i
$A \sim B \ \& \ B \equiv C \Rightarrow A \sim C$,! 12 (\Rightarrow I: 11)	i
$(A \sim B \ \& \ B \equiv C \Rightarrow A \sim C)$,! 13 (()I: 12)	i
$\forall A \forall B \forall C (A \sim B \ \& \ B \equiv C \Rightarrow A \sim C)$! 14 (\forall I: 1,13)	i
\square		
! 12.		i
$\vdash \forall A \forall B \forall C (A \sim B \ \& \ C \equiv B \Rightarrow A \sim C)$		i
A, B, C	,! 1 (Prem)	i
$A \sim B \ \& \ C \equiv B$,! 2 (Prem)	i
$A \sim B$,! 3 (&E: 2)	i
$C \equiv B$,! 4 (&E: 2)	i
$(C \equiv B \Rightarrow C \sim B)$,! 5 (\forall E: P2)	i
$C \equiv B \Rightarrow C \sim B$,! 6 (()E: 5)	i
$C \sim B$,! 7 (\Rightarrow E: 4,6)	i
$A \sim B \ \& \ C \sim B$,! 8 (&I: 3,7)	i
$(A \sim B \ \& \ C \sim B \Rightarrow A \sim C)$,! 9 (\forall E: P7)	i
$A \sim B \ \& \ C \sim B \Rightarrow A \sim C$,! 10 (()E: 9)	i
$A \sim C$,! 11 (\Rightarrow E: 8,10)	i
$A \sim B \ \& \ C \equiv B \Rightarrow A \sim C$,! 12 (\Rightarrow I: 2,11)	i
$(A \sim B \ \& \ C \equiv B \Rightarrow A \sim C)$,! 13 (()I: 12)	i
$\forall A \forall B \forall C (A \sim B \ \& \ C \equiv B \Rightarrow A \sim C)$! 14 (\forall I: 1,13)	i

□

! 13.

$\vdash \forall A \forall B \forall C \forall D (A \sim B \ \& \ A \equiv C \ \& \ B \equiv D \Rightarrow C \sim D)$		i
A, B, C, D	,! 1 (Prem)	i
$A \sim B \ \& \ A \equiv C \ \& \ B \equiv D$,! 2 (Prem)	i
$A \sim B$,! 3 (&E: 2)	i
$A \equiv C$,! 4 (&E: 2)	i
$A \sim B \ \& \ A \equiv C$,! 5 (&I: 3,4)	i
$(A \sim B \ \& \ A \equiv C \Rightarrow C \sim B)$,! 6 (\forall E: P9)	i
$A \sim B \ \& \ A \equiv C \Rightarrow C \sim B$,! 7 (()E: 6)	i
$C \sim B$,! 8 (\Rightarrow E: 5,7)	i
$B \equiv D$,! 9 (&E: 2)	i
$C \sim B \ \& \ B \equiv D$,! 10 (&I: 8,9)	i
$(C \sim B \ \& \ B \equiv D \Rightarrow C \sim D)$,! 11 (\forall E: P11)	i
$C \sim B \ \& \ B \equiv D \Rightarrow C \sim D$,! 12 (()E: 11)	i
$C \sim D$,! 13 (\Rightarrow E: 10,12)	i
$A \sim B \ \& \ A \equiv C \ \& \ B \equiv D \Rightarrow C \sim D$,! 14 (\Rightarrow I: 2,13)	i
$(A \sim B \ \& \ A \equiv C \ \& \ B \equiv D \Rightarrow C \sim D)$,! 15 (()I: 14)	i
$\forall A \forall B \forall C \forall D (A \sim B \ \& \ A \equiv C \ \& \ B \equiv D \Rightarrow C \sim D)$! 16 (\forall I: 1,15)	i

□

! 14.

$\vdash \forall A \forall B \forall C \forall D (A \sim B \ \& \ C \equiv A \ \& \ D \equiv B \Rightarrow C \sim D)$		i
A, B, C, D	,! 1 (Prem)	i
$A \sim B \ \& \ C \equiv A \ \& \ D \equiv B$,! 2 (Prem)	i
$C \equiv A$,! 3 (&E: 2)	i
$(C \equiv A \Rightarrow A \equiv C)$,! 4 (\forall E: III1.10)	i
$C \equiv A \Rightarrow A \equiv C$,! 5 (()E: 4)	i

$A \equiv C$,! 6 (\Rightarrow E: 3,5)	i
$D \equiv B$,! 7 ($\&$ E: 2)	i
$(D \equiv B \Rightarrow B \equiv D)$,! 8 (\forall E: III1.10)	i
$D \equiv B \Rightarrow B \equiv D$,! 9 ($(\)$ E: 8)	i
$B \equiv D$,! 10 (\Rightarrow E: 7,9)	i
$A \equiv C \& B \equiv D$,! 11 ($\&$ I: 6,10)	i
$A \sim B$,! 12 ($\&$ E: 2)	i
$A \sim B \& A \equiv C \& B \equiv D$,! 13 ($\&$ I: 11,12)	i
$(A \sim B \& A \equiv C \& B \equiv D \Rightarrow C \sim D)$,! 14 (\forall E: P13)	i
$A \sim B \& A \equiv C \& B \equiv D \Rightarrow C \sim D$,! 15 ($(\)$ E: 14)	i
$C \sim D$,! 16 (\Rightarrow E: 13,15)	i
$A \sim B \& C \equiv A \& D \equiv B \Rightarrow C \sim D$,! 17 (\Rightarrow I: 2,16)	i
$(A \sim B \& C \equiv A \& D \equiv B \Rightarrow C \sim D)$,! 18 ($(\)$ I: 17)	i
$\forall A \forall B \forall C \forall D (A \sim B \& C \equiv A \& D \equiv B \Rightarrow C \sim D)$! 19 (\forall I: 1,18)	i

□

! 15. The unions of disjoint, corresponding predicates themselves correspond. i

$\vdash \forall A \forall B \forall C \forall D (A \sim B \& C \sim D \& (A \cap C) \equiv \phi \& (B \cap D) \equiv \phi$
 $\Rightarrow (A \cup C) \sim (B \cup D))$ i

A, B, C, D	,! 1 (Prem)	i
$A \sim B \& C \sim D \& (A \cap C) \equiv \phi \& (B \cap D) \equiv \phi$,! 2 (Prem)	i
$A \sim B$,! 3 ($\&$ E: 2)	i
$\exists R (R \Vdash A \& R \Vdash B)$,! 4 (\exists E: P1,3)	i
$(R \Vdash A \& R \Vdash B)$,! 5 (\exists E: 4)	i
$R \Vdash A \& R \Vdash B$,! 6 ($(\)$ E: 5)	i
$R \Vdash A$,! 7 ($\&$ E: 6)	i
$R \Vdash B$,! 8 ($\&$ E: 6)	i
$C \sim D$,! 9 ($\&$ E: 2)	i

$\exists R (R \text{ F } C \ \& \ R \text{ I } D)$,! 10 ($\exists E$: P1,9)	i
$(S \text{ F } C \ \& \ S \text{ I } D)$,! 11 ($\exists E$: 10)	i
$S \text{ F } C \ \& \ S \text{ I } D$,! 12 ($(\)E$: 11)	i
$S \text{ F } C$,! 13 ($\&E$: 12)	i
$S \text{ I } D$,! 14 ($\&E$: 12)	i
$R \text{ F } A \ \& \ S \text{ F } C$,! 15 ($\&I$: 7,13)	i
$(A \cap C) \equiv \phi$,! 16 ($\&E$: 2)	i
$R \text{ F } A \ \& \ S \text{ F } C \ \& \ (A \cap C) \equiv \phi$,! 17 ($\&I$: 15,16)	i
$(R \text{ F } A \ \& \ S \text{ F } C \ \& \ (A \cap C) \equiv \phi \Rightarrow (R \sqcup S) \text{ F } (A \cup C))$,! 18 ($\forall E$: C8.16)	i
$R \text{ F } A \ \& \ S \text{ F } C \ \& \ (A \cap C) \equiv \phi \Rightarrow (R \sqcup S) \text{ F } (A \cup C)$,! 19 ($(\)E$: 18)	i
$(R \sqcup S) \text{ F } (A \cup C)$,! 20 ($\Rightarrow E$: 17,19)	i
$R \text{ I } B \ \& \ S \text{ I } D$,! 21 ($\&I$: 8,14)	i
$B \cap D \equiv \phi$,! 22 ($\&E$: 2)	i
$R \text{ I } B \ \& \ S \text{ I } D \ \& \ (B \cap D) \equiv \phi$,! 23 ($\&I$: 21,22)	i
$(R \text{ I } B \ \& \ S \text{ I } D \ \& \ (B \cap D) \equiv \phi \Rightarrow (R \sqcup S) \text{ I } (B \cup D))$,! 24 ($\forall E$: C9.31)	i
$R \text{ I } B \ \& \ S \text{ I } D \ \& \ (B \cap D) \equiv \phi \Rightarrow (R \sqcup S) \text{ I } (B \cup D)$,! 25 ($(\)E$: 24)	i
$(R \sqcup S) \text{ I } (B \cup D)$,! 26 ($\Rightarrow E$: 23,25)	i
$(R \sqcup S) \text{ F } (A \cup C) \ \& \ (R \sqcup S) \text{ I } (B \cup D)$,! 27 ($\&I$: 20,26)	i
$((R \sqcup S) \text{ F } (A \cup C) \ \& \ (R \sqcup S) \text{ I } (B \cup D))$,! 28 ($(\)I$: 27)	i
$\exists R (R \text{ F } (A \cup C) \ \& \ R \text{ I } (B \cup D))$,! 29 ($\exists I$: 28)	i
$(A \cup C) \sim (B \cup D)$,! 30 ($\exists I$: P1,29)	i
$A \sim B \ \& \ C \sim D \ \& \ (A \cap C) \equiv \phi \ \& \ (B \cap D) \equiv \phi$ $\Rightarrow (A \cup C) \sim (B \cup D)$,! 31 ($\Rightarrow I$: 2,30)	i
$(A \sim B \ \& \ C \sim D \ \& \ (A \cap C) \equiv \phi \ \& \ (B \cap D) \equiv \phi$ $\Rightarrow (A \cup C) \sim (B \cup D))$		

,! 32 ((I: 31) i

$\forall A \forall B \forall C \forall D (A \sim B \ \& \ C \sim D \ \& \ (A \cap C) \equiv \phi \ \& \ (B \cap D) \equiv \phi$
 $\Rightarrow (A \cup C) \sim (B \cup D))$

! 33 ($\forall I$: 1,32) i

□

! 16. P16 is a lemma for P17. i

$\vdash \forall R \forall A (A \subseteq (R^D) \ \& \ f R \ \& \ 1 R \Rightarrow A \sim ((R \uparrow A)^I))$ i

R, A ,! 1 (Prem) i

A $\subseteq (R^D)$ & f R & 1 R ,! 2 (Prem) i

A $\subseteq (R^D)$,! 3 (&E: 2) i

f R ,! 4 (&E: 2) i

f R & A $\subseteq (R^D)$,! 5 (&I: 3,4) i

(f R & A $\subseteq (R^D)$ $\Rightarrow (R \uparrow A) \mathbb{F} A$) ,! 6 ($\forall E$: C8.15) i

f R & A $\subseteq (R^D)$ $\Rightarrow (R \uparrow A) \mathbb{F} A$,! 7 ((E: 6) i

(R \uparrow A) $\mathbb{F} A$,! 8 ($\Rightarrow E$: 5,7) i

1 R ,! 9 (&E: 2) i

(1 R $\Rightarrow (R \uparrow A) \mathbb{1} ((R \uparrow A)^I)$) ,! 10 ($\forall E$: C9.30) i

1 R $\Rightarrow (R \uparrow A) \mathbb{1} ((R \uparrow A)^I)$,! 11 ((E: 10) i

(R \uparrow A) $\mathbb{1} ((R \uparrow A)^I)$,! 12 ($\Rightarrow E$: 9,11) i

(R \uparrow A) $\mathbb{F} A$ & (R \uparrow A) $\mathbb{1} ((R \uparrow A)^I)$,! 13 (&I: 8,12) i

((R \uparrow A) $\mathbb{F} A$ & (R \uparrow A) $\mathbb{1} ((R \uparrow A)^I)$)
,! 14 ((I: 13) i

$\exists R (R \mathbb{F} A \ \& \ R \mathbb{1} ((R \uparrow A)^I))$,! 15 ($\exists I$: 14) i

A $\sim ((R \uparrow A)^I)$,! 16 ($\mathbb{S}I$: P1,15) i

A $\subseteq (R^D)$ & f R & 1 R $\Rightarrow A \sim ((R \uparrow A)^I)$,! 17 ($\Rightarrow I$: 2,16) i

(A $\subseteq (R^D)$ & f R & 1 R $\Rightarrow A \sim ((R \uparrow A)^I)$)
,! 18 ((I: 17) i

$\forall R \forall A (A \subseteq (R^D) \ \& \ f R \ \& \ 1 R \Rightarrow A \sim ((R \uparrow A)^I))$

! 19 ($\forall I$: 1,18) i

□

! 17. i

$\vdash \forall A \forall B \forall C ((A \cup B) \sim C \ \& \ (A \cap B) \equiv \phi$
 $\Rightarrow \exists D \exists E ((D \cup E) \equiv C \ \& \ (D \cap E) \equiv \phi \ \& \ A \sim D$
 $\ \& \ B \sim E))$ i

A, B, C , ! 1 (Prem) i

$(A \cup B) \sim C \ \& \ (A \cap B) \equiv \phi$, ! 2 (Prem) i

! Unpack the premise. i

$(A \cup B) \sim C$, ! 3 (&E: 2) i

$(A \cap B) \equiv \phi$, ! 4 (&E: 2) i

$\exists R (R \ \mathbb{F} \ (A \cup B) \ \& \ R \ \mathbb{1} \ C)$, ! 5 (\exists E: P1,3) i

$(R \ \mathbb{F} \ (A \cup B) \ \& \ R \ \mathbb{1} \ C)$, ! 6 (\exists E: 5) i

$R \ \mathbb{F} \ (A \cup B) \ \& \ R \ \mathbb{1} \ C$, ! 7 (()E: 6) i

$R \ \mathbb{F} \ (A \cup B)$, ! 8 (&E: 7) i

$(R^D) \equiv (A \cup B) \ \& \ \mathbf{f} \ R$, ! 9 (\mathbb{S} E: C8.10,8) i

$(R^D) \equiv (A \cup B)$, ! 10 (&E: 9) i

$\mathbf{f} \ R$, ! 11 (&E: 9) i

$R \ \mathbb{1} \ C$, ! 12 (&E: 7) i

$(R^I) \equiv C \ \& \ \mathbf{1} \ R$, ! 13 (\mathbb{S} E: C9.19,12) i

$(R^I) \equiv C$, ! 14 (&E: 13) i

$\mathbf{1} \ R$, ! 15 (&E: 14) i

! D will equal $((R \ \mathbb{1} \ A)^I)$ and E will equal $((R \ \mathbb{1} \ B)^I)$ i

! To show: $(((R \ \mathbb{1} \ A)^I) \cup ((R \ \mathbb{1} \ B)^I)) \equiv C$ i

$((R^D) \equiv (A \cup B) \Rightarrow (((R \ \mathbb{1} \ A)^I) \cup ((R \ \mathbb{1} \ B)^I)) \equiv (R^I))$
, ! 16 (\forall E: C7.42) i

$(R^D) \equiv (A \cup B) \Rightarrow (((R \ \mathbb{1} \ A)^I) \cup ((R \ \mathbb{1} \ B)^I)) \equiv (R^I)$
, ! 17 (()E: 16) i

$(((R \ \mathbb{1} \ A)^I) \cup ((R \ \mathbb{1} \ B)^I)) \equiv (R^I)$, ! 18 (\Rightarrow E: 10,17) i

$(((R \ \mathbb{1} \ A)^I) \cup ((R \ \mathbb{1} \ B)^I)) \equiv (R^I) \ \& \ (R^I) \equiv C$
, ! 19 (&I: 14,18) i

$$\begin{aligned} & (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I) \ \& \ (R^I) \equiv C \\ & \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv C \end{aligned} \quad ,! \ 20 \ (\forall E: \text{II1.15}) \quad ;$$

$$\begin{aligned} & (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv (R^I) \ \& \ (R^I) \equiv C \\ & \Rightarrow (((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv C \end{aligned} \quad ,! \ 21 \ ({}E: 20) \quad ;$$

$$(((R \uparrow A)^I) \cup ((R \uparrow B)^I)) \equiv C \quad ,! \ 22 \ (\Rightarrow E: 19,21) \quad ;$$

! To show: $(((R \uparrow A)^I) \cap ((R \uparrow B)^I)) \equiv \phi$
i

$$\mathbf{1} \ R \ \& \ (A \cap B) \equiv \phi \quad ,! \ 23 \ (\&I: 4,15) \quad ;$$

$$\begin{aligned} & (\mathbf{1} \ R \ \& \ (A \cap B) \equiv \phi \Rightarrow (((R \uparrow A)^I) \cap ((R \uparrow B)^I)) \equiv \phi) \\ & \quad ,! \ 24 \ (\forall E: \text{C9.17}) \quad ; \end{aligned}$$

$$\mathbf{1} \ R \ \& \ (A \cap B) \equiv \phi \Rightarrow (((R \uparrow A)^I) \cap ((R \uparrow B)^I)) \equiv \phi \quad ,! \ 25 \ ({}E: 24) \quad ;$$

$$(((R \uparrow A)^I) \cap ((R \uparrow B)^I)) \equiv \phi \quad ,! \ 26 \ (\Rightarrow E: 23,25) \quad ;$$

! To show: $A \sim ((R \uparrow A)^I)$
i

$$A \subseteq (A \cup B) \quad ,! \ 27 \ (\forall E: \text{I2.12}) \quad ;$$

$$(R^D) \equiv (A \cup B) \ \& \ A \subseteq (A \cup B) \quad ,! \ 28 \ (\&I: 10,27) \quad ;$$

$$\begin{aligned} & ((R^D) \equiv (A \cup B) \ \& \ A \subseteq (A \cup B) \Rightarrow A \subseteq (R^D)) \\ & \quad ,! \ 29 \ (\forall E: \text{II1.31}) \quad ; \end{aligned}$$

$$(R^D) \equiv (A \cup B) \ \& \ A \subseteq (A \cup B) \Rightarrow A \subseteq (R^D) \quad ,! \ 30 \ ({}E: 29) \quad ;$$

$$A \subseteq (R^D) \quad ,! \ 31 \ (\Rightarrow E: 28,30) \quad ;$$

$$A \subseteq (R^D) \ \& \ \mathbf{f} \ R \quad ,! \ 32 \ (\&I: 11,31) \quad ;$$

$$A \subseteq (R^D) \ \& \ \mathbf{f} \ R \ \& \ \mathbf{1} \ R \quad ,! \ 33 \ (\&I: 15,32) \quad ;$$

$$\begin{aligned} & (A \subseteq (R^D) \ \& \ \mathbf{f} \ R \ \& \ \mathbf{1} \ R \Rightarrow A \sim ((R \uparrow A)^I)) \\ & \quad ,! \ 34 \ (\forall E: \text{P16}) \quad ; \end{aligned}$$

$$A \subseteq (R^D) \ \& \ \mathbf{f} \ R \ \& \ \mathbf{1} \ R \Rightarrow A \sim ((R \uparrow A)^I) \quad ,! \ 35 \ ({}E: 34) \quad ;$$

$$A \sim ((R \uparrow A)^I) \quad ,! \ 36 \ (\Rightarrow E: 33,35) \quad ;$$

! To show: $B \sim ((R \uparrow B)^I)$
i

$$B \subseteq (A \cup B) \quad ,! \ 37 \ (\forall E: \text{II2.13}) \quad ;$$

$$(\mathbf{R}^D) \equiv (\mathbf{A} \cup \mathbf{B}) \ \& \ \mathbf{B} \subseteq (\mathbf{A} \cup \mathbf{B}) \quad ,! \ 38 \ (\&I: 10,37) \quad ;$$

$$(\ (\mathbf{R}^D) \equiv (\mathbf{A} \cup \mathbf{B}) \ \& \ \mathbf{B} \subseteq (\mathbf{A} \cup \mathbf{B}) \Rightarrow \mathbf{B} \subseteq (\mathbf{R}^D) \) \quad ,! \ 39 \ (\forall E: \text{II1.31}) \quad ;$$

$$(\mathbf{R}^D) \equiv (\mathbf{A} \cup \mathbf{B}) \ \& \ \mathbf{B} \subseteq (\mathbf{A} \cup \mathbf{B}) \Rightarrow \mathbf{B} \subseteq (\mathbf{R}^D) \quad ,! \ 40 \ ({}E: 39) \quad ;$$

$$\mathbf{B} \subseteq (\mathbf{R}^D) \quad ,! \ 41 \ (\Rightarrow E: 38,40) \quad ;$$

$$\mathbf{B} \subseteq (\mathbf{R}^D) \ \& \ \mathbf{f} \ \mathbf{R} \quad ,! \ 42 \ (\&I: 11,41) \quad ;$$

$$\mathbf{B} \subseteq (\mathbf{R}^D) \ \& \ \mathbf{f} \ \mathbf{R} \ \& \ \mathbf{1} \ \mathbf{R} \quad ,! \ 43 \ (\&I: 15,42) \quad ;$$

$$(\ \mathbf{B} \subseteq (\mathbf{R}^D) \ \& \ \mathbf{f} \ \mathbf{R} \ \& \ \mathbf{1} \ \mathbf{R} \Rightarrow \mathbf{B} \sim ((\mathbf{R} \lceil \mathbf{B})^I) \) \quad ,! \ 44 \ (\forall E: \text{P16}) \quad ;$$

$$\mathbf{B} \subseteq (\mathbf{R}^D) \ \& \ \mathbf{f} \ \mathbf{R} \ \& \ \mathbf{1} \ \mathbf{R} \Rightarrow \mathbf{B} \sim ((\mathbf{R} \lceil \mathbf{B})^I) \quad ,! \ 45 \ ({}E: 44) \quad ;$$

$$\mathbf{B} \sim ((\mathbf{R} \lceil \mathbf{B})^I) \quad ,! \ 46 \ (\Rightarrow E: 43,45) \quad ;$$

! Conclusion i

$$\begin{aligned} & (((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \mathbf{C} \\ & \& \ (((\mathbf{R} \lceil \mathbf{A})^I) \cap ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \phi \end{aligned} \quad ,! \ 47 \ (\&I: 22,26) \quad ;$$

$$\begin{aligned} & (((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \mathbf{C} \\ & \& \ (((\mathbf{R} \lceil \mathbf{A})^I) \cap ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \phi \\ & \& \ \mathbf{A} \sim ((\mathbf{R} \lceil \mathbf{A})^I) \end{aligned} \quad ,! \ 48 \ (\&I: 36,47) \quad ;$$

$$\begin{aligned} & (((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \mathbf{C} \\ & \& \ (((\mathbf{R} \lceil \mathbf{A})^I) \cap ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \phi \\ & \& \ \mathbf{A} \sim ((\mathbf{R} \lceil \mathbf{A})^I) \ \& \ \mathbf{B} \sim ((\mathbf{R} \lceil \mathbf{B})^I) \end{aligned} \quad ,! \ 49 \ (\&I: 46,48) \quad ;$$

$$\begin{aligned} & ((((\mathbf{R} \lceil \mathbf{A})^I) \cup ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \mathbf{C} \\ & \ \& \ (((\mathbf{R} \lceil \mathbf{A})^I) \cap ((\mathbf{R} \lceil \mathbf{B})^I)) \equiv \phi \\ & \ \& \ \mathbf{A} \sim ((\mathbf{R} \lceil \mathbf{A})^I) \ \& \ \mathbf{B} \sim ((\mathbf{R} \lceil \mathbf{B})^I) \) \end{aligned} \quad ,! \ 50 \ ({}I: 49) \quad ;$$

$$\begin{aligned} \exists E \ (\ (((\mathbf{R} \lceil \mathbf{A})^I) \cup \mathbf{E}) \equiv \mathbf{C} \ \& \ (((\mathbf{R} \lceil \mathbf{A})^I) \cap \mathbf{E}) \equiv \phi \\ \ \& \ \mathbf{A} \sim ((\mathbf{R} \lceil \mathbf{A})^I) \ \& \ \mathbf{B} \sim \mathbf{E} \) \end{aligned} \quad ,! \ 51 \ (\exists I: 50) \quad ;$$

$$\exists D \exists E \ (\ (\mathbf{D} \cup \mathbf{E}) \equiv \mathbf{C} \ \& \ (\mathbf{D} \cap \mathbf{E}) \equiv \phi \ \& \ \mathbf{A} \sim \mathbf{D} \ \& \ \mathbf{B} \sim \mathbf{E} \) \quad ,! \ 52 \ (\exists I: 51) \quad ;$$

$$\begin{aligned} & (\mathbf{A} \cup \mathbf{B}) \sim \mathbf{C} \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \\ \Rightarrow \exists D \exists E \ (\ (\mathbf{D} \cup \mathbf{E}) \equiv \mathbf{C} \ \& \ (\mathbf{D} \cap \mathbf{E}) \equiv \phi \ \& \ \mathbf{A} \sim \mathbf{D} \ \& \ \mathbf{B} \sim \mathbf{E} \) \end{aligned}$$

,! 53 (\Rightarrow I: 52) ;

((A \cup B) \sim C & (A \cap B) \equiv ϕ
 $\Rightarrow \exists D \exists E$ ((D \cup E) \equiv C & (D \cap E) \equiv ϕ & A \sim D & B \sim E))

,! 54 ((I: 53) ;

$\forall A \forall B \forall C$ ((A \cup B) \sim C & (A \cap B) \equiv ϕ

$\Rightarrow \exists D \exists E$ ((D \cup E) \equiv C & (D \cap E) \equiv ϕ & A \sim D
& B \sim E))

! 55 (\forall I: 1,54) ;

□

! P18 through P25 are on the same theme. P18, 19, P21, and P22 establish that correspondence with an empty predicate implies that oneself is empty. P20 states the equivalence of correspondence with an empty predicate and being empty. And P23 through P25 assert that correspondence with a non-empty predicate implies non-emptiness. ;

! 18. ;

$\vdash \forall A$ (A \sim $\phi \Rightarrow A \equiv \phi$) ;

A ,! 1 (Prem) ;

A \sim ϕ ,! 2 (Prem) ;

$\exists R$ (R \Vdash A & R \Vdash ϕ) ,! 3 (\exists E: P1,2) ;

(R \Vdash A & R \Vdash ϕ) ,! 4 (\exists E: 3) ;

R \Vdash A & R \Vdash ϕ ,! 5 ((E: 4) ;

R \Vdash A ,! 6 (&E: 5) ;

(R^D) \equiv A & R \Vdash A ,! 7 (\exists E: C8.10,6) ;

(R^D) \equiv A ,! 8 (&E: 7) ;

R \Vdash ϕ ,! 9 (&E: 5) ;

(R^I) \equiv ϕ & R \Vdash ϕ ,! 10 (\exists E: C9.19,9) ;

(R^I) \equiv ϕ ,! 11 (&E: 10) ;

((R^I) \equiv $\phi \Rightarrow R \equiv \Phi$) ,! 12 (\forall E: C6.30) ;

(R^I) \equiv $\phi \Rightarrow R \equiv \Phi$,! 13 ((E: 12) ;

R \equiv Φ ,! 14 (\Rightarrow E: 11,13) ;

(R \equiv $\Phi \Rightarrow$ (R^D) \equiv ϕ) ,! 15 (\forall E: C5.23) ;

R \equiv $\Phi \Rightarrow$ (R^D) \equiv ϕ ,! 16 ((E: 15) ;

$(\mathbf{R}^D) \equiv \phi$,! 17 (\Rightarrow E: 14,16) i
$(\mathbf{R}^D) \equiv \mathbf{A} \ \& \ (\mathbf{R}^D) \equiv \phi$,! 18 ($\&$ I: 8,17) i
$((\mathbf{R}^D) \equiv \mathbf{A} \ \& \ (\mathbf{R}^D) \equiv \phi \Rightarrow \mathbf{A} \equiv \phi)$,! 19 (\forall E: III.19) i
$(\mathbf{R}^D) \equiv \mathbf{A} \ \& \ (\mathbf{R}^D) \equiv \phi \Rightarrow \mathbf{A} \equiv \phi$,! 20 ($(())$ E: 19) i
$\mathbf{A} \equiv \phi$,! 21 (\Rightarrow E: 18,20) i
$\mathbf{A} \sim \phi \Rightarrow \mathbf{A} \equiv \phi$,! 22 (\Rightarrow I: 2,21) i
$(\mathbf{A} \sim \phi \Rightarrow \mathbf{A} \equiv \phi)$,! 23 ($(())$ I: 22) i
$\forall \mathbf{A} (\mathbf{A} \sim \phi \Rightarrow \mathbf{A} \equiv \phi)$! 24 (\forall I: 1,23) i

□

! 19.

$\vdash \forall \mathbf{A} (\phi \sim \mathbf{A} \Rightarrow \mathbf{A} \equiv \phi)$	i
\mathbf{A}	,! 1 (Prem) i
$\phi \sim \mathbf{A}$,! 2 (Prem) i
$(\phi \sim \mathbf{A} \Rightarrow \mathbf{A} \sim \phi)$,! 3 (\forall E: P4) i
$\phi \sim \mathbf{A} \Rightarrow \mathbf{A} \sim \phi$,! 4 ($(())$ E: 3) i
$\mathbf{A} \sim \phi$,! 5 (\Rightarrow E: 2,4) i
$(\mathbf{A} \sim \phi \Rightarrow \mathbf{A} \equiv \phi)$,! 6 (\forall E: P18) i
$\mathbf{A} \sim \phi \Rightarrow \mathbf{A} \equiv \phi$,! 7 ($(())$ E: 6) i
$\mathbf{A} \equiv \phi$,! 8 (\Rightarrow E: 5,7) i
$\phi \sim \mathbf{A} \Rightarrow \mathbf{A} \equiv \phi$,! 9 (\Rightarrow I: 2,8) i
$(\phi \sim \mathbf{A} \Rightarrow \mathbf{A} \equiv \phi)$,! 10 ($(())$ I: 9) i
$\forall \mathbf{A} (\phi \sim \mathbf{A} \Rightarrow \mathbf{A} \equiv \phi)$! 11 (\forall I: 1,10) i

□

! 20.

$\vdash \forall \mathbf{A} (\mathbf{A} \sim \phi \Leftrightarrow \mathbf{A} \equiv \phi)$	i
\mathbf{A}	,! 1 (Prem) i
$(\mathbf{A} \equiv \phi \Rightarrow \mathbf{A} \sim \phi)$,! 2 (\forall E: P2) i

$A \equiv \phi \Rightarrow A \sim \phi$,! 3 (()E: 2)	i
$(A \sim \phi \Rightarrow A \equiv \phi)$,! 4 (\forall E: P18)	i
$A \sim \phi \Rightarrow A \equiv \phi$,! 5 (()E: 4)	i
$A \sim \phi \Leftrightarrow A \equiv \phi$,! 6 (\Leftrightarrow I: 3,5)	i
$(A \sim \phi \Leftrightarrow A \equiv \phi)$,! 7 (()I: 6)	i
$\forall A (A \sim \phi \Leftrightarrow A \equiv \phi)$! 8 (\forall I: 1,7)	i

□

! 21.

$\vdash \forall A \forall B (A \sim B \ \& \ A \equiv \phi \Rightarrow B \equiv \phi)$		i
A, B	,! 1 (Prem)	i
$A \sim B \ \& \ A \equiv \phi$,! 2 (Prem)	i
$(A \sim B \ \& \ A \equiv \phi \Rightarrow \phi \sim B)$,! 3 (\forall E: P9)	i
$A \sim B \ \& \ A \equiv \phi \Rightarrow \phi \sim B$,! 4 (()E: 3)	i
$\phi \sim B$,! 5 (\Rightarrow E: 2,4)	i
$(\phi \sim B \Rightarrow B \equiv \phi)$,! 6 (\forall E: P19)	i
$\phi \sim B \Rightarrow B \equiv \phi$,! 7 (()E: 6)	i
$B \equiv \phi$,! 8 (\Rightarrow E: 5,7)	i
$A \sim B \ \& \ A \equiv \phi \Rightarrow B \equiv \phi$,! 9 (\Rightarrow I: 2,8)	i
$(A \sim B \ \& \ A \equiv \phi \Rightarrow B \equiv \phi)$,! 10 (()I: 9)	i
$\forall A \forall B (A \sim B \ \& \ A \equiv \phi \Rightarrow B \equiv \phi)$! 11 (\forall I: 1,10)	i

□

! 22.

$\vdash \forall A \forall B (A \sim B \ \& \ B \equiv \phi \Rightarrow A \equiv \phi)$		i
A, B	,! 1 (Prem)	i
$A \sim B \ \& \ B \equiv \phi$,! 2 (Prem)	i
$(A \sim B \ \& \ B \equiv \phi \Rightarrow A \sim \phi)$,! 3 (\forall E: P11)	i
$A \sim B \ \& \ B \equiv \phi \Rightarrow A \sim \phi$,! 4 (()E: 3)	i
$A \sim \phi$,! 5 (\Rightarrow E: 2,4)	i

$(A \sim \phi \Rightarrow A \equiv \phi)$,! 6 ($\forall E$: P18)	i
$A \sim \phi \Rightarrow A \equiv \phi$,! 7 ($(\)E$: 6)	i
$A \equiv \phi$,! 8 ($\Rightarrow E$: 5,7)	i
$A \sim B \ \& \ B \equiv \phi \Rightarrow A \equiv \phi$,! 9 ($\Rightarrow I$: 2,8)	i
$(A \sim B \ \& \ B \equiv \phi \Rightarrow A \equiv \phi)$,! 10 ($(\)I$: 9)	i
$\forall A \forall B (A \sim B \ \& \ B \equiv \phi \Rightarrow A \equiv \phi)$! 11 ($\forall I$: 1,10)	i
\square		

! 23. i

$\vdash \forall A \forall B (A \sim B \ \& \ \neg A \equiv \phi \Rightarrow \neg B \equiv \phi)$		i
A, B	,! 1 (Prem)	i
$A \sim B \ \& \ \neg A \equiv \phi$,! 2 (Prem)	i
$A \sim B$,! 3 ($\&E$: 2)	i
$\neg A \equiv \phi$,! 4 ($\&E$: 2)	i
$B \equiv \phi$,! 5 (Prem)	i
$A \sim B \ \& \ B \equiv \phi$,! 6 ($\&I$: 3,5)	i
$(A \sim B \ \& \ B \equiv \phi \Rightarrow A \equiv \phi)$,! 7 ($\forall E$: P22)	i
$A \sim B \ \& \ B \equiv \phi \Rightarrow A \equiv \phi$,! 8 ($(\)E$: 7)	i
$A \equiv \phi$,! 9 ($\Rightarrow E$: 6,8)	i
\mathfrak{F}	,! 10 ($\mathfrak{F}I$: 4,9)	i
$B \equiv \phi \Rightarrow \mathfrak{F}$,! 11 ($\Rightarrow I$: 5,10)	i
$\neg B \equiv \phi$,! 12 ($\neg I$: 11)	i
$A \sim B \ \& \ \neg A \equiv \phi \Rightarrow \neg B \equiv \phi$,! 13 ($\Rightarrow I$: 2,12)	i
$(A \sim B \ \& \ \neg A \equiv \phi \Rightarrow \neg B \equiv \phi)$,! 14 ($(\)I$: 13)	i
$\forall A \forall B (A \sim B \ \& \ \neg A \equiv \phi \Rightarrow \neg B \equiv \phi)$! 15 ($\forall I$: 1,14)	i
\square		

! 24. i

$\vdash \forall A \forall B (A \sim B \ \& \ \neg B \equiv \phi \Rightarrow \neg A \equiv \phi)$		i
A, B	,! 1 (Prem)	i

$A \sim B \ \& \ \neg B \equiv \phi$,! 2 (Prem)	i
$A \sim B$,! 3 (&E: 2)	i
$\neg B \equiv \phi$,! 4 (&E: 2)	i
$(A \sim B \Rightarrow B \sim A)$,! 5 (\forall E: P4)	i
$A \sim B \Rightarrow B \sim A$,! 6 (()E: 5)	i
$B \sim A$,! 7 (\Rightarrow E: 3,6)	i
$B \sim A \ \& \ \neg B \equiv \phi$,! 8 (\Rightarrow E: 4,7)	i
$(B \sim A \ \& \ \neg B \equiv \phi \Rightarrow \neg A \equiv \phi)$,! 9 (\forall E: P23)	i
$B \sim A \ \& \ \neg B \equiv \phi \Rightarrow \neg A \equiv \phi$,! 10 (()E: 9)	i
$\neg A \equiv \phi$,! 11 (\Rightarrow E: 8,10)	i
$A \sim B \ \& \ \neg B \equiv \phi \Rightarrow \neg A \equiv \phi$,! 12 (\Rightarrow I: 2,11)	i
$(A \sim B \ \& \ \neg B \equiv \phi \Rightarrow \neg A \equiv \phi)$,! 13 (()I: 12)	i
$\forall A \forall B (A \sim B \ \& \ \neg B \equiv \phi \Rightarrow \neg A \equiv \phi)$! 14 (\forall I: 1,13)	i
\square		
! 25.		i
$\vdash \forall A \forall B (A \sim B \ \& \ \exists x A[x] \Rightarrow \exists x B[x])$		i
A, B	,! 1 (Prem)	i
$A \sim B \ \& \ \exists x A[x]$,! 2 (Prem)	i
$A \sim B$,! 3 (&E: 2)	i
$\exists x A[x]$,! 4 (&E: 2)	i
$(\exists x A[x] \Rightarrow \neg A \equiv \phi)$,! 5 (\forall E: II5.7)	i
$\exists x A[x] \Rightarrow \neg A \equiv \phi$,! 6 (()E: 5)	i
$\neg A \equiv \phi$,! 7 (\Rightarrow E: 4,6)	i
$A \sim B \ \& \ \neg A \equiv \phi$,! 8 (&I: 3,7)	i
$(A \sim B \ \& \ \neg A \equiv \phi \Rightarrow \neg B \equiv \phi)$,! 9 (\forall E: P23)	i
$A \sim B \ \& \ \neg A \equiv \phi \Rightarrow \neg B \equiv \phi$,! 10 (()E: 9)	i
$\neg B \equiv \phi$,! 11 (\Rightarrow E: 8,10)	i

$(\neg B \equiv \phi \Rightarrow \exists x B[x])$,! 12 ($\forall E$: II5.16)	i
$\neg B \equiv \phi \Rightarrow \exists x B[x]$,! 13 ($(())E$: 12)	i
$\exists x B[x]$,! 14 ($\Rightarrow E$: 11,13)	i
$A \sim B \ \& \ \exists x A[x] \Rightarrow \exists x B[x]$,! 15 ($\Rightarrow I$: 2,14)	i
$(A \sim B \ \& \ \exists x A[x] \Rightarrow \exists x B[x])$,! 16 ($(())I$: 15)	i
$\forall A \forall B (A \sim B \ \& \ \exists x A[x] \Rightarrow \exists x B[x])$! 17 ($\forall I$: 1,16)	i

□

! 26. Singleton predicates correspond with each other. i

$\vdash \forall a \forall b (a^\bullet \sim (b^\bullet))$		i
a, b	,! 1 (Prem)	i
$(a \ \square \ b) \ \mathbb{F} \ (a^\bullet)$,! 2 ($\forall E$: C12.22)	i
$(a \ \square \ b) \ \mathbb{I} \ (b^\bullet)$,! 3 ($\forall E$: C12.26)	i
$(a \ \square \ b) \ \mathbb{F} \ (a^\bullet) \ \& \ (a \ \square \ b) \ \mathbb{I} \ (b^\bullet)$,! 4 ($\&I$: 2,3)	i
$((a \ \square \ b) \ \mathbb{F} \ (a^\bullet) \ \& \ (a \ \square \ b) \ \mathbb{I} \ (b^\bullet))$,! 5 ($(())I$: 4)	i
$\exists R (R \ \mathbb{F} \ (a^\bullet) \ \& \ R \ \mathbb{I} \ (b^\bullet))$,! 6 ($\exists I$: 5)	i
$(a^\bullet) \sim (b^\bullet)$,! 7 ($\mathbb{S}I$: P1,6)	i
$\forall a \forall b (a^\bullet \sim (b^\bullet))$! 8 ($\forall I$: 1,7)	i

□

! P27 and P28 assert that correspondence with a singleton predicate implies oneself is singleton. i

! 27. i

$\vdash \forall A \forall b ((b^\bullet) \sim A \Rightarrow \exists a A \equiv (a^\bullet))$		i
A, b	,! 1 (Prem)	i
$(b^\bullet) \sim A$,! 2 (Prem)	i
$\exists R (R \ \mathbb{F} \ (b^\bullet) \ \& \ R \ \mathbb{I} \ A)$,! 3 ($\mathbb{S}E$: P1,2)	i
$(R \ \mathbb{F} \ (b^\bullet) \ \& \ R \ \mathbb{I} \ A)$,! 4 ($\exists E$: 3)	i
$R \ \mathbb{F} \ (b^\bullet) \ \& \ R \ \mathbb{I} \ A$,! 5 ($(())E$: 4)	i
$R \ \mathbb{F} \ (b^\bullet)$,! 6 ($\&E$: 5)	i

$(R \text{ F } (b^\bullet) \Rightarrow \exists b R \equiv (b \cdot b))$,! 7 ($\forall E$: C12.23)	i
$R \text{ F } (b^\bullet) \Rightarrow \exists b R \equiv (b \cdot b)$,! 8 ($()E$: 7)	i
$\exists b R \equiv (b \cdot b)$,! 9 ($\Rightarrow E$: 6,8)	i
$R \equiv (b \cdot a)$,! 10 ($\exists E$: 9)	i
$R \text{ I } A$,! 11 ($\&E$: 5)	i
$R \text{ I } A \ \& \ R \equiv (b \cdot a)$,! 12 ($\&I$: 10,11)	i
$(R \text{ I } A \ \& \ R \equiv (b \cdot a) \Rightarrow (b \cdot a) \text{ I } A)$,! 13 ($\forall E$: C9.27)	i
$R \text{ I } A \ \& \ R \equiv (b \cdot a) \Rightarrow (b \cdot a) \text{ I } A$,! 14 ($()E$: 13)	i
$(b \cdot a) \text{ I } A$,! 15 ($\Rightarrow E$: 12,14)	i
$((b \cdot a)^I) \equiv A \ \& \ \text{I } (b \cdot a)$,! 16 ($\E: C9.19,15)	i
$((b \cdot a)^I) \equiv A$,! 17 ($\&E$: 16)	i
$((b \cdot a)^I) \equiv (a^\bullet)$,! 18 ($\forall E$: C12.19)	i
$((b \cdot a)^I) \equiv A \ \& \ ((b \cdot a)^I) \equiv (a^\bullet)$,! 19 ($\&I$: 17,18)	i
$(((b \cdot a)^I) \equiv A \ \& \ ((b \cdot a)^I) \equiv (a^\bullet) \Rightarrow A \equiv (a^\bullet))$,! 20 ($\forall E$: III.19)	i
$((b \cdot a)^I) \equiv A \ \& \ ((b \cdot a)^I) \equiv (a^\bullet) \Rightarrow A \equiv (a^\bullet)$,! 21 ($()E$: 20)	i
$A \equiv (a^\bullet)$,! 22 ($\Rightarrow E$: 19,21)	i
$\exists a A \equiv (a^\bullet)$,! 23 ($\exists I$: 22)	i
$(b^\bullet) \sim A \Rightarrow \exists a A \equiv (a^\bullet)$,! 24 ($\Rightarrow I$: 2,23)	i
$((b^\bullet) \sim A \Rightarrow \exists a A \equiv (a^\bullet))$,! 25 ($()I$: 24)	i
$\forall A \forall b ((b^\bullet) \sim A \Rightarrow \exists a A \equiv (a^\bullet))$! 26 ($\forall I$: 1,25)	i

□

! 28.

$\vdash \forall A \forall b (A \sim (b^\bullet) \Rightarrow \exists a A \equiv (a^\bullet))$		i
A, b	,! 1 (Prem)	i
$A \sim (b^\bullet)$,! 2 (Prem)	i

$(A \sim (b^\bullet) \Rightarrow (b^\bullet) \sim A)$,! 3 ($\forall E$: P4)	i
$A \sim (b^\bullet) \Rightarrow (b^\bullet) \sim A$,! 4 ($(\)E$: 3)	i
$(b^\bullet) \sim A$,! 5 ($\Rightarrow E$: 2,4)	i
$((b^\bullet) \sim A \Rightarrow \exists a A \equiv (a^\bullet))$,! 6 ($\forall E$: P27)	i
$(b^\bullet) \sim A \Rightarrow \exists a A \equiv (a^\bullet)$,! 7 ($(\)E$: 6)	i
$\exists a A \equiv (a^\bullet)$,! 8 ($\Rightarrow E$: 5,7)	i
$A \sim (b^\bullet) \Rightarrow \exists a A \equiv (a^\bullet)$,! 9 ($\Rightarrow I$: 2,8)	i
$(A \sim (b^\bullet) \Rightarrow \exists a A \equiv (a^\bullet))$,! 10 ($(\)I$: 9)	i
$\forall A \forall b (A \sim (b^\bullet) \Rightarrow \exists a A \equiv (a^\bullet))$! 11 ($\forall I$: 1,10)	i

□

! 29.

$\vdash \forall A \forall B \forall a \forall b (A \sim B \ \& \ \neg A[a] \ \& \ \neg B[b] \Rightarrow (A \cup (a^\bullet)) \sim (B \cup (b^\bullet)))$		i
A, B, a, b	,! 1 (Prem)	i
$A \sim B \ \& \ \neg A[a] \ \& \ \neg B[b]$,! 2 (Prem)	i
$A \sim B$,! 3 ($\&E$: 2)	i
$\neg A[a]$,! 4 ($\&E$: 2)	i
$\neg B[b]$,! 5 ($\&E$: 2)	i
$(a^\bullet) \sim (b^\bullet)$,! 6 ($\forall E$: P26)	i
$A \sim B \ \& \ (a^\bullet) \sim (b^\bullet)$,! 7 ($\&I$: 3,6)	i
$(\neg A[a] \Rightarrow (A \cap (a^\bullet)) \equiv \phi)$,! 8 ($\forall E$: II8.40)	i
$\neg A[a] \Rightarrow (A \cap (a^\bullet)) \equiv \phi$,! 9 ($(\)E$: 8)	i
$(A \cap (a^\bullet)) \equiv \phi$,! 10 ($\Rightarrow E$: 4,9)	i
$A \sim B \ \& \ (a^\bullet) \sim (b^\bullet) \ \& \ (A \cap (a^\bullet)) \equiv \phi$,! 11 ($\&I$: 7,10)	i
$(\neg B[b] \Rightarrow (B \cap (b^\bullet)) \equiv \phi)$,! 12 ($\forall E$: II8.40)	i
$\neg B[b] \Rightarrow (B \cap (b^\bullet)) \equiv \phi$,! 13 ($(\)E$: 12)	i
$(B \cap (b^\bullet)) \equiv \phi$,! 14 ($\Rightarrow E$: 5,13)	i
$A \sim B \ \& \ (a^\bullet) \sim (b^\bullet) \ \& \ (A \cap (a^\bullet)) \equiv \phi \ \& \ (B \cap (b^\bullet)) \equiv \phi$		

,! 15 (&I: 11,14) i

$(A \sim B \ \& \ (a^\bullet) \sim (b^\bullet) \ \& \ (A \cap (a^\bullet)) \equiv \phi \ \& \ (B \cap (b^\bullet)) \equiv \phi$
 $\Rightarrow (A \cup (a^\bullet)) \sim (B \cup (b^\bullet))$

,! 16 (\forall E: P15) i

$A \sim B \ \& \ (a^\bullet) \sim (b^\bullet) \ \& \ (A \cap (a^\bullet)) \equiv \phi \ \& \ (B \cap (b^\bullet)) \equiv \phi$
 $\Rightarrow (A \cup (a^\bullet)) \sim (B \cup (b^\bullet))$

,! 17 ((E): 16) i

$(A \cup (a^\bullet)) \sim (B \cup (b^\bullet))$

,! 18 (\Rightarrow E: 15,17) i

$A \sim B \ \& \ \neg A[a] \ \& \ \neg B[b] \Rightarrow (A \cup (a^\bullet)) \sim (B \cup (b^\bullet))$

,! 19 (\Rightarrow I: 2,18) i

$(A \sim B \ \& \ \neg A[a] \ \& \ \neg B[b] \Rightarrow (A \cup (a^\bullet)) \sim (B \cup (b^\bullet)))$

,! 20 ((I): 19) i

$\forall A \forall B \forall a \forall b (A \sim B \ \& \ \neg A[a] \ \& \ \neg B[b] \Rightarrow (A \cup (a^\bullet)) \sim (B \cup (b^\bullet)))$

! 21 (\forall I: 1,20) i

□

! 30.

i

$\vdash \forall A \forall a \forall b (\neg A[a] \ \& \ \neg A[b] \Rightarrow (A \cup (a^\bullet)) \sim (A \cup (b^\bullet)))$

i

A, a, b

,! 1 (Prem) i

$\neg A[a] \ \& \ \neg A[b]$

,! 2 (Prem) i

$A \sim A$

,! 3 (\forall E: P3) i

$A \sim A \ \& \ \neg A[a] \ \& \ \neg A[b]$

,! 4 (&I: 2,3) i

$(A \sim A \ \& \ \neg A[a] \ \& \ \neg A[b] \Rightarrow (A \cup (a^\bullet)) \sim (A \cup (b^\bullet)))$

,! 5 (\forall E: P29) i

$A \sim A \ \& \ \neg A[a] \ \& \ \neg A[b] \Rightarrow (A \cup (a^\bullet)) \sim (A \cup (b^\bullet))$

,! 6 ((E): 5) i

$(A \cup (a^\bullet)) \sim (A \cup (b^\bullet))$

,! 7 (\Rightarrow E: 4,6) i

$\neg A[a] \ \& \ \neg A[b] \Rightarrow (A \cup (a^\bullet)) \sim (A \cup (b^\bullet))$

,! 8 (\Rightarrow I: 2,7) i

$(\neg A[a] \ \& \ \neg A[b] \Rightarrow (A \cup (a^\bullet)) \sim (A \cup (b^\bullet)))$

,! 9 ((I): 8) i

$\forall A \forall a \forall b (\neg A[a] \ \& \ \neg A[b] \Rightarrow (A \cup (a^\bullet)) \sim (A \cup (b^\bullet)))$

! 10 (\forall I: 1,9) i

□

! 31. i

$\vdash \forall A \forall B \forall a \forall b ((A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& A[a] \& B[b] \Rightarrow A \sim B)$;

A, B, a, b ,! 1 (Prem) i

$(A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& A[a] \& B[b]$,! 2 (Prem) i

$(A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))$,! 3 (&E: 2) i

A[a] ,! 4 (&E: 2) i

B[b] ,! 5 (&E: 2) i

$\neg (A \setminus (a^\bullet))[a]$,! 6 (\forall E: II8.47) i

$(A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& \neg (A \setminus (a^\bullet))[a]$,! 7 (&I: 3,6) i

$\neg (B \setminus (b^\bullet))[b]$,! 8 (\forall E: CII8.47) i

$(A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& \neg (A \setminus (a^\bullet))[a] \& \neg (B \setminus (b^\bullet))[b]$,! 9 (&I: 7,8) i

$((A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& \neg (A \setminus (a^\bullet))[a]$
 $\& \neg (B \setminus (b^\bullet))[b]$
 $\Rightarrow ((A \setminus (a^\bullet)) \cup (a^\bullet)) \sim ((B \setminus (b^\bullet)) \cup (b^\bullet))$)
, ! 10 (\forall E: P29) i

$(A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& \neg (A \setminus (a^\bullet))[a] \& \neg (B \setminus (b^\bullet))[b]$
 $\Rightarrow ((A \setminus (a^\bullet)) \cup (a^\bullet)) \sim ((B \setminus (b^\bullet)) \cup (b^\bullet))$
, ! 11 (()E: 10) i

$((A \setminus (a^\bullet)) \cup (a^\bullet)) \sim ((B \setminus (b^\bullet)) \cup (b^\bullet))$
, ! 12 (\Rightarrow E: 9,11) i

$(A[a] \Rightarrow ((A \setminus (a^\bullet)) \cup (a^\bullet)) \equiv A)$,! 13 (\forall E: II8.56) i

A[a] $\Rightarrow ((A \setminus (a^\bullet)) \cup (a^\bullet)) \equiv A$,! 14 (()E: 13) i

$((A \setminus (a^\bullet)) \cup (a^\bullet)) \equiv A$,! 15 (\Rightarrow E: 4,14) i

$((A \setminus (a^\bullet)) \cup (a^\bullet)) \sim ((B \setminus (b^\bullet)) \cup (b^\bullet))$
 $\& ((A \setminus (a^\bullet)) \cup (a^\bullet)) \equiv A$
, ! 16 (&I: 12,15) i

$(B[b] \Rightarrow ((B \setminus (b^\bullet)) \cup (b^\bullet)) \equiv B)$,! 17 (\forall E: II8.56) i

B[b] $\Rightarrow ((B \setminus (b^\bullet)) \cup (b^\bullet)) \equiv B$,! 18 (()E: 17) i

$((B \setminus (b^\bullet)) \cup (b^\bullet)) \equiv B$,! 19 (\Rightarrow E: 5,18) i

$((A \setminus (a^\bullet)) \cup (a^\bullet)) \sim ((B \setminus (b^\bullet)) \cup (b^\bullet))$

$\& ((A \setminus (a^\bullet)) \cup (a^\bullet)) \equiv A$
 $\& ((B \setminus (b^\bullet)) \cup (b^\bullet)) \equiv B$
 ,! 20 (&I: 16,19) i

$((A \setminus (a^\bullet)) \cup (a^\bullet)) \sim ((B \setminus (b^\bullet)) \cup (b^\bullet))$
 $\& ((A \setminus (a^\bullet)) \cup (a^\bullet)) \equiv A$
 $\& ((B \setminus (b^\bullet)) \cup (b^\bullet)) \equiv B$
 $\Rightarrow A \sim B$
 ,! 21 (\forall E: P13) i

$((A \setminus (a^\bullet)) \cup (a^\bullet)) \sim ((B \setminus (b^\bullet)) \cup (b^\bullet))$
 $\& ((A \setminus (a^\bullet)) \cup (a^\bullet)) \equiv A$
 $\& ((B \setminus (b^\bullet)) \cup (b^\bullet)) \equiv B$
 $\Rightarrow A \sim B$
 ,! 22 (()E: 21) i

$A \sim B$,! 23 (\Rightarrow E: 20,22) i

$(A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& A[a] \& B[b] \Rightarrow A \sim B$
 ,! 24 (\Rightarrow I: 2,23) i

$((A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& A[a] \& B[b] \Rightarrow A \sim B)$
 ,! 25 (()I: 24) i

$\forall A \forall B \forall a \forall b ((A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)) \& A[a] \& B[b] \Rightarrow A \sim B)$
 ! 26 (\forall I: 1,25) i

□

! 32. i

$\vdash \forall A \forall a \forall b (A[a] \& A[b] \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet)))$ i

A, a, b ,! 1 (Prem) i

$A[a] \& A[b]$,! 2 (Prem) i

$(a = b \vee \neg a = b)$,! 3 (\forall E: I3.4) i

$a = b \vee \neg a = b$,! 4 (()E: 3) i

! Case 1 : $a = b$ i

$a = b$,! 5 (Prem) i

$(A \setminus (a^\bullet)) \sim (A \setminus (a^\bullet))$,! 6 (\forall E: P3) i

$(A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 7 (=E: 5,6) i

$a = b \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 8 (\Rightarrow I: 5,7) i

! Case 2 : $\neg a = b$

We will use P30 to prove that:

$$\begin{aligned} & ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \\ & \sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \end{aligned}$$

But the first term is equivalent to $(A \setminus (a^\bullet))$ and the second to $(A \setminus (b^\bullet))$.

In order to use P30, both

$$\neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[b] \text{ and}$$

$$\neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[a]$$

need to be established.

$$\neg a = b \quad ,! 9 \text{ (Prem)} \quad i$$

$$((a^\bullet) \cup (b^\bullet))[b] \quad ,! 10 \text{ } (\forall E: \text{II}8.32) \quad i$$

$$\begin{aligned} & (((a^\bullet) \cup (b^\bullet))[b] \Rightarrow \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[b]) \\ & \quad ,! 11 \text{ } (\forall E: \text{II}7.8) \quad i \end{aligned}$$

$$\begin{aligned} & ((a^\bullet) \cup (b^\bullet))[b] \Rightarrow \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[b] \\ & \quad ,! 12 \text{ } ({}E: 11) \quad i \end{aligned}$$

$$\neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[b] \quad ,! 13 \text{ } (\Rightarrow E: 10,12) \quad i$$

$$((a^\bullet) \cup (b^\bullet))[a] \quad ,! 14 \text{ } (\forall E: \text{II}8.33) \quad i$$

$$\begin{aligned} & (((a^\bullet) \cup (b^\bullet))[a] \Rightarrow \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[a]) \\ & \quad ,! 15 \text{ } (\forall E: \text{II}7.8) \quad i \end{aligned}$$

$$\begin{aligned} & ((a^\bullet) \cup (b^\bullet))[a] \Rightarrow \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[a] \\ & \quad ,! 16 \text{ } ({}E: 15) \quad i \end{aligned}$$

$$\neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[a] \quad ,! 17 \text{ } (\Rightarrow E: 14,16) \quad i$$

$$\neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[b] \ \& \ \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[a] \quad ,! 18 \text{ } (\&I: 13,17) \quad i$$

$$\begin{aligned} & (\neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[b] \ \& \ \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[a] \\ & \Rightarrow ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \\ & \quad \sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet))) \\ & \quad ,! 19 \text{ } (\forall E: \text{P}30) \quad i \end{aligned}$$

$$\begin{aligned} & \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[b] \ \& \ \neg (A \setminus ((a^\bullet) \cup (b^\bullet)))[a] \\ & \Rightarrow ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \\ & \quad \sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \\ & \quad ,! 20 \text{ } ({}E: 19) \quad i \end{aligned}$$

$$\begin{aligned} & ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \\ & \sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \\ & \quad ,! 21 \text{ } (\Rightarrow E: 18,20) \quad i \end{aligned}$$

$$A[a] \quad ,! 22 \text{ } (\&E: 2) \quad i$$

$\neg a = b \ \& \ A[a]$,! 23 (&I: 9,22) i

($\neg a = b \ \& \ A[a]$
 $\Rightarrow ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (A \setminus (a^\bullet)))$
 ,! 24 ($\forall E$: II8.64) i

$\neg a = b \ \& \ A[a]$
 $\Rightarrow ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (A \setminus (a^\bullet))$
 ,! 25 ($()E$: 24) i

($(A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (A \setminus (a^\bullet))$
 ,! 26 ($\Rightarrow E$: 23,25) i

($(A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet))$
 $\sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet))$
 $\& ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (A \setminus (a^\bullet))$
 ,! 27 (&I: 21, 26) i

$A[b]$,! 28 (&E: 2) i

$\neg a = b \ \& \ A[b]$,! 29 (&I: 9,28) i

($\neg a = b \ \& \ A[b]$
 $\Rightarrow ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \equiv (A \setminus (b^\bullet)))$
 ,! 30 ($\forall E$: II8.65) i

$\neg a = b \ \& \ A[b]$
 $\Rightarrow ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \equiv (A \setminus (b^\bullet))$
 ,! 31 ($()E$: 30) i

($(A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \equiv (A \setminus (b^\bullet))$
 ,! 32 ($\Rightarrow E$: 29,31) i

($(A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet))$
 $\sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet))$
 $\& ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (A \setminus (a^\bullet))$
 $\& ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \equiv (A \setminus (b^\bullet))$
 ,! 33 (&I: 27,32) i

(($(A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet))$
 $\sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet))$
 $\& ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (A \setminus (a^\bullet))$
 $\& ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \equiv (A \setminus (b^\bullet))$
 $\Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet)))$
 ,! 34 ($\forall E$: P13) i

($(A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet))$
 $\sim ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet))$
 $\& ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (b^\bullet)) \equiv (A \setminus (a^\bullet))$
 $\& ((A \setminus ((a^\bullet) \cup (b^\bullet))) \cup (a^\bullet)) \equiv (A \setminus (b^\bullet))$

$\Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 35 ((E: 34)	i
$(A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 36 (\Rightarrow E: 33,35)	i
$\neg a = b \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 37 (\Rightarrow I: 9,36)	i
$(A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 38 (\forall E: 4,8,37)	i
$A[a] \ \& \ A[b] \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 39 (\Rightarrow I: 2,38)	i
$(A[a] \ \& \ A[b] \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet)))$,! 40 ((I: 39)	i
$\forall A \forall a \forall b (A[a] \ \& \ A[b] \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet)))$! 41 (\forall I: 1,40)	i

□

! 33. Lemma for the next proposition. i

$\vdash \forall A \forall B \forall a (A \sim B \ \& \ A[a]$		
$\Rightarrow \exists b (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)))$		i
A, B, a	,! 1 (Prem)	i
A ~ B & A[a]	,! 2 (Prem)	i
A ~ B	,! 3 (&E: 2)	i
A[a]	,! 4 (&E: 2)	i
$\exists x \ A[x]$,! 5 (\exists I: 4)	i
A ~ B & $\exists x \ A[x]$,! 6 (&I: 3,5)	i
$(A \sim B \ \& \ \exists x \ A[x] \Rightarrow \exists x \ B[x])$,! 7 (\forall E: P25)	i
A ~ B & $\exists x \ A[x] \Rightarrow \exists x \ B[x]$,! 8 ((E: 7)	i
$\exists x \ B[x]$,! 9 (\Rightarrow E: 6,8)	i
B[b]	,! 10 (\exists E: 9)	i
A[a] & B[b]	,! 11 (&I: 4,10)	i
$(A[a] \ \& \ A[b] \Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet)))$,! 12 (\forall E: P32)	i
A[a] & A[b] $\Rightarrow (A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 13 ((E: 12)	i
$(A \setminus (a^\bullet)) \sim (A \setminus (b^\bullet))$,! 14 (\Rightarrow E: 11,13)	i
B[b] & $(A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))$,! 15 (&I: 10,14)	i

$(B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))) \quad ,! \ 16 \ (()I: 15) \quad i$
 $\exists b \ (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))) \quad ,! \ 17 \ (\exists I: 16) \quad i$
 $A \sim B \ \& \ A[a] \Rightarrow \exists b \ (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))) \quad ,! \ 18 \ (\Rightarrow I: 2,17) \quad i$
 $(A \sim B \ \& \ A[a] \Rightarrow \exists b \ (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)))) \quad ,! \ 19 \ (()I: 18) \quad i$
 $\forall A \forall B \forall a \ (A \sim B \ \& \ A[a] \Rightarrow \exists b \ (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)))) \quad ! \ 20 \ (\forall I: 1,19) \quad i$
 \square
! 34. i
 $\vdash \forall A \forall B \forall a \forall b \ (A \sim B \ \& \ A[a] \ \& \ B[b] \Rightarrow (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))) \quad i$
 $A, B, a, b \quad ,! \ 1 \ (\text{Prem}) \quad i$
 $A \sim B \ \& \ A[a] \ \& \ B[b] \quad ,! \ 2 \ (\text{Prem}) \quad i$
 $A \sim B \ \& \ A[a] \quad ,! \ 3 \ (\&E: 2) \quad i$
 $B[b] \quad ,! \ 4 \ (\&E: 2) \quad i$
 $(A \sim B \ \& \ A[a] \Rightarrow \exists b \ (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet)))) \quad ,! \ 5 \ (\forall E: P33) \quad i$
 $A \sim B \ \& \ A[a] \Rightarrow \exists b \ (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))) \quad ,! \ 6 \ (()E: 5) \quad i$
 $\exists b \ (B[b] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (b^\bullet))) \quad ,! \ 7 \ (\Rightarrow E: 3,6) \quad i$
 $(B[c] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (c^\bullet))) \quad ,! \ 8 \ (\exists I: 7) \quad i$
 $B[c] \ \& \ (A \setminus (a^\bullet)) \sim (B \setminus (c^\bullet)) \quad ,! \ 9 \ (()E: 8) \quad i$
 $B[c] \quad ,! \ 10 \ (\&E: 9) \quad i$
 $(A \setminus (a^\bullet)) \sim (B \setminus (c^\bullet)) \quad ,! \ 11 \ (\&E: 9) \quad i$
 $B[b] \ \& \ B[c] \quad ,! \ 12 \ (\&I: 4,10) \quad i$
 $(B[b] \ \& \ B[c] \Rightarrow (B \setminus (b^\bullet)) \sim (B \setminus (c^\bullet))) \quad ,! \ 13 \ (\forall E: P32) \quad i$
 $B[b] \ \& \ B[c] \Rightarrow (B \setminus (b^\bullet)) \sim (B \setminus (c^\bullet)) \quad ,! \ 14 \ (()E: 13) \quad i$
 $(B \setminus (b^\bullet)) \sim (B \setminus (c^\bullet)) \quad ,! \ 15 \ (\Rightarrow E: 12,14) \quad i$

$$\Rightarrow ((A \cup (a^\bullet)) \setminus (a^\bullet)) \sim ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

,! 9 ((E: 8) i

$$((A \cup (a^\bullet)) \setminus (a^\bullet)) \sim ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

,! 10 (\Rightarrow E: 7,9) i

$$\neg A[a]$$

,! 11 (&E: 2) i

$$(\neg A[a] \Rightarrow A \equiv ((A \cup (a^\bullet)) \setminus (a^\bullet)))$$

,! 12 (\forall E: II8.59) i

$$\neg A[a] \Rightarrow A \equiv ((A \cup (a^\bullet)) \setminus (a^\bullet))$$

,! 13 ((E: 12) i

$$A \equiv ((A \cup (a^\bullet)) \setminus (a^\bullet))$$

,! 14 (\Rightarrow E: 11,13) i

$$((A \cup (a^\bullet)) \setminus (a^\bullet)) \sim ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

$$\& A \equiv ((A \cup (a^\bullet)) \setminus (a^\bullet))$$

,! 15 (&I: 10,14) i

$$\neg B[b]$$

,! 16 (&E: 2) i

$$(\neg B[b] \Rightarrow B \equiv ((B \cup (b^\bullet)) \setminus (b^\bullet)))$$

,! 17 (\forall E: II8.59) i

$$\neg B[b] \Rightarrow B \equiv ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

,! 18 ((E: 17) i

$$B \equiv ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

,! 19 (\Rightarrow E: 16,18) i

$$((A \cup (a^\bullet)) \setminus (a^\bullet)) \sim ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

$$\& A \equiv ((A \cup (a^\bullet)) \setminus (a^\bullet))$$

$$\& B \equiv ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

,! 20 (&I: 15,19) i

$$(((A \cup (a^\bullet)) \setminus (a^\bullet)) \sim ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

$$\& A \equiv ((A \cup (a^\bullet)) \setminus (a^\bullet))$$

$$\& B \equiv ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

$$\Rightarrow A \sim B)$$

,! 21 (\forall E: P14) i

$$((A \cup (a^\bullet)) \setminus (a^\bullet)) \sim ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

$$\& A \equiv ((A \cup (a^\bullet)) \setminus (a^\bullet))$$

$$\& B \equiv ((B \cup (b^\bullet)) \setminus (b^\bullet))$$

$$\Rightarrow A \sim B$$

,! 22 ((E: 21) i

$$A \sim B$$

,! 23 (\Rightarrow E: 20,22) i

$$(A \cup (a^\bullet)) \sim (B \cup (b^\bullet)) \& \neg A[a] \& \neg B[b] \Rightarrow A \sim B$$

,! 24 (\Rightarrow I: 2,23) i

$$((A \cup (a^\bullet)) \sim (B \cup (b^\bullet)) \& \neg A[a] \& \neg B[b] \Rightarrow A \sim B)$$

,! 25 ((I: 24) i

$\forall A \forall B \forall a \forall b ((A \cup (a^\bullet)) \sim (B \cup (b^\bullet)) \ \& \ \neg A[a] \ \& \ \neg B[b] \Rightarrow A \sim B)$
! 26 (\forall I: 1,25) i

□

! 36. i

⊢ $\forall P \forall Q \forall R \forall S \forall a \forall b$

($R[a] \ \& \ R \subseteq P \ \& \ S[b] \ \& \ S \subseteq Q$
& $(P \setminus (R \setminus (a^\bullet))) \sim (Q \setminus (S \setminus (b^\bullet)))$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S)$) i

P, Q, R, S, a, b ,! 1 (Prem) i

R[a] & R ⊆ P & S[b] & S ⊆ Q
& **(P \ (R \ (a[•]))) ~ (Q \ (S \ (b[•])))**
,! 2 (Prem) i

R[a] ,! 3 (&E: 2) i

R ⊆ P ,! 4 (&E: 2) i

S[b] ,! 5 (&E: 2) i

S ⊆ Q ,! 6 (&E: 2) i

(P \ (R \ (a[•]))) ~ (Q \ (S \ (b[•]))) ,! 7 (&E: 2) i

R[a] & R ⊆ P ,! 8 (&I: 3,4) i

(**R[a] & R ⊆ P**
 $\Rightarrow ((P \setminus R) \cup (a^\bullet)) \equiv (P \setminus (R \setminus (a^\bullet)))$)
,! 9 (\forall E: II8.66) i

R[a] & R ⊆ P
 $\Rightarrow ((P \setminus R) \cup (a^\bullet)) \equiv (P \setminus (R \setminus (a^\bullet)))$
,! 10 ((E: 9) i

((P \ R) ∪ (a[•])) ≡ (P \ (R \ (a[•]))) ,! 11 (\Rightarrow E: 8,10) i

S[b] & S ⊆ Q ,! 12 (&I: 5,6) i

(**S[b] & S ⊆ Q**
 $\Rightarrow ((Q \setminus S) \cup (b^\bullet)) \equiv (Q \setminus (S \setminus (b^\bullet)))$)
,! 13 (\forall E: II8.66) i

S[b] & S ⊆ Q
 $\Rightarrow ((Q \setminus S) \cup (b^\bullet)) \equiv (Q \setminus (S \setminus (b^\bullet)))$
,! 14 ((E: 13) i

((Q \ S) ∪ (b[•])) ≡ (Q \ (S \ (b[•]))) ,! 15 (\Rightarrow E: 12,14) i

$(P \setminus R) \sim (Q \setminus S)$,! 31 (\Rightarrow E: 28,30) ;

$R[a] \ \& \ R \subseteq P \ \& \ S[b] \ \& \ S \subseteq Q$
 $\& \ (P \setminus (R \setminus (a^\bullet))) \sim (Q \setminus (S \setminus (b^\bullet)))$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S)$
,! 32 (\Rightarrow I: 2,31) ;

$(R[a] \ \& \ R \subseteq P \ \& \ S[b] \ \& \ S \subseteq Q$
 $\ \& \ (P \setminus (R \setminus (a^\bullet))) \sim (Q \setminus (S \setminus (b^\bullet)))$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$
,! 33 (()I: 32) ;

$\forall P \forall Q \forall R \forall S \forall a \forall b$

$(R[a] \ \& \ R \subseteq P \ \& \ S[b] \ \& \ S \subseteq Q$
 $\ \& \ (P \setminus (R \setminus (a^\bullet))) \sim (Q \setminus (S \setminus (b^\bullet)))$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$
! 34 (\forall I: 1,33) ;

□