

SIMPLE CARTESIAN PRODUCTS;

! This chapter introduces simple cartesian products. In general, Cartesian products of A and B are relationships R such that x bears R to y if and only if x has A and y has B. We consider here the special case, which is slightly simpler, and which is sufficient for our subsequent use for multiplication, of the relationship (a X B). x bears (a X B) to y if and only if x is a and y has B. i

! 1. X represents simple Cartesian product. i

$\mathbb{D} X ; (a X B) ; ; \{a,b : a = a \ \& \ B[b]\}$ i

! 2. Fundamental Proposition of Simple Cartesian Products. i

$\vdash \forall a \forall B \forall x \forall y ((a X B)[x,y] \Leftrightarrow x = a \ \& \ B[y])$ i

a, B ,! 1 (Prem) i

$\forall x \forall y (\{a,b : a = a \ \& \ B[b]\}[x,y] \Leftrightarrow x = a \ \& \ B[y])$,! 2 (Pred) i

$\forall x \forall y ((a X B)[x,y] \Leftrightarrow x = a \ \& \ B[y])$,! 3 ($\mathbb{D}I$: P1,2) i

$\forall a \forall B \forall x \forall y ((a X B)[x,y] \Leftrightarrow x = a \ \& \ B[y])$! 4 ($\forall I$: 1,3) i

□

! 3. Fundamental Proposition of Simple Cartesian Products, First Half. i

$\vdash \forall a \forall B \forall x \forall y ((a X B)[x,y] \Rightarrow x = a \ \& \ B[y])$ i

a, B, x, y ,! 1 (Prem) i

$((a X B)[x,y] \Leftrightarrow x = a \ \& \ B[y])$,! 2 ($\forall E$: P2) i

$(a X B)[x,y] \Leftrightarrow x = a \ \& \ B[y]$,! 3 ($(\)E$: 3) i

$(a X B)[x,y] \Rightarrow x = a \ \& \ B[y]$,! 4 ($\Leftrightarrow E$: 3) i

$((a X B)[x,y] \Rightarrow x = a \ \& \ B[y])$,! 5 ($(\)I$: 4) i

$\forall a \forall B \forall x \forall y ((a X B)[x,y] \Rightarrow x = a \ \& \ B[y])$! 6 ($\forall I$: 1,5) i

□

! 4. i

$\vdash \forall a \forall B \forall x \forall y ((a X B)[x,y] \Rightarrow x = a)$ i

a, B, x, y ,! 1 (Prem) i

$((a X B)[x,y] \Rightarrow x = a \ \& \ B[y])$,! 2 ($\forall E$: P3) i

$(a \times B)[x,y] \Rightarrow x = a \ \& \ B[y]$, ! 3 (()E: 2)	i
$(a \times B)[x,y]$, ! 4 (Prem)	i
$x = a \ \& \ B[y]$, ! 5 (\Rightarrow E: 3,4)	i
$x = a$, ! 6 ($\&$ E: 5)	i
$(a \times B)[x,y] \Rightarrow x = a$, ! 7 (\Rightarrow I: 4,6)	i
$((a \times B)[x,y] \Rightarrow x = a)$, ! 8 (()I: 7)	i

$\forall a \forall B \forall x \forall y ((a \times B)[x,y] \Rightarrow x = a)$! 9 (\forall I: 1,8) i

□

! 5. i

$\vdash \forall a \forall B \forall x \forall y ((a \times B)[x,y] \Rightarrow B[y])$ i

a, B, x, y	, ! 1 (Prem)	i
$((a \times B)[x,y] \Rightarrow x = a \ \& \ B[y])$, ! 2 (\forall E: P3)	i
$(a \times B)[x,y] \Rightarrow x = a \ \& \ B[y]$, ! 3 (()E: 2)	i
$(a \times B)[x,y]$, ! 4 (Prem)	i
$x = a \ \& \ B[y]$, ! 5 (\Rightarrow E: 3,4)	i
$B[y]$, ! 6 ($\&$ E: 5)	i
$(a \times B)[x,y] \Rightarrow B[y]$, ! 7 (\Rightarrow I: 4,6)	i
$((a \times B)[x,y] \Rightarrow B[y])$, ! 8 (()I: 7)	i

$\forall a \forall B \forall x \forall y ((a \times B)[x,y] \Rightarrow B[y])$! 9 (\forall I: 1,8) i

□

! 6. P6 replaces any use of the Second Half of the Fundamental Proposition, which has not been asserted. i

$\vdash \forall a \forall B \forall b (B[b] \Rightarrow (a \times B)[a,b])$ i

a, B, b	, ! 1 (Prem)	i
$B[b]$, ! 2 (Prem)	i
$a = a$, ! 3 (=I)	i
$a = a \ \& \ B[b]$, ! 4 ($\&$ I: 2,3)	i
$((a \times B)[a,b] \Leftrightarrow a = a \ \& \ B[b])$, ! 5 (\forall E: P2)	i
$(a \times B)[a,b] \Leftrightarrow a = a \ \& \ B[b]$, ! 6 (()E: 5)	i

$a = a \ \& \ B[b] \Rightarrow (a \ X \ B)[a,b]$, ! 7 (\Leftrightarrow E: 6)	i
$(a \ X \ B)[a,b]$, ! 8 (\Rightarrow E: 4,7)	i
$B[b] \Rightarrow (a \ X \ B)[a,b]$, ! 9 (\Rightarrow I: 2,8)	i
$(B[b] \Rightarrow (a \ X \ B)[a,b])$, ! 10 ($(())$ I: 9)	i
$\forall a \forall B \forall b (B[b] \Rightarrow (a \ X \ B)[a,b])$! 11 (\forall I: 1,10)	i

□

! 7. i

$\vdash \forall a \forall B \forall b ((a \ X \ B)[a,b] \Leftrightarrow B[b])$		i
a, B, b	, ! 1 (Prem)	i
$((a \ X \ B)[a,b] \Rightarrow B[b])$, ! 2 (\forall E: P5)	i
$(a \ X \ B)[a,b] \Rightarrow B[b]$, ! 3 ($(())$ E: 2)	i
$(B[b] \Rightarrow (a \ X \ B)[a,b])$, ! 4 (\forall E: P6)	i
$(a \ X \ B)[a,b] \Rightarrow B[b]$, ! 5 ($(())$ E: 4)	i
$(a \ X \ B)[a,b] \Leftrightarrow B[b]$, ! 6 (\Leftrightarrow I: 5)	i
$((a \ X \ B)[a,b] \Leftrightarrow B[b])$, ! 7 ($(())$ I: 6)	i
$\forall a \forall B \forall b ((a \ X \ B)[a,b] \Leftrightarrow B[b])$! 8 (\forall I: 1,7)	i

□

! 8. P8 is a lemma for P9. As such, it saves a few steps in P9's proof to use " $A \equiv B$ " rather than " $A \subseteq B$ " in P8's antecedent. i

$\vdash \forall a \forall b \forall A \forall B (a = b \ \& \ A \equiv B \Rightarrow (a \ X \ A) \subseteq (b \ X \ B))$		i
a, b, A, B	, ! 1 (Prem)	i
$a = b \ \& \ A \equiv B$, ! 2 (Prem)	i
$a = b$, ! 3 ($\&$ E: 2)	i
$A \equiv B$, ! 4 ($\&$ E: 2)	i
x, y	, ! 5 (Prem)	i
$(a \ X \ A)[x,y]$, ! 6 (Prem)	i
$((a \ X \ A)[x,y] \Rightarrow x = a \ \& \ A[y])$, ! 7 (\forall E: P3)	i

$(a \times A)[x, y] \Rightarrow x = a \ \& \ A[y]$,! 8 (()E: 7)	i
$x = a \ \& \ A[y]$,! 9 (\Rightarrow E: 6,8)	i
$x = a$,! 10 ($\&$ E: 9)	i
$A[y]$,! 11 ($\&$ E: 9)	i
$x = b$,! 12 ($=$ E: 3,10)	i
$A[y] \ \& \ A \equiv B$,! 13 ($\&$ I: 4,11)	i
$(A[y] \ \& \ A \equiv B \Rightarrow B[y])$,! 14 (\forall E III1.35)	i
$A[y] \ \& \ A \equiv B \Rightarrow B[y]$,! 15 (()E: 14)	i
$B[y]$,! 16 (\Rightarrow E: 13,15)	i
$(B[y] \Rightarrow (b \times B)[b, y])$,! 17 (\forall E: P6)	i
$B[y] \Rightarrow (b \times B)[b, y]$,! 18 (()E: 17)	i
$(b \times B)[b, y]$,! 19 (\Rightarrow E: 16,18)	i
$(b \times B)[x, y]$,! 20 ($=$ E: 12,19)	i
$(a \times A)[x, y] \Rightarrow (b \times B)[x, y]$,! 21 (\Rightarrow I: 6,20)	i
$((a \times A)[x, y] \Rightarrow (b \times B)[x, y])$,! 22 (()I: 21)	i
$\forall x \forall y ((a \times A)[x, y] \Rightarrow (b \times B)[x, y])$,! 23 (\forall I: 5,22)	i
$(a \times A) \subseteq (b \times B)$,! 24 ($\$$ I: C1.1,23)	i
$a = b \ \& \ A \equiv B \Rightarrow (a \times A) \subseteq (b \times B)$,! 25 (\Rightarrow I: 2,24)	i
$(a = b \ \& \ A \equiv B \Rightarrow (a \times A) \subseteq (b \times B))$,! 26 (()I: 25)	i
$\forall a \forall b \forall A \forall B (a = b \ \& \ A \equiv B \Rightarrow (a \times A) \subseteq (b \times B))$! 27 (\forall I: 1,26)	i

□

! 9.

$\vdash \forall a \forall b \forall A \forall B (a = b \ \& \ A \equiv B \Rightarrow (a \times A) \equiv (b \times B))$		i
a, b, A, B	,! 1 (Prem)	i
$a = b \ \& \ A \equiv B$,! 2 (Prem)	i
$(a = b \ \& \ A \equiv B \Rightarrow (a \times A) \subseteq (b \times B))$,! 3 (\forall E: P8)	i
$a = b \ \& \ A \equiv B \Rightarrow (a \times A) \subseteq (b \times B)$,! 4 (()E: 3)	i

$(a \times A) \subseteq (b \times B)$,! 5 ($\Rightarrow E$: 2,4)	i
$a = b$,! 6 ($\&E$: 2)	i
$A \equiv B$,! 7 ($\&E$: 2)	i
$a = a$,! 8 ($=I$)	i
$b = a$,! 9 ($=E$: 6,8)	i
$(A \equiv B \Rightarrow B \equiv A)$,! 10 ($\forall E$: III.10)	i
$A \equiv B \Rightarrow B \equiv A$,! 11 ($(\)E$: 10)	i
$B \equiv A$,! 12 ($\Rightarrow E$: 7,11)	i
$b = a \ \& \ B \equiv A$,! 13 ($\&I$: 9,12)	i
$(b = a \ \& \ B \equiv A \Rightarrow (b \times B) \subseteq (a \times A))$,! 14 ($\forall E$: P8)	i
$b = a \ \& \ B \equiv A \Rightarrow (b \times B) \subseteq (a \times A)$,! 15 ($(\)E$: 14)	i
$(b \times B) \subseteq (a \times A)$,! 16 ($\Rightarrow E$: 13,15)	i
$(a \times A) \subseteq (b \times B) \ \& \ (b \times B) \subseteq (a \times A)$,! 17 ($\&I$: 5,16)	i
$((a \times A) \subseteq (b \times B) \ \& \ (b \times B) \subseteq (a \times A) \Rightarrow (a \times A) \equiv (b \times B))$,! 18 ($\forall E$: C1.6)	i
$(a \times A) \subseteq (b \times B) \ \& \ (b \times B) \subseteq (a \times A) \Rightarrow (a \times A) \equiv (b \times B)$,! 19 ($(\)E$: 18)	i
$(a \times A) \equiv (b \times B)$,! 20 ($\Rightarrow E$: 17,19)	i
$a = b \ \& \ A \equiv B \Rightarrow (a \times A) \equiv (b \times B)$,! 21 ($\Rightarrow I$: 2,20)	i
$(a = b \ \& \ A \equiv B \Rightarrow (a \times A) \equiv (b \times B))$,! 22 ($(\)I$: 21)	i
$\forall a \forall b \forall A \forall B (a = b \ \& \ A \equiv B \Rightarrow (a \times A) \equiv (b \times B))$! 23 ($\forall I$: 1,22)	i
\square		
! 10.		i
$\vdash \forall a \forall b \forall A \forall B ((a \times A) \equiv (b \times B) \Rightarrow A \subseteq B)$		i
a, b, A, B	,! 1 (Prem)	i
$(a \times A) \equiv (b \times B)$,! 2 (Prem)	i

x	,! 3 (Prem)	i
$A[x]$,! 4 (Prem)	i
$(A[x] \Rightarrow (a \times A)[a,x])$,! 5 ($\forall E$: P6)	i
$A[x] \Rightarrow (a \times A)[a,x]$,! 6 ($(\Rightarrow)E$: 5)	i
$(a \times A)[a,x]$,! 7 ($\Rightarrow E$: 4,6)	i
$(a \times A)[a,x] \ \& \ (a \times A) \equiv (b \times B)$,! 8 ($\&I$: 2,7)	i
$((a \times A)[a,x] \ \& \ (a \times A) \equiv (b \times B) \Rightarrow (b \times B)[a,x])$,! 9 ($\forall E$: C1.20)	i
$(a \times A)[a,x] \ \& \ (a \times A) \equiv (b \times B) \Rightarrow (b \times B)[a,x]$,! 10 ($(\Rightarrow)E$: 9)	i
$(b \times B)[a,x]$,! 11 ($\Rightarrow E$: 8,10)	i
$((b \times B)[a,x] \Rightarrow B[x])$,! 12 ($\forall E$: P5)	i
$(b \times B)[a,x] \Rightarrow B[x]$,! 13 ($(\Rightarrow)E$: 12)	i
$B[x]$,! 14 ($\Rightarrow E$: 11,13)	i
$A[x] \Rightarrow B[x]$,! 15 ($\Rightarrow I$: 4,14)	i
$(A[x] \Rightarrow B[x])$,! 16 ($(\Rightarrow)I$: 15)	i
$\forall x (A[x] \Rightarrow B[x])$,! 17 ($\forall I$: 3,16)	i
$A \subseteq B$,! 18 ($\S I$: III1.1,17)	i
$(a \times A) \equiv (b \times B) \Rightarrow A \subseteq B$,! 19 ($\Rightarrow I$: 2,18)	i
$((a \times A) \equiv (b \times B) \Rightarrow A \subseteq B)$,! 20 ($(\Rightarrow)I$: 19)	i
$\forall a \forall b \forall A \forall B ((a \times A) \equiv (b \times B) \Rightarrow A \subseteq B)$! 21 ($\forall I$: 1,20)	i
\square		
! 11.		i
$\vdash \forall a \forall b \forall A \forall B ((a \times A) \equiv (b \times B) \Rightarrow A \equiv B)$		i
a, b, A, B	,! 1 (Prem)	i
$(a \times A) \equiv (b \times B)$,! 2 (Prem)	i
$((a \times A) \equiv (b \times B) \Rightarrow A \subseteq B)$,! 3 ($\forall E$: P10)	i
$(a \times A) \equiv (b \times B) \Rightarrow A \subseteq B$,! 4 ($(\Rightarrow)E$: 3)	i

$A \subseteq B$, ! 5 (\Rightarrow E: 2,4)	i
$((a \times A) \equiv (b \times B) \Rightarrow (b \times B) \equiv (a \times A))$, ! 6 (\forall E: C1.8)	i
$(a \times A) \equiv (b \times B) \Rightarrow (b \times B) \equiv (a \times A)$, ! 7 ($(\)$ E: 6)	i
$(b \times B) \equiv (a \times A)$, ! 8 (\Rightarrow E: 2,7)	i
$((b \times B) \equiv (a \times A) \Rightarrow B \subseteq A)$, ! 9 (\forall E: P10)	i
$(b \times B) \equiv (a \times A) \Rightarrow B \subseteq A$, ! 10 ($(\)$ E: 9)	i
$B \subseteq A$, ! 11 (\Rightarrow E: 8,10)	i
$A \subseteq B \ \& \ B \subseteq A$, ! 12 ($\&$ I: 5,11)	i
$(A \subseteq B \ \& \ B \subseteq A \Rightarrow A \equiv B)$, ! 13 (\forall E: III1.8)	i
$A \subseteq B \ \& \ B \subseteq A \Rightarrow A \equiv B$, ! 14 ($(\)$ E: 13)	i
$A \equiv B$, ! 15 (\Rightarrow E: 12,14)	i
$(a \times A) \equiv (b \times B) \Rightarrow A \equiv B$, ! 16 (\Rightarrow I: 2,15)	i
$((a \times A) \equiv (b \times B) \Rightarrow A \equiv B)$, ! 17 ($(\)$ I: 16)	i
$\forall a \forall b \forall A \forall B ((a \times A) \equiv (b \times B) \Rightarrow A \equiv B)$! 18 (\forall I: 1,17)	i

□

! Remark that in general it is not the case that $(a \times A) \equiv (b \times B)$ implies $a = b$. This is only the case if in addition $\neg A \equiv \phi$ (see P15). In fact, $(a \times \phi) \equiv (b \times \phi)$ for all a and b (see P14).

! 12.

$\vdash \forall a \forall B (B \equiv \phi \Rightarrow (a \times B) \equiv \Phi)$		i
a, B	, ! 1 (Prem)	i
$B \equiv \phi$, ! 2 (Prem)	i
$(B \equiv \phi \Rightarrow \neg \exists x B[x])$, ! 3 (\forall E II5.6)	i
$B \equiv \phi \Rightarrow \neg \exists x B[x]$, ! 4 ($(\)$ E: 3)	i
$\neg \exists x B[x]$, ! 5 (\Rightarrow E: 2,4)	i
$(\forall x \forall y \neg (a \times B)[x,y] \Rightarrow (a \times B) \equiv \Phi)$, ! 6 (\forall E: C4.8)	i

$\forall x \forall y \neg (a \times B)[x,y] \Rightarrow (a \times B) \equiv \Phi$,! 7 (()E: 6)	i
x, y	,! 8 (Prem)	i
$(a \times B)[x,y]$,! 9 (Prem)	i
$((a \times B)[x,y] \Rightarrow B[y])$,! 10 (\forall E: P5)	i
$(a \times B)[x,y] \Rightarrow B[y]$,! 11 (()E: 10)	i
$B[y]$,! 12 (\Rightarrow E: 9,11)	i
$\exists x B[x]$,! 13 (\exists I: 12)	i
\mathfrak{F}	,! 14 (\mathfrak{F} I: 5,13)	i
$(a \times B)[x,y] \Rightarrow \mathfrak{F}$,! 15 (\Rightarrow I: 9,14)	i
$\neg (a \times B)[x,y]$,! 16 (\neg I: 15)	i
$\forall x \forall y \neg (a \times B)[x,y]$,! 17 (\forall I: 8,16)	i
$(a \times B) \equiv \Phi$,! 18 (\Rightarrow E: 7,17)	i
$B \equiv \phi \Rightarrow (a \times B) \equiv \Phi$,! 19 (\Rightarrow I: 2,18)	i
$(B \equiv \phi \Rightarrow (a \times B) \equiv \Phi)$,! 20 (()I: 19)	i
$\forall a \forall B (B \equiv \phi \Rightarrow (a \times B) \equiv \Phi)$! 21 (\forall I: 1,20)	i

□

! 13.

$\vdash \forall a (a \times \phi) \equiv \Phi$		i
a	,! 1 (Prem)	i
$\phi \equiv \phi$,! 2 (\forall E: III1.9)	i
$(\phi \equiv \phi \Rightarrow (a \times \phi) \equiv \Phi)$,! 3 (\forall E: P12)	i
$\phi \equiv \phi \Rightarrow (a \times \phi) \equiv \Phi$,! 4 (()E: 3)	i
$(a \times \phi) \equiv \Phi$,! 5 (\Rightarrow E: 2,4)	i
$\forall a (a \times \phi) \equiv \Phi$! 6 (\forall I: 1,5)	i

□

! 14.

$\vdash \forall a \forall b (a \times \phi) \equiv (b \times \phi)$		i
a, b	,! 1 (Prem)	i

$(a X \phi) \equiv \Phi$,! 2 ($\forall E$: P13)	i
$(b X \phi) \equiv \Phi$,! 3 ($\forall E$: P13)	i
$(a X \phi) \equiv \Phi \ \& \ (b X \phi) \equiv \Phi$,! 4 ($\&I$: 2,3)	i
$((a X \phi) \equiv \Phi \ \& \ (b X \phi) \equiv \Phi \Rightarrow (a X \phi) \equiv (b X \phi))$,! 5 ($\forall E$: C1.17)	i
$(a X \phi) \equiv \Phi \ \& \ (b X \phi) \equiv \Phi \Rightarrow (a X \phi) \equiv (b X \phi)$,! 6 ($()E$: 5)	i
$(a X \phi) \equiv (b X \phi)$,! 7 ($\Rightarrow E$: 4,6)	i
$\forall a \forall b (a X \phi) \equiv (b X \phi)$! 8 ($\forall I$: 1,7)	i
\square		
! 15.		i
$\vdash \forall a \forall b \forall A \forall B (\neg A \equiv \phi \ \& \ (a X A) \equiv (b X B) \Rightarrow a = b)$		i
a, b, A, B	,! 1 (Prem)	i
$\neg A \equiv \phi \ \& \ (a X A) \equiv (b X B)$,! 2 (Prem)	i
$\neg A \equiv \phi$,! 3 ($\&E$: 2)	i
$(a X A) \equiv (b X B)$,! 4 ($\&E$: 2)	i
$(\neg A \equiv \phi \Rightarrow \exists x A[x])$,! 5 ($\forall E$: II5.16)	i
$\neg A \equiv \phi \Rightarrow \exists x A[x]$,! 6 ($()E$: 5)	i
$\exists x A[x]$,! 7 ($\Rightarrow E$: 3,6)	i
A[x]	,! 8 ($\exists E$: 7)	i
$(A[x] \Rightarrow (a X A)[a, x])$,! 9 ($\forall E$: P6)	i
$A[x] \Rightarrow (a X A)[a, x]$,! 10 ($()E$: 9)	i
$(a X A)[a, x]$,! 11 ($\Rightarrow E$: 8,10)	i
$(a X A)[a, x] \ \& \ (a X A) \equiv (b X B)$,! 12 ($\&I$: 4,11)	i
$((a X A)[a, x] \ \& \ (a X A) \equiv (b X B) \Rightarrow (b X B)[a, x])$,! 13 ($\forall E$: C1.20)	i
$(a X A)[a, x] \ \& \ (a X A) \equiv (b X B) \Rightarrow (b X B)[a, x]$,! 14 ($()E$: 13)	i
$(b X B)[a, x]$,! 15 ($\Rightarrow E$: 12,14)	i

$((b \times B)[a, x] \Rightarrow a = b)$,! 16 ($\forall E$: P4) ;
 $(b \times B)[a, x] \Rightarrow a = b$,! 17 ($(\)E$: 16) ;
 $a = b$,! 18 ($\Rightarrow E$: 15,17) ;
 $\neg A \equiv \phi \ \& \ (a \times A) \equiv (b \times B) \Rightarrow a = b$,! 19 ($\Rightarrow I$: 2,18) ;
 $(\neg A \equiv \phi \ \& \ (a \times A) \equiv (b \times B) \Rightarrow a = b)$,! 20 ($(\)I$: 19) ;
 $\forall a \forall b \forall A \forall B (\neg A \equiv \phi \ \& \ (a \times A) \equiv (b \times B) \Rightarrow a = b)$
! 21 ($\forall I$: 1,20) ;

□

! 16.

$\vdash \forall a \forall b \forall A \forall B (\neg A \equiv \phi \Rightarrow (a = b \ \& \ A \equiv B \Leftrightarrow (a \times A) \equiv (b \times B)))$;
 a, b, A, B ,! 1 (Prem) ;
 $(a = b \ \& \ A \equiv B \Rightarrow (a \times A) \equiv (b \times B))$,! 2 ($\forall E$: P9) ;
 $a = b \ \& \ A \equiv B \Rightarrow (a \times A) \equiv (b \times B)$,! 3 ($(\)E$: 2) ;
 $\neg A \equiv \phi$,! 4 (Prem) ;
 $(a \times A) \equiv (b \times B)$,! 5 (Prem) ;
 $((a \times A) \equiv (b \times B) \Rightarrow A \equiv B)$,! 6 ($\forall E$: P11) ;
 $(a \times A) \equiv (b \times B) \Rightarrow A \equiv B$,! 7 ($(\)E$: 6) ;
 $A \equiv B$,! 8 ($\Rightarrow E$: 5,7) ;
 $\neg A \equiv \phi \ \& \ (a \times A) \equiv (b \times B)$,! 9 ($\&I$: 4,5) ;
 $(\neg A \equiv \phi \ \& \ (a \times A) \equiv (b \times B) \Rightarrow a = b)$
,! 10 ($\forall E$: P15) ;
 $\neg A \equiv \phi \ \& \ (a \times A) \equiv (b \times B) \Rightarrow a = b$
,! 11 ($(\)E$: 10) ;
 $a = b$,! 12 ($\Rightarrow E$: 9,11) ;
 $a = b \ \& \ A \equiv B$,! 13 ($\&I$: 8,12) ;
 $(a \times A) \equiv (b \times B) \Rightarrow a = b \ \& \ A \equiv B$,! 14 ($\Rightarrow I$: 5,13) ;
 $a = b \ \& \ A \equiv B \Leftrightarrow (a \times A) \equiv (b \times B)$,! 15 ($\Leftrightarrow I$: 3,14) ;
 $(a = b \ \& \ A \equiv B \Leftrightarrow (a \times A) \equiv (b \times B))$,! 16 ($(\)I$: 15) ;
 $\neg A \equiv \phi \Rightarrow (a = b \ \& \ A \equiv B \Leftrightarrow (a \times A) \equiv (b \times B))$;

,! 17 (\Rightarrow I: 4,16) i

($\neg \mathbf{A} \equiv \phi \Rightarrow (\mathbf{a} = \mathbf{b} \ \& \ \mathbf{A} \equiv \mathbf{B} \Leftrightarrow (\mathbf{a} \ \mathbf{X} \ \mathbf{A}) \equiv (\mathbf{b} \ \mathbf{X} \ \mathbf{B}))$)
,! 18 ($()$ I: 17) i

$\forall \mathbf{a} \forall \mathbf{b} \forall \mathbf{A} \forall \mathbf{B}$ ($\neg \mathbf{A} \equiv \phi \Rightarrow (\mathbf{a} = \mathbf{b} \ \& \ \mathbf{A} \equiv \mathbf{B} \Leftrightarrow (\mathbf{a} \ \mathbf{X} \ \mathbf{A}) \equiv (\mathbf{b} \ \mathbf{X} \ \mathbf{B}))$)
! 19 ($(\forall$ I: 1,18) i

□

! 17. The domain of $(\mathbf{a} \ \mathbf{X} \ \mathbf{B})$ is included in the singleton predicate (\mathbf{a}^\bullet) . It is not always equivalent because, when \mathbf{B} is empty, the domain is empty (see P18). When \mathbf{B} is non-empty, then equivalence in fact holds (see P19). i

$\vdash \forall \mathbf{a} \forall \mathbf{B} ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$ i

\mathbf{a}, \mathbf{B} ,! 1 (Prem) i

($\forall \mathbf{x} \forall \mathbf{y} ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}]) \Rightarrow ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$)
,! 2 (\forall E: C5.11) i

$\forall \mathbf{x} \forall \mathbf{y} ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}]) \Rightarrow ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$
,! 3 ($()$ E: 2) i

\mathbf{x}, \mathbf{y} ,! 4 (Prem) i

$(\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}]$,! 5 (Prem) i

($(\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{a}$) ,! 6 (\forall E: P4) i

$(\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}] \Rightarrow \mathbf{x} = \mathbf{a}$,! 7 ($()$ E: 6) i

$\mathbf{x} = \mathbf{a}$,! 8 (\Rightarrow E: 5,7) i

$(\mathbf{a}^\bullet)[\mathbf{a}]$,! 9 (\forall E: II8.5) i

$(\mathbf{a}^\bullet)[\mathbf{x}]$,! 10 ($=$ E: 8,9) i

$(\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}]$,! 11 (\Rightarrow I: 5,10) i

$((\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}])$,! 12 ($()$ I: 11) i

$\forall \mathbf{x} \forall \mathbf{y} ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})[\mathbf{x}, \mathbf{y}] \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}])$,! 13 (\forall I: 4,12) i

$((\mathbf{a} \ \mathbf{X} \ \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$,! 14 (\Rightarrow E: 3,13) i

$\forall \mathbf{a} \forall \mathbf{B} ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$! 15 (\forall I: 1,14) i

□

! 18. i

$\vdash \forall \mathbf{a} \forall \mathbf{B} (\mathbf{B} \equiv \phi \Rightarrow ((\mathbf{a} \ \mathbf{X} \ \mathbf{B})^D) \equiv \phi)$ i

\mathbf{a}, \mathbf{B}	,! 1 (Prem)	i
$\mathbf{B} \equiv \phi$,! 2 (Prem)	i
$(\mathbf{B} \equiv \phi \Rightarrow (\mathbf{a} \times \mathbf{B}) \equiv \Phi)$,! 3 ($\forall E$: P12)	i
$\mathbf{B} \equiv \phi \Rightarrow (\mathbf{a} \times \mathbf{B}) \equiv \Phi$,! 4 ($(\)E$: 3)	i
$(\mathbf{a} \times \mathbf{B}) \equiv \Phi$,! 5 ($\Rightarrow E$: 2,4)	i
$((\mathbf{a} \times \mathbf{B}) \equiv \Phi \Rightarrow ((\mathbf{a} \times \mathbf{B})^D) \equiv \phi)$,! 6 ($\forall E$: C5.23)	i
$(\mathbf{a} \times \mathbf{B}) \equiv \Phi \Rightarrow ((\mathbf{a} \times \mathbf{B})^D) \equiv \phi$,! 7 ($(\)E$: 6)	i
$((\mathbf{a} \times \mathbf{B})^D) \equiv \phi$,! 8 ($\Rightarrow E$: 5,7)	i
$\mathbf{B} \equiv \phi \Rightarrow ((\mathbf{a} \times \mathbf{B})^D) \equiv \phi$,! 9 ($\Rightarrow I$: 2,8)	i
$(\mathbf{B} \equiv \phi \Rightarrow ((\mathbf{a} \times \mathbf{B})^D) \equiv \phi)$,! 10 ($(\)I$: 9)	i
$\forall \mathbf{a} \forall \mathbf{B} (\mathbf{B} \equiv \phi \Rightarrow ((\mathbf{a} \times \mathbf{B})^D) \equiv \phi)$! 11 ($\forall I$: 1,10)	i

□

! 19.

⊢ $\forall \mathbf{a} \forall \mathbf{B} (\neg \mathbf{B} \equiv \phi \Rightarrow ((\mathbf{a} \times \mathbf{B})^D) \equiv (\mathbf{a}^\bullet))$

\mathbf{a}, \mathbf{B}	,! 1 (Prem)	i
$\neg \mathbf{B} \equiv \phi$,! 2 (Prem)	i

! As P17 already has established that $((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$, it remains only to show that $(\mathbf{a}^\bullet) \subseteq ((\mathbf{a} \times \mathbf{B})^D)$.

$(\forall x((\mathbf{a}^\bullet)[x] \Rightarrow \exists y (\mathbf{a} \times \mathbf{B})[x,y]) \Rightarrow (\mathbf{a}^\bullet) \subseteq ((\mathbf{a} \times \mathbf{B})^D))$
, ! 3 ($\forall E$ C5.12)

$\forall x((\mathbf{a}^\bullet)[x] \Rightarrow \exists y (\mathbf{a} \times \mathbf{B})[x,y]) \Rightarrow (\mathbf{a}^\bullet) \subseteq ((\mathbf{a} \times \mathbf{B})^D)$
, ! 4 ($(\)E$: 3)

\mathbf{x} ,! 5 (Prem) i

$(\mathbf{a}^\bullet)[\mathbf{x}]$,! 6 (Prem) i

$((\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a})$,! 7 ($\forall E$: P3) i

$(\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a}$,! 8 ($(\)E$: 7) i

$\mathbf{x} = \mathbf{a}$,! 9 ($\Rightarrow E$: 6,8) i

$(\neg \mathbf{B} \equiv \phi \Rightarrow \exists x \mathbf{B}[x])$,! 10 ($\forall E$: II5.16) i

$\neg B \equiv \phi \Rightarrow \exists x B[x]$,! 11 ((E: 10)	i
$\exists x B[x]$,! 12 (\Rightarrow E: 2,11)	i
$B[b]$,! 13 (\exists E: 12)	i
$(B[b] \Rightarrow (a \times B)[a,b])$,! 14 (\forall E: P6)	i
$B[b] \Rightarrow (a \times B)[a,b]$,! 15 ((E: 14)	i
$(a \times B)[a,b]$,! 16 (\Rightarrow E: 13,15)	i
$(a \times B)[x,b]$,! 17 (=E: 9,16)	i
$\exists y (a \times B)[x,y]$,! 18 (\exists I: 17)	i
$(a^\bullet)[x] \Rightarrow \exists y (a \times B)[x,y]$,! 19 (\Rightarrow I: 6,18)	i
$((a^\bullet)[x] \Rightarrow \exists y (a \times B)[x,y])$,! 20 ((I: 19)	i
$\forall x((a^\bullet)[x] \Rightarrow \exists y (a \times B)[x,y])$,! 21 (\forall I: 5,20)	i
$(a^\bullet) \subseteq ((a \times B)^D)$,! 22 (\Rightarrow I: 4,21)	i
$((a \times B)^D) \subseteq (a^\bullet)$,! 23 (\forall E: P17)	i
$((a \times B)^D) \subseteq (a^\bullet) \ \& \ (a^\bullet) \subseteq ((a \times B)^D)$,! 24 ($\&$ I: 22,23)	i
$(((a \times B)^D) \subseteq (a^\bullet) \ \& \ (a^\bullet) \subseteq ((a \times B)^D) \Rightarrow ((a \times B)^D) \equiv (a^\bullet))$,! 25 (\forall E: III1.8)	i
$((a \times B)^D) \subseteq (a^\bullet) \ \& \ (a^\bullet) \subseteq ((a \times B)^D) \Rightarrow ((a \times B)^D) \equiv (a^\bullet)$,! 26 ((I: 25)	i
$((a \times B)^D) \equiv (a^\bullet)$,! 27 (\Rightarrow E: 24,26)	i
$\neg B \equiv \phi \Rightarrow ((a \times B)^D) \equiv (a^\bullet)$,! 28 (\Rightarrow I: 2,27)	i
$(\neg B \equiv \phi \Rightarrow ((a \times B)^D) \equiv (a^\bullet))$,! 29 ((I: 28)	i
$\forall a \forall B (\neg B \equiv \phi \Rightarrow ((a \times B)^D) \equiv (a^\bullet))$,! 30 (\forall I: 1,29)	i
\square		
! 20.		i
$\vdash \forall a \forall B \forall R \forall P ((R^D) \subseteq P \Rightarrow ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet)))$		i
a, B, R, P	,! 1 (Prem)	i

$(R^D) \subseteq P$,! 2 (Prem)	i
$((R \sqcup (a \times B))^D) \equiv ((R^D) \cup ((a \times B)^D))$,! 3 ($\forall E$: C5.19)	i
$((a \times B)^D) \subseteq (a^\bullet)$,! 4 ($\forall E$: P17)	i
$(R^D) \subseteq P \ \& \ ((a \times B)^D) \subseteq (a^\bullet)$,! 5 ($\&I$: 2,4)	i
$((R^D) \subseteq P \ \& \ ((a \times B)^D) \subseteq (a^\bullet) \Rightarrow ((R^D) \cup ((a \times B)^D)) \subseteq (P \cup (a^\bullet)))$,! 6 ($\forall E$: II2.32)	i
$(R^D) \subseteq P \ \& \ ((a \times B)^D) \subseteq (a^\bullet) \Rightarrow ((R^D) \cup ((a \times B)^D)) \subseteq (P \cup (a^\bullet))$,! 7 ($()E$: 6)	i
$((R^D) \cup ((a \times B)^D)) \subseteq (P \cup (a^\bullet))$,! 8 ($\Rightarrow E$: 5,7)	i
$((R \sqcup (a \times B))^D) \equiv ((R^D) \cup ((a \times B)^D))$ $\& \ ((R^D) \cup ((a \times B)^D)) \subseteq (P \cup (a^\bullet))$,! 9 ($\&I$: 3,8)	i
$(((R \sqcup (a \times B))^D) \equiv ((R^D) \cup ((a \times B)^D)) \ \& \ ((R^D) \cup ((a \times B)^D)) \subseteq (P \cup (a^\bullet)) \Rightarrow ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet)))$,! 10 ($\forall E$: III1.29)	i
$((R \sqcup (a \times B))^D) \equiv ((R^D) \cup ((a \times B)^D))$ $\& \ ((R^D) \cup ((a \times B)^D)) \subseteq (P \cup (a^\bullet))$ $\Rightarrow ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet))$,! 11 ($()E$: 10)	i
$((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet))$,! 12 ($\Rightarrow E$: 9,11)	i
$(R^D) \subseteq P \Rightarrow ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet))$,! 13 ($\Rightarrow I$: 2,12)	i
$((R^D) \subseteq P \Rightarrow ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet)))$,! 14 ($()I$: 13)	i
$\forall a \forall B \forall R \forall P ((R^D) \subseteq P \Rightarrow ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet)))$! 15 ($\forall I$: 1,14)	i

□

! 21. The image of $(a \times B)$ is equivalent to B . i

$\vdash \forall a \forall B ((a \times B)^I) \equiv B$ i

a, B ,! 1 (Prem) i

$(\forall y(\exists x (a \times B)[x,y] \Leftrightarrow B[y]) \Rightarrow ((a \times B)^I) \equiv B)$,! 2 ($\forall E$: C6.14)	i
$\forall y(\exists x (a \times B)[x,y] \Leftrightarrow B[y]) \Rightarrow ((a \times B)^I) \equiv B$,! 3 ($(\)E$: 2)	i
y	,! 4 (Prem)	i
$\exists x (a \times B)[x,y]$,! 5 (Prem)	i
$(a \times B)[x,y]$,! 6 ($\exists E$: 5)	i
$((a \times B)[x,y] \Rightarrow B[y])$,! 7 ($\forall E$: P5)	i
$(a \times B)[x,y] \Rightarrow B[y]$,! 8 ($(\)E$: 7)	i
$B[y]$,! 9 ($\Rightarrow E$: 6,8)	i
$\exists x (a \times B)[x,y] \Rightarrow B[y]$,! 10 ($\Rightarrow I$: 5,9)	i
$B[y]$,! 11 (Prem)	i
$(B[y] \Rightarrow (a \times B)[a,y])$,! 12 ($\forall E$: P6)	i
$B[y] \Rightarrow (a \times B)[a,y]$,! 13 ($(\)E$: 12)	i
$(a \times B)[a,y]$,! 14 ($\Rightarrow E$: 11,13)	i
$\exists x (a \times B)[x,y]$,! 15 ($\exists I$: 14)	i
$B[y] \Rightarrow \exists x (a \times B)[x,y]$,! 16 ($\Rightarrow I$: 11,15)	i
$\exists x (a \times B)[x,y] \Leftrightarrow B[y]$,! 17 ($\Leftrightarrow I$: 10,16)	i
$(\exists x (a \times B)[x,y] \Leftrightarrow B[y])$,! 18 ($(\)I$: 17)	i
$\forall y(\exists x (a \times B)[x,y] \Leftrightarrow B[y])$,! 19 ($\forall I$: 4,18)	i
$((a \times B)^I) \equiv B$,! 20 ($\Rightarrow I$: 3,19)	i
$\forall a \forall B ((a \times B)^I) \equiv B$! 21 ($\forall I$: 1,20)	i
\square		
! 22.		i
$\vdash \forall a \forall B \forall R (R \equiv (a \times B) \Rightarrow (R^I) \equiv B)$		i
a, B, R	,! 1 (Prem)	i
$R \equiv (a \times B)$,! 2 (Prem)	i
$(R \equiv (a \times B) \Rightarrow (R^I) \equiv ((a \times B)^I))$,! 3 ($\forall E$: C6.22)	i
$R \equiv (a \times B) \Rightarrow (R^I) \equiv ((a \times B)^I)$,! 4 ($(\)E$: 3)	i

$(R^I) \equiv ((a \times B)^I)$,! 5 ($\Rightarrow E$: 2,4) ;
 $((a \times B)^I) \equiv B$,! 6 ($\forall E$: P21) ;
 $(R^I) \equiv ((a \times B)^I) \ \& \ ((a \times B)^I) \equiv B$,! 7 ($\&I$: 5,6) ;
 $((R^I) \equiv ((a \times B)^I) \ \& \ ((a \times B)^I) \equiv B \Rightarrow (R^I) \equiv B)$
, ! 8 ($\forall E$: III.15) ;
 $(R^I) \equiv ((a \times B)^I) \ \& \ ((a \times B)^I) \equiv B \Rightarrow (R^I) \equiv B$
, ! 9 ($()E$: 8) ;
 $(R^I) \equiv B$,! 10 ($\Rightarrow E$: 7,9) ;
 $R \equiv (a \times B) \Rightarrow (R^I) \equiv B$,! 11 ($\Rightarrow I$: 2,10) ;
 $(R \equiv (a \times B) \Rightarrow (R^I) \equiv B)$,! 12 ($()I$: 11) ;
 $\forall a \forall B \forall R (R \equiv (a \times B) \Rightarrow (R^I) \equiv B)$! 13 ($\forall I$: 1,12) ;
 \square
! 23. ;
 $\vdash \forall a \forall B \forall R ((R \sqcup (a \times B))^I) \equiv ((R^I) \cup B)$;
 a, B, R ,! 1 (Prem) ;
 $((R \sqcup (a \times B))^I) \equiv ((R^I) \cup ((a \times B)^I))$
, ! 2 ($\forall E$: C6.22) ;
 $((a \times B)^I) \equiv B$,! 3 ($\forall E$: P21) ;
 $(((a \times B)^I) \equiv B \Rightarrow ((R^I) \cup ((a \times B)^I)) \equiv ((R^I) \cup B))$
, ! 4 ($\forall E$: II2.37) ;
 $((a \times B)^I) \equiv B \Rightarrow ((R^I) \cup ((a \times B)^I)) \equiv ((R^I) \cup B)$
, ! 5 ($()E$: 4) ;
 $((R^I) \cup ((a \times B)^I)) \equiv ((R^I) \cup B)$,! 6 ($\Rightarrow E$: 3,5) ;
 $((R \sqcup (a \times B))^I) \equiv ((R^I) \cup ((a \times B)^I))$
 $\& ((R^I) \cup ((a \times B)^I)) \equiv ((R^I) \cup B)$
, ! 7 ($\&I$: 2,6) ;
 $(((R \sqcup (a \times B))^I) \equiv ((R^I) \cup ((a \times B)^I))$
 $\& ((R^I) \cup ((a \times B)^I)) \equiv ((R^I) \cup B)$
 $\Rightarrow ((R \sqcup (a \times B))^I) \equiv ((R^I) \cup B))$
, ! 8 ($\forall E$: III.15) ;
 $((R \sqcup (a \times B))^I) \equiv ((R^I) \cup ((a \times B)^I))$

$\& ((\mathbf{R}^I) \cup ((\mathbf{a} \times \mathbf{B})^I)) \equiv ((\mathbf{R}^I) \cup \mathbf{B})$
 $\Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^I) \equiv ((\mathbf{R}^I) \cup \mathbf{B})$,! 9 ((E: 8) i
 $((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^I) \equiv ((\mathbf{R}^I) \cup \mathbf{B})$,! 10 (\Rightarrow E: 7,9) i
 $\forall a \forall B \forall R ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^I) \equiv ((\mathbf{R}^I) \cup \mathbf{B})$! 11 (\forall I: 1,10) i

□

! P24 through P27 make various assertions about the restrictions of simple Cartesian products to singletons. i

! 24. i

$\vdash \forall a \forall B ((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$ i
 \mathbf{a}, \mathbf{B} ,! 1 (Prem) i
 $((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$,! 2 (\forall E: P17) i
 $((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet) \Rightarrow ((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$,! 3 (\forall E: C7.24) i
 $((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet) \Rightarrow ((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$,! 4 ((E: 3) i
 $((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$,! 5 (\Rightarrow E: 2,4) i
 $\forall a \forall B ((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$! 6 (\forall I: 1,5) i

□

! 25. i

$\vdash \forall a \forall x \forall B (\neg x = a \Rightarrow ((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{x}^\bullet)) \equiv \Phi)$ i
 $\mathbf{a}, \mathbf{x}, \mathbf{B}$,! 1 (Prem) i
 $\neg \mathbf{x} = \mathbf{a}$,! 2 (Prem) i
 $(\neg ((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}] \Rightarrow ((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{x}^\bullet)) \equiv \Phi)$,! 3 (\forall E: C7.37) i
 $\neg ((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}] \Rightarrow ((\mathbf{a} \times \mathbf{B}) \upharpoonright (\mathbf{x}^\bullet)) \equiv \Phi$,! 4 ((E: 3) i
 $((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}]$,! 5 (Prem) i
 $((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$,! 6 (\forall E: P17) i
 $((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}] \& ((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet)$,! 7 ($\&$ I: 5,6) i
 $((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}] \& ((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet) \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}]$,! 8 (\forall E: III.2) i

$((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}] \ \& \ ((\mathbf{a} \times \mathbf{B})^D) \subseteq (\mathbf{a}^\bullet) \Rightarrow (\mathbf{a}^\bullet)[\mathbf{x}]$,! 9 (()E: 8)	i
$(\mathbf{a}^\bullet)[\mathbf{x}]$,! 10 (\Rightarrow E: 7,9)	i
$((\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a})$,! 11 (\forall E: II8.3)	i
$(\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a}$,! 12 (()E: 11)	i
$\mathbf{x} = \mathbf{a}$,! 13 (\Rightarrow E: 10,12)	i
\mathfrak{F}	,! 14 (\mathfrak{F} I: 2,13)	i
$((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}] \Rightarrow \mathfrak{F}$,! 15 (\Rightarrow I: 5,14)	i
$\neg ((\mathbf{a} \times \mathbf{B})^D)[\mathbf{x}]$,! 16 (\neg I: 15)	i
$((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{x}^\bullet)) \equiv \Phi$,! 17 (\Rightarrow E: 4,16)	i
$\neg \mathbf{x} = \mathbf{a} \Rightarrow ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{x}^\bullet)) \equiv \Phi$,! 18 (\Rightarrow I: 2,17)	i
$(\neg \mathbf{x} = \mathbf{a} \Rightarrow ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{x}^\bullet)) \equiv \Phi)$,! 19 (()I: 18)	i
$\forall \mathbf{a} \forall \mathbf{x} \forall \mathbf{B} (\neg \mathbf{x} = \mathbf{a} \Rightarrow ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{x}^\bullet)) \equiv \Phi)$! 20 (\forall I: 1,19)	i
\square		
! 26.		i
$\vdash \forall \mathbf{a} \forall \mathbf{B} \forall \mathbf{R} (\neg (\mathbf{R}^D)[\mathbf{a}] \Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \vdash (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B}))$		i
$\mathbf{a}, \mathbf{B}, \mathbf{R}$,! 1 (Prem)	i
$\neg (\mathbf{R}^D)[\mathbf{a}]$,! 2 (Prem)	i
$(\neg (\mathbf{R}^D)[\mathbf{a}] \Rightarrow (\mathbf{R} \vdash (\mathbf{a}^\bullet)) \equiv \Phi)$,! 3 (\forall E: C7.37)	i
$\neg (\mathbf{R}^D)[\mathbf{a}] \Rightarrow (\mathbf{R} \vdash (\mathbf{a}^\bullet)) \equiv \Phi$,! 4 (()E: 3)	i
$(\mathbf{R} \vdash (\mathbf{a}^\bullet)) \equiv \Phi$,! 5 (\Rightarrow E: 2,4)	i
$((\mathbf{R} \vdash (\mathbf{a}^\bullet)) \equiv \Phi$ $\Rightarrow ((\mathbf{R} \vdash (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{a}^\bullet))) \equiv ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{a}^\bullet)))$,! 6 (\forall E: C4.13)	i
$(\mathbf{R} \vdash (\mathbf{a}^\bullet)) \equiv \Phi$ $\Rightarrow ((\mathbf{R} \vdash (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{a}^\bullet))) \equiv ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{a}^\bullet))$,! 7 (()E: 6)	i
$(\mathbf{R} \vdash (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{a}^\bullet)) \equiv ((\mathbf{a} \times \mathbf{B}) \vdash (\mathbf{a}^\bullet))$		

, ! 8 (\Rightarrow E: 5,7) i

$((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$, ! 9 (\forall E: P24) i

$((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))) \equiv ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))$
 $\& ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$
, ! 10 ($\&$ I: 8,9) i

$((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv ((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)))$
, ! 11 (\forall E: C7.19) i

$((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv ((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)))$
 $\& ((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))) \equiv ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))$
 $\& ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$
, ! 12 ($\&$ I: 10,11) i

$((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv ((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)))$
 $\& ((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))) \equiv ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))$
 $\& ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$
 $\Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$
, ! 13 (\forall E: C1.19) i

$((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv ((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)))$
 $\& ((\mathbf{R} \uparrow (\mathbf{a}^\bullet)) \sqcup ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))) \equiv ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet))$
 $\& ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$
 $\Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$
, ! 14 ($(\)$ E: 13) i

$((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$, ! 15 (\Rightarrow E: 12,14) i

$\neg (\mathbf{R}^D)[\mathbf{a}] \Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B})$
, ! 16 (\Rightarrow I: 2,15) i

$(\neg (\mathbf{R}^D)[\mathbf{a}] \Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B}))$
, ! 17 ($(\)$ I: 16) i

$\forall \mathbf{a} \forall \mathbf{B} \forall \mathbf{R} (\neg (\mathbf{R}^D)[\mathbf{a}] \Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \times \mathbf{B}))$
! 18 (\forall I: 1,17) i

□

! 27.

$\vdash \forall \mathbf{a} \forall \mathbf{x} \forall \mathbf{B} \forall \mathbf{R} (\neg \mathbf{x} = \mathbf{a} \Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \uparrow (\mathbf{x}^\bullet)) \equiv (\mathbf{R} \uparrow (\mathbf{x}^\bullet)))$ i

$\mathbf{a}, \mathbf{x}, \mathbf{B}, \mathbf{R}$, ! 1 (Prem) i

$\neg \mathbf{x} = \mathbf{a}$, ! 2 (Prem) i

$(\neg \mathbf{x} = \mathbf{a} \Rightarrow ((\mathbf{a} \times \mathbf{B}) \uparrow (\mathbf{x}^\bullet)) \equiv \Phi)$, ! 3 (\forall E: P25) i

$$\neg x = a \Rightarrow ((a X B) \lceil (x^\bullet)) \equiv \Phi \quad ,! 4 ((\text{E}: 3) \quad ;$$

$$((a X B) \lceil (x^\bullet)) \equiv \Phi \quad ,! 5 (\Rightarrow\text{E}: 2,4) \quad ;$$

$$\begin{aligned} & ((a X B) \lceil (x^\bullet)) \equiv \Phi \\ \Rightarrow & ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \equiv (R \lceil (x^\bullet)) \end{aligned} \quad ,! 6 (\forall\text{E}: C4.12) \quad ;$$

$$\begin{aligned} & ((a X B) \lceil (x^\bullet)) \equiv \Phi \\ \Rightarrow & ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \equiv (R \lceil (x^\bullet)) \end{aligned} \quad ,! 7 ((\text{E}: 6) \quad ;$$

$$((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \equiv (R \lceil (x^\bullet)) \quad ,! 8 (\Rightarrow\text{E}: 5,7) \quad ;$$

$$((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \quad ,! 9 (\forall\text{E}: C7.19) \quad ;$$

$$\begin{aligned} & ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \\ & \& ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \equiv (R \lceil (x^\bullet)) \end{aligned} \quad ,! 10 (\&\text{I}: 8,9) \quad ;$$

$$\begin{aligned} & ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \\ & \& ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \equiv (R \lceil (x^\bullet)) \\ \Rightarrow & ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv (R \lceil (x^\bullet)) \end{aligned} \quad ,! 11 (\forall\text{E}: C1.15) \quad ;$$

$$\begin{aligned} & ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \\ & \& ((R \lceil (x^\bullet)) \sqcup ((a X B) \lceil (x^\bullet))) \equiv (R \lceil (x^\bullet)) \\ \Rightarrow & ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv (R \lceil (x^\bullet)) \end{aligned} \quad ,! 12 ((\text{E}: 11) \quad ;$$

$$((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv (R \lceil (x^\bullet)) \quad ,! 13 (\Rightarrow\text{E}: 10,12) \quad ;$$

$$\neg x = a \Rightarrow ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv (R \lceil (x^\bullet)) \quad ,! 14 (\Rightarrow\text{I}: 2,13) \quad ;$$

$$(\neg x = a \Rightarrow ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv (R \lceil (x^\bullet))) \quad ,! 15 ((\text{I}: 14) \quad ;$$

$$\forall a \forall x \forall B \forall R (\neg x = a \Rightarrow ((R \sqcup (a X B)) \lceil (x^\bullet)) \equiv (R \lceil (x^\bullet))) \quad ! 16 (\forall\text{I}: 1,15) \quad ;$$

□

! 28. i

⊢ $\forall a \forall B \mathbf{1} (a X B)$ i

a, B ,! 1 (Prem) i

x, y, z	,! 2 (Prem)	i
$(a \times B)[x, y] \ \& \ (a \times B)[z, y]$,! 3 (Prem)	i
$(a \times B)[x, y]$,! 4 (&E: 3)	i
$(a \times B)[z, y]$,! 5 (&E: 3)	i
$((a \times B)[x, y] \Rightarrow x = a)$,! 6 (\forall E: P4)	i
$(a \times B)[x, y] \Rightarrow x = a$,! 7 (()E: 6)	i
$x = a$,! 8 (\Rightarrow E: 4,7)	i
$((a \times B)[z, y] \Rightarrow z = a)$,! 9 (\forall E: P4)	i
$(a \times B)[z, y] \Rightarrow z = a$,! 10 (()E: 9)	i
$z = a$,! 11 (\Rightarrow E: 5,10)	i
$x = z$,! 12 (=E: 8,11)	i
$(a \times B)[x, y] \ \& \ (a \times B)[z, y] \Rightarrow x = z$,! 13 (\Rightarrow I: 3,12)	i
$((a \times B)[x, y] \ \& \ (a \times B)[z, y] \Rightarrow x = z)$,! 14 (()I: 13)	i
$\forall x \forall y \forall z ((a \times B)[x, y] \ \& \ (a \times B)[z, y] \Rightarrow x = z)$,! 15 (\forall I: 2,14)	i
1 $(a \times B)$,! 16 ($\$$ I: C9.1,15)	i
$\forall a \forall B$ 1 $(a \times B)$! 17 (\forall I: 1,16)	i
\square		
! 29.		i
$\vdash \forall a \forall B (a \times B) \ \mathbb{1} \ B$		i
a, B	,! 1 (Prem)	i
$((a \times B)^{\mathbb{I}}) \equiv B$,! 2 (\forall E: P21)	i
1 $(a \times B)$,! 3 (\forall E: P28)	i
$((a \times B)^{\mathbb{I}}) \equiv B \ \& \ \mathbf{1} \ (a \times B)$,! 4 (&I: 2,3)	i
$(a \times B) \ \mathbb{1} \ B$,! 5 ($\$$ I: C9.19)	i
$\forall a \forall B (a \times B) \ \mathbb{1} \ B$! 6 (\forall I: 1,5)	i
\square		
! 30.		i

$$\vdash \forall a \forall A \forall B \forall R (R \Vdash A \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup (a \times B)) \Vdash (A \cup B)) \quad i$$

a, A, B, R ,! 1 (Prem) i
 $R \Vdash A \ \& \ (A \cap B) \equiv \phi$,! 2 (Prem) i
 $(a \times B) \Vdash B$,! 3 ($\forall E$: P29) i
 $R \Vdash A \ \& \ (a \times B) \Vdash B \ \& \ (A \cap B) \equiv \phi$,! 4 ($\&I$: 2,3) i
 $(R \Vdash A \ \& \ (a \times B) \Vdash B \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup (a \times B)) \Vdash (A \cup B))$,! 5 ($\forall E$: C9.31) i
 $R \Vdash A \ \& \ (a \times B) \Vdash B \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup (a \times B)) \Vdash (A \cup B)$,! 6 ($(\Rightarrow)E$: 5) i
 $(R \sqcup (a \times B)) \Vdash (A \cup B)$,! 7 ($\Rightarrow E$: 4,6) i
 $R \Vdash A \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup (a \times B)) \Vdash (A \cup B)$,! 8 ($\Rightarrow I$: 2,7) i
 $(R \Vdash A \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup (a \times B)) \Vdash (A \cup B))$,! 9 ($(\Rightarrow)I$: 8) i
 $\forall a \forall A \forall B \forall R (R \Vdash A \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup (a \times B)) \Vdash (A \cup B))$! 10 ($\forall I$: 1,9) i

□

! 31. P31 is positioned here since it follows easily as a corollary to P30. i

$$\vdash \forall a \forall B \forall R (\mathbf{1} R \ \& \ ((R^I) \cap B) \equiv \phi \Rightarrow \mathbf{1} (R \sqcup (a \times B))) \quad i$$

a, B, R ,! 1 (Prem) i
 $\mathbf{1} R \ \& \ ((R^I) \cap B) \equiv \phi$,! 2 (Prem) i
 $\mathbf{1} R$,! 3 ($\&E$: 2) i
 $((R^I) \cap B) \equiv \phi$,! 4 ($\&E$: 2) i
 $(\mathbf{1} R \Rightarrow R \Vdash (R^I))$,! 5 ($\forall E$: C9.29) i
 $\mathbf{1} R \Rightarrow R \Vdash (R^I)$,! 6 ($(\Rightarrow)E$: 5) i
 $R \Vdash (R^I)$,! 7 ($\Rightarrow E$: 3,6) i
 $R \Vdash (R^I) \ \& \ ((R^I) \cap B) \equiv \phi$,! 8 ($\&I$: 4,7) i
 $(R \Vdash (R^I) \ \& \ ((R^I) \cap B) \equiv \phi \Rightarrow (R \sqcup (a \times B)) \Vdash ((R^I) \cup B))$,! 9 ($\forall E$: P30) i

$R \Vdash (R^I) \ \& \ ((R^I) \cap B) \equiv \phi \Rightarrow (R \sqcup (a \ X \ B)) \Vdash ((R^I) \cup B)$
, ! 10 (()E: 9) i

$(R \sqcup (a \ X \ B)) \Vdash ((R^I) \cup B)$, ! 11 (\Rightarrow E: 8,10) i

$((R \sqcup (a \ X \ B))^I) \equiv ((R^I) \cup B) \ \& \ \mathbf{1} \ (R \sqcup (a \ X \ B))$
, ! 12 (\S E: C9.19,11) i

$\mathbf{1} \ (R \sqcup (a \ X \ B))$, ! 13 ($\&$ E: 12) i

$\mathbf{1} \ R \ \& \ ((R^I) \cap B) \equiv \phi \Rightarrow \mathbf{1} \ (R \sqcup (a \ X \ B))$
, ! 14 (\Rightarrow I: 2,13) i

$(\mathbf{1} \ R \ \& \ ((R^I) \cap B) \equiv \phi \Rightarrow \mathbf{1} \ (R \sqcup (a \ X \ B)))$
, ! 15 (()I: 14) i

$\forall a \forall B \forall R \ (\mathbf{1} \ R \ \& \ ((R^I) \cap B) \equiv \phi \Rightarrow \mathbf{1} \ (R \sqcup (a \ X \ B)))$
! 16 (\forall I: 1,15) i

□