

! CHAPTER 9

1-1 PREDICATES;

! This chapter introduces the notion of one-to-one relationships, and the argument of a one-to-one relationship given its value. Another \mathbb{T} -definition--for $(R \backslash y)$ --appears.

1 R is used to say that R is a one-to-one relationship,

$R \mathbb{1} B$ means that R is a one-to-one relationship with image B , and

$(R \backslash y)$ refers to the (unique) thing which bears R to y , a definition which pre-supposes that R is one-to-one and y belongs to the image of R .

There is evidently duality between one-to-one and functional relationships through the inverse. That is, R is one-to-one if and only if the inverse is functional. This fact is established almost immediately, in P3-P8, so it can be used in subsequent proofs. i

! 1. **1** R says that R is a one-to-one relationship. i

\S **1** ; **1** R ; $\forall x \forall y \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$ i

! 2. i

$\vdash \forall R \forall x \forall y \forall z (\mathbf{1} R \ \& \ R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$ i

R, x, y, z ,! 1 (Prem) i

1 $R \ \& \ R[x,y] \ \& \ R[z,y]$,! 2 (Prem) i

1 R ,! 3 (&E: 2) i

$R[x,y] \ \& \ R[z,y]$,! 4 (&E: 2) i

$\forall x \forall y \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$,! 5 ($\mathbb{S}E$: P1,3) i

$(R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$,! 6 ($\forall E$: 5) i

$R[x,y] \ \& \ R[z,y] \Rightarrow x = z$,! 7 ((\Rightarrow)E: 6) i

$x = z$,! 8 ($\Rightarrow E$: 4,7) i

1 $R \ \& \ R[x,y] \ \& \ R[z,y] \Rightarrow x = z$,! 9 ($\Rightarrow I$: 2,8) i

$(\mathbf{1} R \ \& \ R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$,! 10 ((\Rightarrow)I: 9) i

$\forall R \forall x \forall y \forall z (\mathbf{1} R \ \& \ R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$

! 11 ($\forall I$: 1,10) i

\square

! P3-P8 establish duality between one-to-one and functional relationships. Almost all this chapter's subsequent propositions are proven, directly or indirectly, using these duality results. i

! 3. i

$\vdash \forall R (\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$		i
R	,! 1 (Prem)	i
$\mathbf{1} R$,! 2 (Prem)	i
$\mathbf{x}, \mathbf{y}, \mathbf{z}$,! 3 (Prem)	i
$(R^*)[\mathbf{x}, \mathbf{y}] \ \& \ (R^*)[\mathbf{x}, \mathbf{z}]$,! 4 (Prem)	i
$(R^*)[\mathbf{x}, \mathbf{y}]$,! 5 (&E)	i
$((R^*)[\mathbf{x}, \mathbf{y}] \Rightarrow R[\mathbf{y}, \mathbf{x}])$,! 6 (\forall E: C3.3)	i
$(R^*)[\mathbf{x}, \mathbf{y}] \Rightarrow R[\mathbf{y}, \mathbf{x}]$,! 7 (()E: 6)	i
$R[\mathbf{y}, \mathbf{x}]$,! 8 (\Rightarrow E: 5,7)	i
$(R^*)[\mathbf{x}, \mathbf{z}]$,! 9 (&E: 4)	i
$((R^*)[\mathbf{x}, \mathbf{z}] \Rightarrow R[\mathbf{z}, \mathbf{x}])$,! 10 (\forall E: C3.3)	i
$(R^*)[\mathbf{x}, \mathbf{z}] \Rightarrow R[\mathbf{z}, \mathbf{x}]$,! 11 (()E: 10)	i
$R[\mathbf{z}, \mathbf{x}]$,! 12 (\Rightarrow E: 9,11)	i
$R[\mathbf{y}, \mathbf{x}] \ \& \ R[\mathbf{z}, \mathbf{x}]$,! 13 (&I: 8,12)	i
$\mathbf{1} R \ \& \ R[\mathbf{y}, \mathbf{x}] \ \& \ R[\mathbf{z}, \mathbf{x}]$,! 14 (&I: 2,13)	i
$(\mathbf{1} R \ \& \ R[\mathbf{y}, \mathbf{x}] \ \& \ R[\mathbf{z}, \mathbf{x}] \Rightarrow \mathbf{y} = \mathbf{z})$,! 15 (\forall E: P2)	i
$\mathbf{1} R \ \& \ R[\mathbf{y}, \mathbf{x}] \ \& \ R[\mathbf{z}, \mathbf{x}] \Rightarrow \mathbf{y} = \mathbf{z}$,! 16 (()E: 15)	i
$\mathbf{y} = \mathbf{z}$,! 17 (\Rightarrow E: 14,16)	i
$(R^*)[\mathbf{x}, \mathbf{y}] \ \& \ (R^*)[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{y} = \mathbf{z}$,! 18 (\Rightarrow I: 4,17)	i
$((R^*)[\mathbf{x}, \mathbf{y}] \ \& \ (R^*)[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{y} = \mathbf{z})$,! 19 (()I: 18)	i
$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} ((R^*)[\mathbf{x}, \mathbf{y}] \ \& \ (R^*)[\mathbf{x}, \mathbf{z}] \Rightarrow \mathbf{y} = \mathbf{z})$,! 20 (\forall I: 3,19)	i
$\mathbf{f} (R^*)$,! 21 (\mathbb{S} I: C8.1,20)	i
$\mathbf{1} R \Rightarrow \mathbf{f} (R^*)$,! 22 (\Rightarrow I: 2,21)	i
$(\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$,! 23 (()I: 22)	i
$\forall R (\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$! 24 (\forall I: 1,23)	i
\square		
! 4.		i

$\vdash \forall R (f (R^*) \Rightarrow 1 R)$		i
R	,! 1 (Prem)	i
$f (R^*)$,! 2 (Prem)	i
x, y, z	,! 3 (Prem)	i
$R[x, y] \ \& \ R[z, y]$,! 4 (Prem)	i
$R[x, y]$,! 5 (&E: 4)	i
$(R[x, y] \Rightarrow (R^*)[y, x])$,! 6 (\forall E: C3.4)	i
$R[x, y] \Rightarrow (R^*)[y, x]$,! 7 (()E: 6)	i
$(R^*)[y, x]$,! 8 (\Rightarrow E: 5,7)	i
$R[z, y]$,! 9 (&E: 4)	i
$(R[z, y] \Rightarrow (R^*)[y, z])$,! 10 (\forall E: C3.4)	i
$R[z, y] \Rightarrow (R^*)[y, z]$,! 11 (()E: 10)	i
$(R^*)[y, z]$,! 12 (\Rightarrow E: 9,11)	i
$(R^*)[y, x] \ \& \ (R^*)[y, z]$,! 13 (&I: 8,12)	i
$f (R^*) \ \& \ (R^*)[y, x] \ \& \ (R^*)[y, z]$,! 14 (&I: 2,13)	i
$(f (R^*) \ \& \ (R^*)[y, x] \ \& \ (R^*)[y, z] \Rightarrow x = z)$,! 15 (\forall E: C8.2)	i
$f (R^*) \ \& \ (R^*)[y, x] \ \& \ (R^*)[y, z] \Rightarrow x = z$,! 16 (()E: 15)	i
$x = z$,! 17 (\Rightarrow E: 14,16)	i
$R[x, y] \ \& \ R[z, y] \Rightarrow x = z$,! 18 (\Rightarrow I: 4,17)	i
$(R[x, y] \ \& \ R[z, y] \Rightarrow x = z)$,! 19 (()I: 18)	i
$\forall x \forall y \forall z (R[x, y] \ \& \ R[z, y] \Rightarrow x = z)$,! 20 (\forall I: 19)	i
$1 R$,! 21 (\S I: P1,20)	i
$f (R^*) \Rightarrow 1 R$,! 22 (\Rightarrow I: 2,21)	i
$(f (R^*) \Rightarrow 1 R)$,! 23 (()I: 22)	i
$\forall R (f (R^*) \Rightarrow 1 R)$! 24 (\forall I: 1,23)	i

□

! 5.		i
$\vdash \forall R (\mathbf{1} R \Leftrightarrow \mathbf{f} (R^*))$		i
R	,! 1 (Prem)	i
$(\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$,! 2 ($\forall E$: P3)	i
$\mathbf{1} R \Rightarrow \mathbf{f} (R^*)$,! 3 ($()E$: 2)	i
$(\mathbf{f} (R^*) \Rightarrow \mathbf{1} R)$,! 4 ($\forall E$: P4)	i
$\mathbf{f} (R^*) \Rightarrow \mathbf{1} R$,! 5 ($()E$: 4)	i
$\mathbf{1} R \Leftrightarrow \mathbf{f} (R^*)$,! 6 ($\Leftrightarrow E$: 3,5)	i
$(\mathbf{1} R \Leftrightarrow \mathbf{f} (R^*))$,! 7 ($()I$: 6)	i
$\forall R (\mathbf{1} R \Leftrightarrow \mathbf{f} (R^*))$! 8 ($\forall I$: 1,7)	i

□

! 6.		i
$\vdash \forall R (\mathbf{f} R \Leftrightarrow \mathbf{1} (R^*))$		i
R	,! 1 (Prem)	i
$(\mathbf{1} (R^*) \Leftrightarrow \mathbf{f} ((R^*)^*))$,! 2 ($\forall E$: P5)	i
$\mathbf{1} (R^*) \Leftrightarrow \mathbf{f} ((R^*)^*)$,! 3 ($()E$: 2)	i
$R \equiv ((R^*)^*)$,! 4 ($\forall E$: C3.18)	i
$\mathbf{f} R$,! 5 (Prem)	i
$\mathbf{f} R \ \& \ R \equiv ((R^*)^*)$,! 6 ($\&I$: 4,5)	i
$(\mathbf{f} R \ \& \ R \equiv ((R^*)^*) \Rightarrow \mathbf{f} ((R^*)^*))$,! 7 ($\forall E$: C8.5)	i
$\mathbf{f} R \ \& \ R \equiv ((R^*)^*) \Rightarrow \mathbf{f} ((R^*)^*)$,! 8 ($()E$: 7)	i
$\mathbf{f} ((R^*)^*)$,! 9 ($\Rightarrow E$: 6,8)	i
$\mathbf{f} ((R^*)^*) \Rightarrow \mathbf{1} (R^*)$,! 10 ($\Leftrightarrow E$: 3)	i
$\mathbf{1} (R^*)$,! 11 ($\Rightarrow E$: 9,10)	i
$\mathbf{f} R \Rightarrow \mathbf{1} (R^*)$,! 12 ($\Rightarrow I$: 5,11)	i
$\mathbf{1} (R^*)$,! 13 (Prem)	i
$\mathbf{1} (R^*) \Rightarrow \mathbf{f} ((R^*)^*)$,! 14 ($\Leftrightarrow E$: 3)	i

$\mathbf{f} ((\mathbf{R}^*)^*)$,!	15	(\Rightarrow E: 13,14)	i
$\mathbf{f} ((\mathbf{R}^*)^*) \ \& \ \mathbf{R} \equiv ((\mathbf{R}^*)^*)$,!	16	(&I: 4,15)	i
$(\mathbf{f} ((\mathbf{R}^*)^*) \ \& \ \mathbf{R} \equiv ((\mathbf{R}^*)^*) \Rightarrow \mathbf{f} \mathbf{R})$,!	17	(\forall E: C8.6)	i
$\mathbf{f} ((\mathbf{R}^*)^*) \ \& \ \mathbf{R} \equiv ((\mathbf{R}^*)^*) \Rightarrow \mathbf{f} \mathbf{R}$,!	19	($()$ E: 17)	i
$\mathbf{f} \mathbf{R}$,!	20	(\Rightarrow E: 16,19)	i
$\mathbf{1} (\mathbf{R}^*) \Rightarrow \mathbf{f} \mathbf{R}$,!	21	(\Rightarrow I: 13,20)	i
$\mathbf{f} \mathbf{R} \Leftrightarrow \mathbf{1} (\mathbf{R}^*)$,!	22	(\Leftrightarrow I: 12,21)	i
$(\mathbf{f} \mathbf{R} \Leftrightarrow \mathbf{1} (\mathbf{R}^*))$,!	23	($()$ I: 22)	i
$\forall \mathbf{R} (\mathbf{f} \mathbf{R} \Leftrightarrow \mathbf{1} (\mathbf{R}^*))$!	24	(\forall I: 1,23)	i
\square				
! 7.				i
$\vdash \forall \mathbf{R} (\mathbf{f} \mathbf{R} \Rightarrow \mathbf{1} (\mathbf{R}^*))$				i
\mathbf{R}	,!	1	(Prem)	i
$(\mathbf{f} \mathbf{R} \Leftrightarrow \mathbf{1} (\mathbf{R}^*))$,!	2	(\forall E: P6)	i
$\mathbf{f} \mathbf{R} \Leftrightarrow \mathbf{1} (\mathbf{R}^*)$,!	3	($()$ E: 2)	i
$\mathbf{f} \mathbf{R} \Rightarrow \mathbf{1} (\mathbf{R}^*)$,!	4	(\Leftrightarrow E: 3)	i
$(\mathbf{f} \mathbf{R} \Rightarrow \mathbf{1} (\mathbf{R}^*))$,!	5	($()$ I: 4)	i
$\forall \mathbf{R} (\mathbf{f} \mathbf{R} \Rightarrow \mathbf{1} (\mathbf{R}^*))$!	6	(\forall I: 1,5)	i
\square				
! 8.				i
$\vdash \forall \mathbf{R} (\mathbf{1} (\mathbf{R}^*) \Rightarrow \mathbf{f} \mathbf{R})$				i
\mathbf{R}	,!	1	(Prem)	i
$(\mathbf{f} \mathbf{R} \Leftrightarrow \mathbf{1} (\mathbf{R}^*))$,!	2	(\forall E: P6)	i
$\mathbf{f} \mathbf{R} \Leftrightarrow \mathbf{1} (\mathbf{R}^*)$,!	3	($()$ E: 2)	i
$\mathbf{1} (\mathbf{R}^*) \Rightarrow \mathbf{f} \mathbf{R}$,!	4	(\Leftrightarrow E: 3)	i
$(\mathbf{1} (\mathbf{R}^*) \Rightarrow \mathbf{f} \mathbf{R})$,!	5	($()$ I: 4)	i
$\forall \mathbf{R} (\mathbf{1} (\mathbf{R}^*) \Rightarrow \mathbf{f} \mathbf{R})$!	6	(\forall I: 1,5)	i

□

! 9. Sub-relations of one-to-one relationships are also one-to-one. This is proven by appealing to duality and the corresponding proposition for functional relationships. i

$\vdash \forall R \forall S (\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S)$ i

R, S ,! 1 (Prem) i

$\mathbf{1} R \ \& \ S \subseteq R$,! 2 (Prem) i

$\mathbf{1} R$,! 3 (&E: 2) i

$(\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$,! 4 (\forall E: P3) i

$\mathbf{1} R \Rightarrow \mathbf{f} (R^*)$,! 5 (()E: 4) i

$\mathbf{f} (R^*)$,! 6 (\Rightarrow E: 3,5) i

$S \subseteq R$,! 7 (&E: 2) i

$(S \subseteq R \Rightarrow (S^*) \subseteq (R^*))$,! 8 (\forall E: C3.19) i

$S \subseteq R \Rightarrow (S^*) \subseteq (R^*)$,! 9 (()E: 8) i

$(S^*) \subseteq (R^*)$,! 10 (\Rightarrow E: 7,9) i

$\mathbf{f} (R^*) \ \& \ (S^*) \subseteq (R^*)$,! 11 (&I: 6,10) i

$(\mathbf{f} (R^*) \ \& \ (S^*) \subseteq (R^*) \Rightarrow \mathbf{f} (S^*))$,! 12 (\forall E: C8.3) i

$\mathbf{f} (R^*) \ \& \ (S^*) \subseteq (R^*) \Rightarrow \mathbf{f} (S^*)$,! 13 (()E: 12) i

$\mathbf{f} (S^*)$,! 14 (\Rightarrow E: 11,13) i

$(\mathbf{f} (S^*) \Rightarrow \mathbf{1} S)$,! 15 (\forall E: P4) i

$\mathbf{f} (S^*) \Rightarrow \mathbf{1} S$,! 16 (()E: 15) i

$\mathbf{1} S$,! 17 (\Rightarrow E: 14,16) i

$\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S$,! 18 (\Rightarrow I: 2,17) i

$(\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S)$,! 19 (()I: 18) i

$\forall R \forall S (\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S)$! 20 (\forall I: 1,19) i

□

! 10. Restrictions of one-to-one relationships are also one-to-one. i

$\vdash \forall R \forall A (\mathbf{1} R \Rightarrow \mathbf{1} (R \upharpoonright A))$ i

R, A	,! 1 (Prem)	i
$\mathbf{1} R$,! 2 (Prem)	i
$(R \sqsupset A) \subseteq R$,! 3 ($\forall E$: C7.7)	i
$\mathbf{1} R \ \& \ (R \sqsupset A) \subseteq R$,! 4 ($\&I$: 2,3)	i
$(\mathbf{1} R \ \& \ (R \sqsupset A) \subseteq R \Rightarrow \mathbf{1} (R \sqsupset A))$,! 5 ($\forall E$: P9)	i
$\mathbf{1} R \ \& \ (R \sqsupset A) \subseteq R \Rightarrow \mathbf{1} (R \sqsupset A)$,! 6 ($(\)E$: 5)	i
$\mathbf{1} (R \sqsupset A)$,! 7 ($\Rightarrow E$: 4,6)	i
$\mathbf{1} R \Rightarrow \mathbf{1} (R \sqsupset A)$,! 8 ($\Rightarrow I$: 2,7)	i
$(\mathbf{1} R \Rightarrow \mathbf{1} (R \sqsupset A))$,! 9 ($(\)I$: 8)	i
$\forall R \forall A (\mathbf{1} R \Rightarrow \mathbf{1} (R \sqsupset A))$! 10 ($\forall I$: 1,9)	i
\square		
! 11.		i
$\vdash \forall R \forall S (\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S)$		i
R, S	,! 1 (Prem)	i
$\mathbf{1} R \ \& \ R \equiv S$,! 2 (Prem)	i
$\mathbf{1} R$,! 3 ($\&E$: 2)	i
$R \equiv S$,! 4 ($\&E$: 2)	i
$(R \equiv S \Rightarrow S \subseteq R)$,! 5 ($\forall E$: C1.10)	i
$R \equiv S \Rightarrow S \subseteq R$,! 6 ($(\)E$: 5)	i
$S \subseteq R$,! 7 ($\Rightarrow E$: 4,6)	i
$\mathbf{1} R \ \& \ S \subseteq R$,! 8 ($\&I$: 3,7)	i
$(\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S)$,! 9 ($\forall E$: P9)	i
$\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S$,! 10 ($(\)E$: 9)	i
$\mathbf{1} S$,! 11 ($\Rightarrow E$: 8,10)	i
$\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S$,! 12 ($\Rightarrow I$: 2,11)	i
$(\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S)$,! 13 ($(\)I$: 12)	i
$\forall R \forall S (\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S)$! 14 ($\forall I$: 1,13)	i
\square		

! 12.		i
$\vdash \forall R \forall S (\mathbf{1} R \ \& \ S \equiv R \Rightarrow \mathbf{1} S)$		i
R, S	,! 1 (Prem)	i
$\mathbf{1} R \ \& \ S \equiv R$,! 2 (Prem)	i
$\mathbf{1} R$,! 3 (&E: 2)	i
$S \equiv R$,! 4 (&E: 2)	i
$(S \equiv R \Rightarrow R \equiv S)$,! 5 (\forall E: C1.8)	i
$S \equiv R \Rightarrow R \equiv S$,! 6 (()E: 5)	i
$R \equiv S$,! 7 (\Rightarrow E: 4,6)	i
$\mathbf{1} R \ \& \ R \equiv S$,! 8 (&I: 3,7)	i
$(\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S)$,! 9 (\forall E: P11)	i
$\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S$,! 10 (()E: 9)	i
$\mathbf{1} S$,! 11 (\Rightarrow E: 8,10)	i
$\mathbf{1} R \ \& \ S \equiv R \Rightarrow \mathbf{1} S$,! 12 (\Rightarrow I: 2,11)	i
$(\mathbf{1} R \ \& \ S \equiv R \Rightarrow \mathbf{1} S)$,! 13 (()I: 12)	i
$\forall R \forall S (\mathbf{1} R \ \& \ S \equiv R \Rightarrow \mathbf{1} S)$! 14 (\forall I: 1,13)	i

□

! P13-P17 are a return to the theme of duality. i

! 13. i

$\vdash \forall R \forall S (\mathbf{1} R \ \& \ (R^*) \equiv S \Rightarrow \mathbf{f} S)$		i
R, S	,! 1 (Prem)	i
$\mathbf{1} R \ \& \ (R^*) \equiv S$,! 2 (Prem)	i
$\mathbf{1} R$,! 3 (&E: 2)	i
$(R^*) \equiv S$,! 4 (&E: 2)	i
$(\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$,! 5 (\forall E: P3)	i
$\mathbf{1} R \Rightarrow \mathbf{f} (R^*)$,! 6 (()E: 5)	i
$\mathbf{f} (R^*)$,! 7 (\Rightarrow E: 3,6)	i
$\mathbf{f} (R^*) \ \& \ (R^*) \equiv S$,! 8 (&I: 4,7)	i

$(f (R^*) \ \& \ (R^*) \equiv S \Rightarrow f S)$,! 9 ($\forall E$: C8.5)	i
$f (R^*) \ \& \ (R^*) \equiv S \Rightarrow f S$,! 10 ($()E$: 9)	i
$f S$,! 11 ($\Rightarrow E$: 8,10)	i
$\mathbf{1} \ R \ \& \ (R^*) \equiv S \Rightarrow f S$,! 12 ($\Rightarrow I$: 2,11)	i
$(\mathbf{1} \ R \ \& \ (R^*) \equiv S \Rightarrow f S)$,! 13 ($()I$: 12)	i
$\forall R \forall S (\mathbf{1} \ R \ \& \ (R^*) \equiv S \Rightarrow f S)$! 14 ($\forall I$: 1,13)	i
\square		

! 14.

$\vdash \forall R \forall S (\mathbf{1} \ R \ \& \ (S^*) \equiv R \Rightarrow f S)$	i	
R, S	,! 1 (Prem)	i
$\mathbf{1} \ R \ \& \ (S^*) \equiv R$,! 2 (Prem)	i
$\mathbf{1} \ R$,! 3 ($\&E$: 2)	i
$(S^*) \equiv R$,! 4 ($\&E$: 2)	i
$((S^*) \equiv R \Rightarrow (R^*) \equiv S)$,! 5 ($\forall E$: C3.23)	i
$(S^*) \equiv R \Rightarrow (R^*) \equiv S$,! 6 ($()E$: 5)	i
$(R^*) \equiv S$,! 7 ($\Rightarrow E$: 4,6)	i
$\mathbf{1} \ R \ \& \ (R^*) \equiv S$,! 8 ($\&I$: 3,7)	i
$(\mathbf{1} \ R \ \& \ (R^*) \equiv S \Rightarrow f S)$,! 9 ($\forall E$: P13)	i
$\mathbf{1} \ R \ \& \ (R^*) \equiv S \Rightarrow f S$,! 10 ($()E$: 9)	i
$f S$,! 11 ($\Rightarrow E$: 8,10)	i
$\mathbf{1} \ R \ \& \ (S^*) \equiv R \Rightarrow f S$,! 12 ($\Rightarrow I$: 2,11)	i
$(\mathbf{1} \ R \ \& \ (S^*) \equiv R \Rightarrow f S)$,! 13 ($()I$: 12)	i
$\forall R \forall S (\mathbf{1} \ R \ \& \ (S^*) \equiv R \Rightarrow f S)$! 14 ($\forall I$: 1,13)	i
\square		

! 15.

$\vdash \forall R \forall S (f R \ \& \ (R^*) \equiv S \Rightarrow \mathbf{1} S)$	i	
R, S	,! 1 (Prem)	i

$\mathbf{f R \ \& \ (R^*) \equiv S}$,! 2 (Prem)	i
$\mathbf{f R}$,! 3 (&E: 2)	i
$\mathbf{(R^*) \equiv S}$,! 4 (&E: 2)	i
$\mathbf{(f R \Rightarrow 1 (R^*))}$,! 5 (\forall E: P7)	i
$\mathbf{f R \Rightarrow 1 (R^*)}$,! 6 (()E: 5)	i
$\mathbf{1 (R^*)}$,! 7 (\Rightarrow E: 3,6)	i
$\mathbf{1 (R^*) \ \& \ (R^*) \equiv S}$,! 8 (&I: 4,7)	i
$\mathbf{(1 (R^*) \ \& \ (R^*) \equiv S \Rightarrow 1 S)}$,! 9 (\forall E: P11)	i
$\mathbf{1 (R^*) \ \& \ (R^*) \equiv S \Rightarrow 1 S}$,! 10 (()E: 9)	i
$\mathbf{1 S}$,! 11 (\Rightarrow E: 8,10)	i
$\mathbf{f R \ \& \ (R^*) \equiv S \Rightarrow 1 S}$,! 12 (\Rightarrow I: 2,11)	i
$\mathbf{(f R \ \& \ (R^*) \equiv S \Rightarrow 1 S)}$,! 13 (()I: 12)	i
$\forall R \forall S (f R \ \& \ (R^*) \equiv S \Rightarrow 1 S)$! 14 (\forall I: 1,13)	i
\square		
! 16.		i
$\vdash \forall R \forall S (f R \ \& \ (S^*) \equiv R \Rightarrow 1 S)$		i
$\mathbf{R, S}$,! 1 (Prem)	i
$\mathbf{f R \ \& \ (S^*) \equiv R}$,! 2 (Prem)	i
$\mathbf{f R}$,! 3 (&E: 2)	i
$\mathbf{(S^*) \equiv R}$,! 4 (&E: 2)	i
$\mathbf{((S^*) \equiv R \Rightarrow (R^*) \equiv S)}$,! 5 (\forall E: C3.23)	i
$\mathbf{(S^*) \equiv R \Rightarrow (R^*) \equiv S}$,! 6 (()E: 5)	i
$\mathbf{(R^*) \equiv S}$,! 7 (\Rightarrow E: 4,6)	i
$\mathbf{f R \ \& \ (R^*) \equiv S}$,! 8 (&I: 3,7)	i
$\mathbf{(f R \ \& \ (R^*) \equiv S \Rightarrow 1 S)}$,! 9 (\forall E: P15)	i
$\mathbf{f R \ \& \ (R^*) \equiv S \Rightarrow 1 S}$,! 10 (()E: 9)	i
$\mathbf{1 S}$,! 11 (\Rightarrow E: 8,10)	i

$\mathbf{f R \& (S^*) \equiv R \Rightarrow 1 S}$,! 12 (\Rightarrow I: 2,11) ;
 $(\mathbf{f R \& (S^*) \equiv R \Rightarrow 1 S })$,! 13 ($(\)$ I: 12) ;
 $\forall R \forall S (\mathbf{f R \& (S^*) \equiv R \Rightarrow 1 S })$! 14 (\forall I: 1,13) ;
 \square

! 17. One-to-relationships, restricted to disjoint domains, have disjoint images. ;

$\vdash \forall R \forall A \forall B (\mathbf{1 R \& (A \cap B) \equiv \phi}$
 $\Rightarrow (((R \upharpoonright A)^I) \cap ((R \upharpoonright B)^I)) \equiv \phi)$;
 $\mathbf{R, A, B}$,! 1 (Prem) ;
 $\mathbf{1 R \& (A \cap B) \equiv \phi}$,! 2 (Prem) ;
 $(\neg \exists x (((R \upharpoonright A)^I)[x] \& ((R \upharpoonright B)^I)[x])$
 $\Rightarrow (((R \upharpoonright A)^I) \cap ((R \upharpoonright B)^I)) \equiv \phi)$
, ! 3 (\forall E II5.29) ;
 $\neg \exists x (((R \upharpoonright A)^I)[x] \& ((R \upharpoonright B)^I)[x])$
 $\Rightarrow (((R \upharpoonright A)^I) \cap ((R \upharpoonright B)^I)) \equiv \phi$
, ! 4 ($(\)$ E: 3) ;

! To show: $\neg \exists x (((R \upharpoonright A)^I)[x] \& ((R \upharpoonright B)^I)[x])$. The proof proceeds by contradiction. ;

$\exists x (((R \upharpoonright A)^I)[x] \& ((R \upharpoonright B)^I)[x])$,! 5 (Prem) ;
 $(((R \upharpoonright A)^I)[y] \& ((R \upharpoonright B)^I)[y])$,! 6 (\exists E: 5) ;
 $((R \upharpoonright A)^I)[y] \& ((R \upharpoonright B)^I)[y]$,! 7 ($(\)$ E: 6) ;
 $((R \upharpoonright A)^I)[y]$,! 8 ($\&$ E: 7) ;
 $(((R \upharpoonright A)^I)[y] \Rightarrow \exists x (R \upharpoonright A[x,y]))$,! 9 (\forall E: C6.3) ;
 $((R \upharpoonright A)^I)[y] \Rightarrow \exists x (R \upharpoonright A[x,y])$,! 10 ($(\)$ E: 9) ;
 $\exists x (R \upharpoonright A[x,y])$,! 11 (\Rightarrow E: 8,10) ;
 $(R \upharpoonright A[x,y])$,! 12 (\exists E: 11) ;
 $((R \upharpoonright A[x,y] \Rightarrow R[x,y] \& A[x]))$,! 13 (\forall E: C7.3) ;
 $(R \upharpoonright A[x,y] \Rightarrow R[x,y] \& A[x])$,! 14 ($(\)$ E: 13) ;
 $\mathbf{R[x,y] \& A[x]}$,! 15 (\Rightarrow E: 12,14) ;
 $\mathbf{R[x,y]}$,! 16 ($\&$ E: 15) ;
 $\mathbf{A[x]}$,! 17 ($\&$ E: 15) ;

$((R \uparrow B)^I)[y]$,!	18 (&E: 7)	i
$(((R \uparrow B)^I)[y] \Rightarrow \exists x (R \uparrow B)[x,y])$,!	19 (\forall E: C6.3)	i
$((R \uparrow B)^I)[y] \Rightarrow \exists x (R \uparrow B)[x,y]$,!	20 (())E: 19)	i
$\exists x (R \uparrow B)[x,y]$,!	21 (\Rightarrow E: 18,20)	i
$(R \uparrow B)[z,y]$,!	22 (\exists E: 21)	i
$((R \uparrow B)[z,y] \Rightarrow R[z,y] \& B[z])$,!	23 (\forall E: C7.3)	i
$(R \uparrow B)[z,y] \Rightarrow R[z,y] \& B[z]$,!	24 (())E: 23)	i
$R[z,y] \& B[z]$,!	25 (\Rightarrow E: 22,24)	i
$R[z,y]$,!	26 (&E: 25)	i
$B[z]$,!	27 (&E: 25)	i
$R[x,y] \& R[z,y]$,!	28 (&I: 16,26)	i
$\mathbf{1} R$,!	29 (&E: 2)	i
$\mathbf{1} R \& R[x,y] \& R[z,y]$,!	30 (&E: 28,29)	i
$(\mathbf{1} R \& R[x,y] \& R[z,y] \Rightarrow x = z)$,!	31 (\forall E: P2)	i
$\mathbf{1} R \& R[x,y] \& R[z,y] \Rightarrow x = z$,!	32 (())E: 31)	i
$x = z$,!	33 (\Rightarrow E: 30,32)	i
$B[x]$,!	34 (=E: 27,33)	i
$(A \cap B) \equiv \phi$,!	35 (&E: 2)	i
$B[x] \& (A \cap B) \equiv \phi$,!	36 (&I: 34,35)	i
$(B[x] \& (A \cap B) \equiv \phi \Rightarrow \neg A[x])$,!	37 (\forall E: II5.24)	i
$B[x] \& (A \cap B) \equiv \phi \Rightarrow \neg A[x]$,!	38 (())E: 37)	i
$\neg A[x]$,!	39 (\Rightarrow E: 36,38)	i
\mathfrak{F}	,!	40 (\mathfrak{F} I: 17,39)	i
$\exists x (((R \uparrow A)^I)[x] \& ((R \uparrow B)^I)[x]) \Rightarrow \mathfrak{F}$,!	41 (\Rightarrow I: 5,40)	i
$\neg \exists x (((R \uparrow A)^I)[x] \& ((R \uparrow B)^I)[x])$,!	42 (\neg I: 41)	i
$(((R \uparrow A)^I) \cap ((R \uparrow B)^I)) \equiv \phi$,!	43 (\Rightarrow E: 4,42)	i
$\mathbf{1} R \& (A \cap B) \equiv \phi \Rightarrow (((R \uparrow A)^I) \cap ((R \uparrow B)^I)) \equiv \phi$			

,! 44 (\Rightarrow I: 2,43) i

($\mathbf{1} \ R \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi \Rightarrow (((\mathbf{R} \uparrow \mathbf{A})^{\mathbf{I}}) \cap ((\mathbf{R} \uparrow \mathbf{B})^{\mathbf{I}})) \equiv \phi)$

,! 45 ($(\)$ I: 44) i

$\forall R \forall A \forall B (\mathbf{1} \ R \ \& \ (\mathbf{A} \cap \mathbf{B}) \equiv \phi$

$\Rightarrow (((\mathbf{R} \uparrow \mathbf{A})^{\mathbf{I}}) \cap ((\mathbf{R} \uparrow \mathbf{B})^{\mathbf{I}})) \equiv \phi)$

! 46 (\forall I: 1,45) i

□

! 18. Our empty relationship is one-to-one. i

⊢ $\mathbf{1} \ \Phi$ i

($\mathbf{f} \ \Phi \ \& \ \Phi \equiv (\Phi^*) \Rightarrow \mathbf{f} \ (\Phi^*)$)

,! 1 (\forall E: C8.5) i

$\mathbf{f} \ \Phi \ \& \ \Phi \equiv (\Phi^*) \Rightarrow \mathbf{f} \ (\Phi^*)$

,! 2 ($(\)$ E: 1) i

$\mathbf{f} \ \Phi \ \& \ \Phi \equiv (\Phi^*)$

,! 3 ($\&$ I: C8.9,C4.14) i

$\mathbf{f} \ (\Phi^*)$

,! 4 (\Rightarrow E: 2,3) i

($\mathbf{f} \ (\Phi^*) \Rightarrow \mathbf{1} \ \Phi$)

,! 5 (\forall E: P4) i

$\mathbf{f} \ (\Phi^*) \Rightarrow \mathbf{1} \ \Phi$

,! 6 ($(\)$ E: 5) i

$\mathbf{1} \ \Phi$

! 7 (\Rightarrow E: 4,6) i

□

! 19. $\mathbf{R} \uparrow \mathbf{B}$ means that \mathbf{R} is a one-to-one relationship with image \mathbf{B} . i

$\mathbb{S} \uparrow ; \mathbf{R} \uparrow \mathbf{B} ; (\mathbf{R}^{\mathbf{I}}) \equiv \mathbf{B} \ \& \ \mathbf{1} \ \mathbf{R}$ i

! P20-P25 are more propositions concerning duality. i

! 20. i

⊢ $\forall R \forall B (\mathbf{R} \uparrow \mathbf{B} \Rightarrow (\mathbf{R}^*) \uparrow \mathbf{B})$ i

\mathbf{R}, \mathbf{B}

,! 1 (Prem) i

$\mathbf{R} \uparrow \mathbf{B}$

,! 2 (Prem) i

$(\mathbf{R}^{\mathbf{I}}) \equiv \mathbf{B} \ \& \ \mathbf{1} \ \mathbf{R}$

,! 3 (\mathbb{S} E: P19,2) i

$(\mathbf{R}^{\mathbf{I}}) \equiv \mathbf{B}$

,! 4 ($\&$ E: 3) i

($(\mathbf{R}^{\mathbf{I}}) \equiv \mathbf{B} \Rightarrow ((\mathbf{R}^*)^{\mathbf{D}}) \equiv \mathbf{B}$)

,! 5 (\forall E: C6.17) i

$(\mathbf{R}^{\mathbf{I}}) \equiv \mathbf{B} \Rightarrow ((\mathbf{R}^*)^{\mathbf{D}}) \equiv \mathbf{B}$

,! 6 ($(\)$ E: 5) i

$((R^*)^D) \equiv B$,!	7	(\Rightarrow E: 4,6)	i
$\mathbf{1} R$,!	8	(&E: 3)	i
$(\mathbf{1} R \Rightarrow \mathbf{f}(R^*))$,!	9	(\forall E: P3)	i
$\mathbf{1} R \Rightarrow \mathbf{f}(R^*)$,!	10	($()$ E: 9)	i
$\mathbf{f}(R^*)$,!	11	(\Rightarrow E: 8,10)	i
$((R^*)^D) \equiv B \ \& \ \mathbf{f}(R^*)$,!	12	(&I: 7,11)	i
$(R^*) \mathbf{F} B$,!	13	(\S I: C8.10,12)	i
$R \mathbf{1} B \Rightarrow (R^*) \mathbf{F} B$,!	14	(\Rightarrow I: 2,13)	i
$(R \mathbf{1} B \Rightarrow (R^*) \mathbf{F} B)$,!	15	($()$ I: 14)	i
$\forall R \forall B (R \mathbf{1} B \Rightarrow (R^*) \mathbf{F} B)$!	16	(\forall I: 1,15)	i
\square				
! 21.				i
$\vdash \forall R \forall B ((R^*) \mathbf{F} B \Rightarrow R \mathbf{1} B)$				i
R, B	,!	1	(Prem)	i
$(R^*) \mathbf{F} B$,!	2	(Prem)	i
$((R^*)^D) \equiv B \ \& \ \mathbf{f}(R^*)$,!	3	(\S E: C8.10,2)	i
$((R^*)^D) \equiv B$,!	4	(&E: 3)	i
$((R^*)^D) \equiv B \Rightarrow (R^I) \equiv B$,!	5	(\forall E: C6.18)	i
$((R^*)^D) \equiv B \Rightarrow (R^I) \equiv B$,!	6	($()$ E: 5)	i
$(R^I) \equiv B$,!	7	(\Rightarrow E: 4,6)	i
$\mathbf{f}(R^*)$,!	8	(&E: 3)	i
$(\mathbf{f}(R^*) \Rightarrow \mathbf{1} R)$,!	9	(\forall E: P4)	i
$\mathbf{f}(R^*) \Rightarrow \mathbf{1} R$,!	10	($()$ E: 9)	i
$\mathbf{1} R$,!	11	(\Rightarrow E: 8,10)	i
$(R^I) \equiv B \ \& \ \mathbf{1} R$,!	12	(&I: 7,11)	i
$R \mathbf{1} B$,!	13	(\S I: P19,12)	i

$(R^*) \mathbb{F} B \Rightarrow R \mathbb{1} B$,! 14 (\Rightarrow I: 2,13)	i
$((R^*) \mathbb{F} B \Rightarrow R \mathbb{1} B)$,! 15 ($(())$ I: 14)	i
$\forall R \forall B ((R^*) \mathbb{F} B \Rightarrow R \mathbb{1} B)$! 16 (\forall I: 1,15)	i
\square		

! 22.

$\vdash \forall R \forall B (R \mathbb{1} B \Leftrightarrow (R^*) \mathbb{F} B)$		i
R, B	,! 1 (Prem)	i
$(R \mathbb{1} B \Rightarrow (R^*) \mathbb{F} B)$,! 2 (\forall E: P2)	i
$R \mathbb{1} B \Rightarrow (R^*) \mathbb{F} B$,! 3 ($(())$ E: 2)	i
$((R^*) \mathbb{F} B \Rightarrow R \mathbb{1} B)$,! 4 (\forall E: P21)	i
$(R^*) \mathbb{F} B \Rightarrow R \mathbb{1} B$,! 5 ($(())$ E: 4)	i
$R \mathbb{1} B \Leftrightarrow (R^*) \mathbb{F} B$,! 6 (\Leftrightarrow I: 3,5)	i
$(R \mathbb{1} B \Leftrightarrow (R^*) \mathbb{F} B)$,! 7 ($(())$ I: 6)	i
$\forall R \forall B (R \mathbb{1} B \Leftrightarrow (R^*) \mathbb{F} B)$! 8 (\forall I: 1,7)	i
\square		

! 23.

$\vdash \forall R \forall A (R \mathbb{F} A \Rightarrow (R^*) \mathbb{1} A)$		i
R, A	,! 1 (Prem)	i
$R \mathbb{F} A$,! 2 (Prem)	i
$R \equiv ((R^*)^*)$,! 3 (\forall E: C3.18)	i
$R \mathbb{F} A \ \& \ R \equiv ((R^*)^*)$,! 4 ($\&$ I: 2,3)	i
$(R \mathbb{F} A \ \& \ R \equiv ((R^*)^*) \Rightarrow ((R^*)^*) \mathbb{F} A)$,! 5 (\forall E: C8.12)	i
$R \mathbb{F} A \ \& \ R \equiv ((R^*)^*) \Rightarrow ((R^*)^*) \mathbb{F} A$,! 6 ($(())$ E: 5)	i
$((R^*)^*) \mathbb{F} A$,! 7 (\Rightarrow E: 4,6)	i
$(((R^*)^*) \mathbb{F} A \Rightarrow (R^*) \mathbb{1} A)$,! 8 (\forall E: P21)	i
$((R^*)^*) \mathbb{F} A \Rightarrow (R^*) \mathbb{1} A$,! 9 ($(())$ E: 8)	i
$(R^*) \mathbb{1} A$,! 10 (\Rightarrow E: 7,9)	i

$R \text{ F } A \Rightarrow (R^*) \text{ I } A$,! 11 (\Rightarrow I: 2,10)	i
$(R \text{ F } A \Rightarrow (R^*) \text{ I } A)$,! 12 ($(\)$ I: 11)	i
$\forall R \forall A (R \text{ F } A \Rightarrow (R^*) \text{ I } A)$! 13 (\forall I: 1,12)	i
\square		

! 24. i

$\vdash \forall R \forall A ((R^*) \text{ I } A \Rightarrow R \text{ F } A)$		i
R, A	,! 1 (Prem)	i
$(R^*) \text{ I } A$,! 2 (Prem)	i
$((R^*) \text{ I } A \Rightarrow ((R^*)^*) \text{ F } A)$,! 3 (\forall E: P20)	i
$(R^*) \text{ I } A \Rightarrow ((R^*)^*) \text{ F } A$,! 4 ($(\)$ E: 3)	i
$((R^*)^*) \text{ F } A$,! 5 (\Rightarrow E: 2,4)	i
$((R^*)^*) \equiv R$,! 6 (\forall E: C3.17)	i
$((R^*)^*) \text{ F } A \ \& \ ((R^*)^*) \equiv R$,! 7 ($\&$ I: 5,6)	i
$(((R^*)^*) \text{ F } A \ \& \ ((R^*)^*) \equiv R \Rightarrow R \text{ F } A)$,! 8 (\forall E: C8.12)	i
$((R^*)^*) \text{ F } A \ \& \ ((R^*)^*) \equiv R \Rightarrow R \text{ F } A$,! 9 ($(\)$ E: 8)	i
$R \text{ F } A$,! 10 (\Rightarrow E: 7,9)	i
$(R^*) \text{ I } A \Rightarrow R \text{ F } A$,! 11 (\Rightarrow I: 2,10)	i
$((R^*) \text{ I } A \Rightarrow R \text{ F } A)$,! 12 ($(\)$ I: 11)	i
$\forall R \forall A ((R^*) \text{ I } A \Rightarrow R \text{ F } A)$! 13 (\forall I: 1,12)	i
\square		

! 25. i

$\vdash \forall R \forall A (R \text{ F } A \Leftrightarrow (R^*) \text{ I } A)$		i
R, A	,! 1 (Prem)	i
$(R \text{ F } A \Rightarrow (R^*) \text{ I } A)$,! 2 (\forall E: P23)	i
$R \text{ F } A \Rightarrow (R^*) \text{ I } A$,! 3 ($(\)$ E: 2)	i
$((R^*) \text{ I } A \Rightarrow R \text{ F } A)$,! 4 (\forall E: P24)	i

$(R^*) \Vdash A \Rightarrow R \Vdash A$,! 5 ((E: 4)	i
$R \Vdash A \Leftrightarrow (R^*) \Vdash A$,! 6 (\Leftrightarrow I: 3,5)	i
$(R \Vdash A \Leftrightarrow (R^*) \Vdash A)$,! 7 ((I: 6)	i
$\forall R \forall A (R \Vdash A \Leftrightarrow (R^*) \Vdash A)$! 8 (\forall I: 1,7)	i

□

! P26-P28 are on the same theme: equivalence maintains one-to-oneness.

! 26.

$\vdash \forall R \forall S \forall A \forall B (R \Vdash A \ \& \ R \equiv S \ \& \ A \equiv B \Rightarrow S \Vdash B)$		i
R, S, A, B	,! 1 (Prem)	i
$R \Vdash A \ \& \ R \equiv S \ \& \ A \equiv B$,! 2 (Prem)	i
$R \Vdash A$,! 3 ($\&$ E: 2)	i
$(R \Vdash A \Rightarrow (R^*) \Vdash A)$,! 4 (\forall E: P20)	i
$R \Vdash A \Rightarrow (R^*) \Vdash A$,! 5 ((E: 4)	i
$(R^*) \Vdash A$,! 6 (\Rightarrow E: 3,5)	i
$R \equiv S$,! 7 ($\&$ E: 2)	i
$(R \equiv S \Rightarrow (R^*) \equiv (S^*))$,! 8 (\forall E: C3.20)	i
$R \equiv S \Rightarrow (R^*) \equiv (S^*)$,! 9 ((E: 8)	i
$(R^*) \equiv (S^*)$,! 10 (\Rightarrow E: 7,9)	i
$(R^*) \Vdash A \ \& \ (R^*) \equiv (S^*)$,! 11 ($\&$ I: 6,10)	i
$A \equiv B$,! 12 ($\&$ E: 2)	i
$(R^*) \Vdash A \ \& \ (R^*) \equiv (S^*) \ \& \ A \equiv B$,! 13 ($\&$ I: 11,12)	i
$((R^*) \Vdash A \ \& \ (R^*) \equiv (S^*) \ \& \ A \equiv B \Rightarrow (S^*) \Vdash B)$,! 14 (\forall E: C8.11)	i
$(R^*) \Vdash A \ \& \ (R^*) \equiv (S^*) \ \& \ A \equiv B \Rightarrow (S^*) \Vdash B$,! 15 ((E: 14)	i
$(S^*) \Vdash B$,! 16 (\Rightarrow E: 13,15)	i
$((S^*) \Vdash B \Rightarrow S \Vdash B)$,! 17 (\forall E: P21)	i
$(S^*) \Vdash B \Rightarrow S \Vdash B$,! 18 ((E: 17)	i

$S \perp B$,! 19 ($\Rightarrow E$: 16,18)	i
$R \perp A \ \& \ R \equiv S \ \& \ A \equiv B \Rightarrow S \perp B$,! 20 ($\Rightarrow I$: 2,19)	i
$(R \perp A \ \& \ R \equiv S \ \& \ A \equiv B \Rightarrow S \perp B)$,! 21 ($(\)I$: 20)	i
$\forall R \forall S \forall A \forall B (R \perp A \ \& \ R \equiv S \ \& \ A \equiv B \Rightarrow S \perp B)$! 22 ($\forall I$: 1,21)	i

□

! 27.

$\vdash \forall R \forall S \forall A (R \perp A \ \& \ R \equiv S \Rightarrow S \perp A)$		i
R, S, A	,! 1 (Prem)	i
$R \perp A \ \& \ R \equiv S$,! 2 (Prem)	i
$A \equiv A$,! 3 ($\forall E$: III.1.9)	i
$R \perp A \ \& \ R \equiv S \ \& \ A \equiv A$,! 4 ($\&I$: 2,3)	i
$(R \perp A \ \& \ R \equiv S \ \& \ A \equiv A \Rightarrow S \perp A)$,! 5 ($\forall E$: P26)	i
$R \perp A \ \& \ R \equiv S \ \& \ A \equiv A \Rightarrow S \perp A$,! 6 ($(\)E$: 5)	i
$S \perp A$,! 7 ($\Rightarrow E$: 4,6)	i
$R \perp A \ \& \ R \equiv S \Rightarrow S \perp A$,! 8 ($\Rightarrow I$: 2,7)	i
$(R \perp A \ \& \ R \equiv S \Rightarrow S \perp A)$,! 9 ($(\)I$: 8)	i
$\forall R \forall S \forall A (R \perp A \ \& \ R \equiv S \Rightarrow S \perp A)$! 10 ($\forall I$: 1,9)	i

□

! 28.

$\vdash \forall R \forall A \forall B (R \perp A \ \& \ A \equiv B \Rightarrow R \perp B)$		i
R, A, B	,! 1 (Prem)	i
$R \perp A \ \& \ A \equiv B$,! 2 (Prem)	i
$R \equiv R$,! 3 ($\forall E$: C1.7)	i
$R \perp A \ \& \ R \equiv R \ \& \ A \equiv B$,! 4 ($\&I$: 2,3)	i
$(R \perp A \ \& \ R \equiv R \ \& \ A \equiv B \Rightarrow R \perp B)$,! 5 ($\forall E$: P26)	i
$R \perp A \ \& \ R \equiv R \ \& \ A \equiv B \Rightarrow R \perp B$,! 6 ($(\)E$: 5)	i
$R \perp B$,! 7 ($\Rightarrow E$: 4,6)	i

$R \Vdash A \ \& \ A \equiv B \Rightarrow R \Vdash B$, ! 8 (\Rightarrow I: 2,7)	i
$(R \Vdash A \ \& \ A \equiv B \Rightarrow R \Vdash B)$, ! 9 ($(())$ I: 8)	i
$\forall R \forall A \forall B (R \Vdash A \ \& \ A \equiv B \Rightarrow R \Vdash B)$! 10 (\forall I: 1,9)	i

□

! 29. A one-to-one relationship is one-to-one on its image. i

$\vdash \forall R (\mathbf{1} R \Rightarrow R \Vdash (R^I))$		
R	, ! 1 (Prem)	i
$\mathbf{1} R$, ! 2 (Prem)	i
$(R^I) \equiv (R^I)$, ! 3 (\forall E: III1.9)	i
$(R^I) \equiv (R^I) \ \& \ \mathbf{1} R$, ! 4 ($\&$ I: 2,3)	i
$R \Vdash (R^I)$, ! 5 (\S I: P19,4)	i
$\mathbf{1} R \Rightarrow R \Vdash (R^I)$, ! 6 (\Rightarrow I: 2,5)	i
$(\mathbf{1} R \Rightarrow R \Vdash (R^I))$, ! 7 ($(())$ I: 6)	i
$\forall R (\mathbf{1} R \Rightarrow R \Vdash (R^I))$! 8 (\forall I: 1,7)	i

□

! 30. P30 is a corollary of P29. i

$\vdash \forall R \forall A (\mathbf{1} R \Rightarrow (R \Vdash A) \Vdash ((R \Vdash A)^I))$		
R, A	, ! 1 (Prem)	i
$\mathbf{1} R$, ! 2 (Prem)	i
$(\mathbf{1} R \Rightarrow \mathbf{1} (R \Vdash A))$, ! 3 (\forall E: P10)	i
$\mathbf{1} R \Rightarrow \mathbf{1} (R \Vdash A)$, ! 4 ($(())$ E: 3)	i
$\mathbf{1} (R \Vdash A)$, ! 5 (\Rightarrow E: 2,4)	i
$(\mathbf{1} (R \Vdash A) \Rightarrow (R \Vdash A) \Vdash ((R \Vdash A)^I))$, ! 6 (\forall E: P29)	i
$\mathbf{1} (R \Vdash A) \Rightarrow (R \Vdash A) \Vdash ((R \Vdash A)^I)$, ! 7 ($(())$ I: 6)	i
$(R \Vdash A) \Vdash ((R \Vdash A)^I)$, ! 8 (\Rightarrow E: 5,7)	i
$\mathbf{1} R \Rightarrow (R \Vdash A) \Vdash ((R \Vdash A)^I)$, ! 9 (\Rightarrow I: 2,8)	i
$(\mathbf{1} R \Rightarrow (R \Vdash A) \Vdash ((R \Vdash A)^I))$, ! 10 ($(())$ I: 9)	i

$\forall R \forall A (\mathbf{1} R \Rightarrow (R \upharpoonright A) \upharpoonright ((R \upharpoonright A)^{\perp})) \quad ! 11 (\forall I: 1,10) \quad ;$

□

! 31. The union of one-to-one relationships with disjoint images are one-to-one, and the image of the union is the union of the images. i

$\vdash \forall R \forall S \forall A \forall B (R \upharpoonright A \ \& \ S \upharpoonright B \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \cup S) \upharpoonright (A \cup B)) \quad ;$

$R, S, A, B \quad ,! 1 \text{ (Prem)} \quad ;$

$R \upharpoonright A \ \& \ S \upharpoonright B \ \& \ (A \cap B) \equiv \phi \quad ,! 2 \text{ (Prem)} \quad ;$

$R \upharpoonright A \quad ,! 3 \text{ (&E: 2)} \quad ;$

$(R \upharpoonright A \Rightarrow (R^*) \upharpoonright A) \quad ,! 4 \text{ (\forall E: P20)} \quad ;$

$R \upharpoonright A \Rightarrow (R^*) \upharpoonright A \quad ,! 5 \text{ (()E: 4)} \quad ;$

$(R^*) \upharpoonright A \quad ,! 6 \text{ (\Rightarrow E: 3,5)} \quad ;$

$S \upharpoonright B \quad ,! 7 \text{ (&E: 2)} \quad ;$

$(S \upharpoonright B \Rightarrow (S^*) \upharpoonright B) \quad ,! 8 \text{ (\forall E: P4)} \quad ;$

$S \upharpoonright B \Rightarrow (S^*) \upharpoonright B \quad ,! 9 \text{ (()E: 8)} \quad ;$

$(S^*) \upharpoonright B \quad ,! 10 \text{ (\Rightarrow E: 7,9)} \quad ;$

$(R^*) \upharpoonright A \ \& \ (S^*) \upharpoonright B \quad ,! 11 \text{ (&I: 6,10)} \quad ;$

$(A \cap B) \equiv \phi \quad ,! 12 \text{ (&E: 2)} \quad ;$

$(R^*) \upharpoonright A \ \& \ (S^*) \upharpoonright B \ \& \ (A \cap B) \equiv \phi \quad ,! 13 \text{ (&I: 11,12)} \quad ;$

$((R^*) \upharpoonright A \ \& \ (S^*) \upharpoonright B \ \& \ (A \cap B) \equiv \phi \Rightarrow ((R^*) \cup (S^*)) \upharpoonright (A \cup B)) \quad ,! 14 \text{ (\forall E: C8.16)} \quad ;$

$(R^*) \upharpoonright A \ \& \ (S^*) \upharpoonright B \ \& \ (A \cap B) \equiv \phi \Rightarrow ((R^*) \cup (S^*)) \upharpoonright (A \cup B) \quad ,! 15 \text{ (()E: 14)} \quad ;$

$((R^*) \cup (S^*)) \upharpoonright (A \cup B) \quad ,! 16 \text{ (\Rightarrow E: 13,15)} \quad ;$

$((R^*) \cup (S^*)) \equiv ((R \cup S)^*) \quad ,! 17 \text{ (\forall E: C3.28)} \quad ;$

$((R^*) \cup (S^*)) \upharpoonright (A \cup B) \ \& \ ((R^*) \cup (S^*)) \equiv ((R \cup S)^*) \upharpoonright (A \cup B) \quad ,! 18 \text{ (&I: 16,17)} \quad ;$

$(((R^*) \cup (S^*)) \upharpoonright (A \cup B) \ \& \ ((R^*) \cup (S^*)) \equiv ((R \cup S)^*) \upharpoonright (A \cup B))$

$\Rightarrow ((R \sqcup S)^*) \mathbb{F} (A \cup B)$,! 19 ($\forall E$: C8.12) ;

$((R^*) \sqcup (S^*)) \mathbb{F} (A \cup B) \ \& \ ((R^*) \sqcup (S^*)) \equiv ((R \sqcup S)^*)$
 $\Rightarrow ((R \sqcup S)^*) \mathbb{F} (A \cup B)$,! 20 ($(\)E$: 19) ;

$((R \sqcup S)^*) \mathbb{F} (A \cup B)$,! 21 ($\Rightarrow E$: 18,20) ;

$(((R \sqcup S)^*) \mathbb{F} (A \cup B) \Rightarrow (R \sqcup S) \mathbb{1} (A \cup B))$
 ,! 22 ($\forall E$: P21) ;

$((R \sqcup S)^*) \mathbb{F} (A \cup B) \Rightarrow (R \sqcup S) \mathbb{1} (A \cup B)$
 ,! 23 ($(\)E$: 22) ;

$(R \sqcup S) \mathbb{1} (A \cup B)$,! 24 ($\Rightarrow E$: 21,23) ;

$R \mathbb{1} A \ \& \ S \mathbb{1} B \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup S) \mathbb{1} (A \cup B)$
 ,! 25 ($\Rightarrow I$: 2,24) ;

$(R \mathbb{1} A \ \& \ S \mathbb{1} B \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup S) \mathbb{1} (A \cup B))$
 ,! 26 ($(\)I$: 25) ;

$\forall R \forall S \forall A \forall B (R \mathbb{1} A \ \& \ S \mathbb{1} B \ \& \ (A \cap B) \equiv \phi \Rightarrow (R \sqcup S) \mathbb{1} (A \cup B))$
 ! 27 ($\forall I$: 1,26) ;

□

! 32. Our empty relationship is one-to-one with our empty (one-place) predicate as image. P32 summarizes previously asserted propositions. ;

$\vdash \Phi \mathbb{1} \phi$;

$(\Phi^I) \equiv \phi \ \& \ \mathbb{1} \ \Phi$,! 1 ($\&I$: C6.28, P18) ;

$\Phi \mathbb{1} \phi$! 2 ($\mathbb{S}I$: P19,1) ;

□

! As their duals C8.18 and C8.19 in Chapter 8, P33 and P34 are needed to introduce the \mathbb{T} -definition of P34. P33 is again unusual, in that it appeals only to a proposition from chapter 6. ;

! 33. ;

$\vdash \forall R \forall y (\mathbb{1} R \ \& \ (R^I)[y] \Rightarrow \exists a R[a,y])$;

R, y ,! 1 (Prem) ;

$\mathbb{1} R \ \& \ (R^I)[y]$,! 2 (Prem) ;

$(R^I)[y]$,! 3 ($\&E$: 2) ;

$((R^I)[y] \Rightarrow \exists x R[x,y])$,! 4 ($\forall E$: C6.3) ;
 $(R^I)[y] \Rightarrow \exists x R[x,y]$,! 5 ($(\Rightarrow)E$: 4) ;
 $\exists x R[x,y]$,! 6 ($\Rightarrow E$: 3,5) ;
 $R[x,y]$,! 7 ($\exists E$: 6) ;
 $\exists a R[a,y]$,! 8 ($\exists I$: 7) ;
1 $R \ \& \ (R^I)[y] \Rightarrow \exists a R[a,y]$,! 9 ($\Rightarrow I$: 2,8) ;
 $(\mathbf{1} \ R \ \& \ (R^I)[y] \Rightarrow \exists a R[a,y])$,! 10 ($(\Rightarrow)I$: 9) ;
 $\forall R \forall y (\mathbf{1} \ R \ \& \ (R^I)[y] \Rightarrow \exists a R[a,y])$! 11 ($\forall I$: 1,10) ;

□

! 34. ;

$\vdash \forall R \forall y (\mathbf{1} \ R \ \& \ (R^I)[y] \Rightarrow \forall x \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z))$;
 R, y ,! 1 (Prem) ;
 $\mathbf{1} \ R \ \& \ (R^I)[y]$,! 2 (Prem) ;
 $\mathbf{1} \ R$,! 3 ($\&E$: 2) ;
 $\forall x \forall y \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$,! 4 ($\E: 3,P1) ;
 x ,! 5 (Prem) ;
 $\forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$,! 6 ($\forall E$: 4) ;
 $\forall x \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$,! 7 ($\forall I$: 5,6) ;
 $\mathbf{1} \ R \ \& \ (R^I)[y] \Rightarrow \forall x \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z)$,! 8 ($\Rightarrow I$: 2,7) ;
 $(\mathbf{1} \ R \ \& \ (R^I)[y] \Rightarrow \forall x \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z))$,! 9 ($(\Rightarrow)I$: 8) ;
 $\forall R \forall y (\mathbf{1} \ R \ \& \ (R^I)[y] \Rightarrow \forall x \forall z (R[x,y] \ \& \ R[z,y] \Rightarrow x = z))$;
! 10 ($\forall I$: 1,9) ;

□

! 35. (R^y) refers to the thing x which bears R to y . It presupposes that R is one-to-one and y is in the image of R , which ensures that x exists (P33) and is unique (P34). (R^y) is dual to (R^x) . ;

$\mathbb{T} \ \` \ ; \ (R^y) \ ; \ \mathbf{1} \ R \ \& \ (R^I)[y] \ ; \ R[x,y]$;! ($\mathbb{T}D$: P33,P34) ;

! 36. ;

$\vdash \forall R \forall x \forall y (\mathbf{1} R \ \& \ R[x,y] \Rightarrow x = (R \backslash y))$		i
R, x, y	, ! 1 (Prem)	i
$\mathbf{1} R \ \& \ R[x,y]$, ! 2 (Prem)	i
$R[(R \backslash y), y]$, ! 3 ($\mathbb{T}I$: P35,2)	i
$\mathbf{1} R \ \& \ R[x,y] \ \& \ R[(R \backslash y), y]$, ! 4 ($\&I$: 2,3)	i
$(\mathbf{1} R \ \& \ R[x,y] \ \& \ R[(R \backslash y), y] \Rightarrow x = (R \backslash y))$, ! 5 ($\forall E$: P2; ($R \backslash y$): P35,2)	i
$\mathbf{1} R \ \& \ R[x,y] \ \& \ R[(R \backslash y), y] \Rightarrow x = (R \backslash y)$, ! 6 ($()E$: 5)	i
$x = (R \backslash y)$, ! 7 ($\Rightarrow E$: 4,6)	i
$\mathbf{1} R \ \& \ R[x,y] \Rightarrow x = (R \backslash y)$, ! 8 ($\Rightarrow I$: 2,7)	i
$(\mathbf{1} R \ \& \ R[x,y] \Rightarrow x = (R \backslash y))$, ! 9 ($()I$: 8)	i
$\forall R \forall x \forall y (\mathbf{1} R \ \& \ R[x,y] \Rightarrow x = (R \backslash y))$! 10 ($\forall I$: 1,9)	i
\square		

! 37. i

$\vdash \forall R \forall x \forall y (\mathbf{1} R \ \& \ R[x,y] \Rightarrow (R \backslash y) = x)$		i
R, x, y	, ! 1 (Prem)	i
$\mathbf{1} R \ \& \ R[x,y]$, ! 2 (Prem)	i
$(\mathbf{1} R \ \& \ R[x,y] \Rightarrow x = (R \backslash y))$, ! 3 ($\forall E$: P36)	i
$\mathbf{1} R \ \& \ R[x,y] \Rightarrow x = (R \backslash y)$, ! 4 ($()E$: 3)	i
$x = (R \backslash y)$, ! 5 ($\Rightarrow E$: 2,4)	i
$x = x$, ! 6 ($=E$: 5,5)	i
$(R \backslash y) = x$, ! 7 ($=E$: 5,6)	i
$\mathbf{1} R \ \& \ R[x,y] \Rightarrow (R \backslash y) = x$, ! 8 ($\Rightarrow I$: 2,7)	i
$(\mathbf{1} R \ \& \ R[x,y] \Rightarrow (R \backslash y) = x)$, ! 9 ($()I$: 8)	i
$\forall R \forall x \forall y (\mathbf{1} R \ \& \ R[x,y] \Rightarrow (R \backslash y) = x)$! 10 ($\forall I$: 1,9)	i
\square		

! 38. i

$\vdash \forall R \forall S \forall y (\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (S \backslash y) = (R \backslash y))$		i
--	--	---

R, S, y	,! 1 (Prem)	i
$\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y]$,! 2 (Prem)	i
$\mathbf{1} R \ \& \ S \subseteq R$,! 3 (&E: 2)	i
$(\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S)$,! 4 (\forall E: P9)	i
$\mathbf{1} R \ \& \ S \subseteq R \Rightarrow \mathbf{1} S$,! 5 ($(\)$ E: 4)	i
$\mathbf{1} S$,! 6 (\Rightarrow E: 3,5)	i
$(S^I)[y]$,! 7 (&E: 2)	i
$\mathbf{1} S \ \& \ (S^I)[y]$,! 8 (&I: 6,7)	i
$S[(S`y), y]$,! 9 (\mathbb{T} I: P35,8)	i
$S \subseteq R$,! 10 (&E: 2)	i
$S[(S`y), y] \ \& \ S \subseteq R$,! 11 (&I: 9,10)	i
$(S[(S`y), y] \ \& \ S \subseteq R \Rightarrow R[(S`y), y])$,! 12 (\forall E: C1.2; (S`y): P35,8)	i
$S[(S`y), y] \ \& \ S \subseteq R \Rightarrow R[(S`y), y]$,! 13 ($(\)$ E: 12)	i
$R[(S`y), y]$,! 14 (\Rightarrow E: 11,13)	i
$\mathbf{1} S \ \& \ R[(S`y), y]$,! 15 (&I: 6,14)	i
$(\mathbf{1} S \ \& \ R[(S`y), y] \Rightarrow (S`y) = (R`y))$,! 16 (\forall E: P36; (S`y): P35,8)	i
$\mathbf{1} S \ \& \ R[(S`y), y] \Rightarrow (S`y) = (R`y)$,! 17 ($(\)$ E: 16)	i
$(S`y) = (R`y)$,! 18 (\Rightarrow E: 15,17)	i
$\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (S`y) = (R`y)$,! 19 (\Rightarrow I: 2,18)	i
$(\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (S`y) = (R`y))$,! 20 ($(\)$ I: 19)	i
$\forall R \forall S \forall y (\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (S`y) = (R`y))$! 21 (\forall I: 1,20)	i
\square		
! 39.		i
$\vdash \forall R \forall S \forall y (\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (R`y) = (S`y))$		i
R, S, y	,! 1 (Prem)	i

$\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y]$,! 2 (Prem) i
 $(\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (S`y) = (R`y))$
, ! 3 ($\forall E$: P38) i
 $\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (S`y) = (R`y)$
, ! 4 ($()E$: 3) i
 $(S`y) = (R`y)$, ! 5 ($\Rightarrow E$: 2,4) i
 $(R`y) = (R`y)$, ! 6 ($=E$: 5,5) i
 $(R`y) = (S`y)$, ! 7 ($=E$: 5,6) i
 $\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (R`y) = (S`y)$, ! 8 ($\Rightarrow I$: 2,7) i
 $(\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (R`y) = (S`y))$
, ! 9 ($()I$: 8) i
 $\forall R \forall S \forall y (\mathbf{1} R \ \& \ S \subseteq R \ \& \ (S^I)[y] \Rightarrow (R`y) = (S`y))$
! 10 ($\forall I$: 1,9) i

□

! 40. i

$\vdash \forall R \forall S \forall y (\mathbf{1} R \ \& \ (R^I)[y] \ \& \ R \equiv S \Rightarrow (R`y) = (S`y))$ i
 R, S, y , ! 1 (Prem) i
 $\mathbf{1} R \ \& \ (R^I)[y] \ \& \ R \equiv S$, ! 2 (Prem) i
 $\mathbf{1} R$, ! 3 ($\&E$: 2) i
 $(R^I)[y]$, ! 4 ($\&E$: 2) i
 $R \equiv S$, ! 5 ($\&E$: 2) i
 $(R \equiv S \Rightarrow R \subseteq S)$, ! 6 ($\forall E$: C1.9) i
 $R \equiv S \Rightarrow R \subseteq S$, ! 7 ($()E$: 6) i
 $R \subseteq S$, ! 8 ($\Rightarrow E$: 5,7) i
 $R \subseteq S \ \& \ (R^I)[y]$, ! 9 ($\&I$: 4,8) i
 $\mathbf{1} R \ \& \ R \equiv S$, ! 10 ($\&I$: 3,5) i
 $(\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S)$, ! 11 ($\forall E$: P11) i
 $\mathbf{1} R \ \& \ R \equiv S \Rightarrow \mathbf{1} S$, ! 12 ($()E$: 11) i
 $\mathbf{1} S$, ! 13 ($\Rightarrow E$: 10,12) i

$\mathbf{1} \ S \ \& \ R \subseteq S \ \& \ (R^I)[y]$,! 14 (&I: 9,13) i
 $(\mathbf{1} \ S \ \& \ R \subseteq S \ \& \ (R^I)[y] \Rightarrow (R`y) = (S`y))$
, ! 15 ($\forall E$: P38) i
 $\mathbf{1} \ R \ \& \ R \subseteq S \ \& \ (R^I)[y] \Rightarrow (R`y) = (S`y)$
, ! 16 ($()E$: 15) i
 $(R`y) = (S`y)$, ! 17 ($\Rightarrow E$: 14,16) i
 $\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ R \equiv S \Rightarrow (R`y) = (S`y)$, ! 18 ($\Rightarrow I$: 2,17) i
 $(\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ R \equiv S \Rightarrow (R`y) = (S`y))$
, ! 19 ($()I$: 18) i
 $\forall R \forall S \forall y (\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ R \equiv S \Rightarrow (R`y) = (S`y))$
! 20 ($\forall I$: 1,19) i
 \square
! 41. i
 $\vdash \forall R \forall S \forall y (\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ S \equiv R \Rightarrow (R`y) = (S`y))$ i
 R, S, y , ! 1 (Prem) i
 $\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ S \equiv R$, ! 2 (Prem) i
 $\mathbf{1} \ R \ \& \ (R^I)[y]$, ! 3 ($\&E$: 2) i
 $S \equiv R$, ! 4 ($\&E$: 2) i
 $(S \equiv R \Rightarrow R \equiv S)$, ! 5 ($\forall E$: C1.8) i
 $S \equiv R \Rightarrow R \equiv S$, ! 6 ($()E$: 5) i
 $R \equiv S$, ! 7 ($\Rightarrow E$: 4,6) i
 $\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ R \equiv S$, ! 8 ($\&I$: 3,7) i
 $(\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ R \equiv S \Rightarrow (R`y) = (S`y))$
, ! 9 ($\forall E$: P40) i
 $\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ R \equiv S \Rightarrow (R`y) = (S`y)$
, ! 10 ($()E$: 9) i
 $(R`y) = (S`y)$, ! 11 ($\Rightarrow E$: 8,10) i
 $\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ S \equiv R \Rightarrow (R`y) = (S`y)$, ! 12 ($\Rightarrow I$: 2,11) i
 $(\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ S \equiv R \Rightarrow (R`y) = (S`y))$
, ! 13 ($()I$: 12) i
 $\forall R \forall S \forall y (\mathbf{1} \ R \ \& \ (R^I)[y] \ \& \ S \equiv R \Rightarrow (R`y) = (S`y))$

! 14 ($\forall I$: 1,13) i

□

! 42.

$\vdash \forall R \forall x (f R \ \& \ \mathbf{1} R \ \& \ (R^D)[x] \Rightarrow (R \setminus (R'x)) = x)$ i

R, x ,! 1 (Prem) i

$f R \ \& \ \mathbf{1} R \ \& \ (R^D)[x]$,! 2 (Prem) i

$f R$,! 3 ($\&E$: 2) i

$\mathbf{1} R$,! 4 ($\&E$: 2) i

$(R^D)[x]$,! 5 ($\&E$: 2) i

$f R \ \& \ (R^D)[x]$,! 6 ($\&I$: 3,5) i

$R[x, (R'x)]$,! 7 ($\mathbb{T}I$: C8.20,6) i

$\mathbf{1} R \ \& \ R[x, (R'x)]$,! 8 ($\&I$: 4,7) i

$(\mathbf{1} R \ \& \ R[x, (R'x)] \Rightarrow (R \setminus (R'x)) = x)$,! 9 ($\forall E$: P37;
($R'x$): C8.20,6) i

$\mathbf{1} R \ \& \ R[x, (R'x)] \Rightarrow (R \setminus (R'x)) = x$,! 10 ($(\)E$: 9) i

$(R \setminus (R'x)) = x$,! 11 ($\Rightarrow E$: 8,10) i

$f R \ \& \ \mathbf{1} R \ \& \ (R^D)[x] \Rightarrow (R \setminus (R'x)) = x$,! 12 ($\Rightarrow I$: 2,11) i

$(f R \ \& \ \mathbf{1} R \ \& \ (R^D)[x] \Rightarrow (R \setminus (R'x)) = x)$
,! 13 ($(\)I$: 12) i

$\forall R \forall x (f R \ \& \ \mathbf{1} R \ \& \ (R^D)[x] \Rightarrow (R \setminus (R'x)) = x)$
! 14 ($\forall I$: 1,13) i

□

! 43.

$\vdash \forall R \forall y (f R \ \& \ \mathbf{1} R \ \& \ (R^I)[y] \Rightarrow (R' \setminus (R'y)) = y)$ i

R, y ,! 1 (Prem) i

$f R \ \& \ \mathbf{1} R \ \& \ (R^I)[y]$,! 2 (Prem) i

$f R$,! 3 ($\&E$: 2) i

$\mathbf{1} R \ \& \ (R^I)[y]$,! 4 ($\&E$: 2) i

$R[(R'y), y]$,! 5 ($\mathbb{T}I$: P35,4) i

$\mathbf{f} R \ \& \ R[(R \backslash \mathbf{y}), \mathbf{y}]$,! 6 (&I: 3,5) ;
 $(\mathbf{f} R \ \& \ R[(R \backslash \mathbf{y}), \mathbf{y}] \Rightarrow (R \backslash (R \backslash \mathbf{y})) = \mathbf{y})$,! 7 (\forall E: C8.22;
 $(R \backslash \mathbf{y})$: P35,4) ;
 $\mathbf{f} R \ \& \ R[(R \backslash \mathbf{y}), \mathbf{y}] \Rightarrow (R \backslash (R \backslash \mathbf{y})) = \mathbf{y}$,! 8 ((E: 7) ;
 $(R \backslash (R \backslash \mathbf{y})) = \mathbf{y}$,! 9 (\Rightarrow E: 6,8) ;
 $\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[\mathbf{y}] \Rightarrow (R \backslash (R \backslash \mathbf{y})) = \mathbf{y}$,! 10 (\Rightarrow I: 2,9) ;
 $(\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[\mathbf{y}] \Rightarrow (R \backslash (R \backslash \mathbf{y})) = \mathbf{y})$
,! 11 ((I: 10) ;
 $\forall R \forall \mathbf{y} (\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[\mathbf{y}] \Rightarrow (R \backslash (R \backslash \mathbf{y})) = \mathbf{y})$
! 12 (\forall I: 1,11) ;
 \square
! 44. ;
 $\vdash \forall R \forall \mathbf{x} \forall \mathbf{y} (\mathbf{1} R \ \& \ (R \backslash \mathbf{x}) = (R \backslash \mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y})$;
 $R, \mathbf{x}, \mathbf{y}$,! 1 (Prem) ;
 $\mathbf{1} R \ \& \ (R \backslash \mathbf{x}) = (R \backslash \mathbf{y})$,! 2 (Prem) ;
 $\mathbf{1} R$,! 3 (&E: 2) ;
 $(R \backslash \mathbf{x}) = (R \backslash \mathbf{y})$,! 4 (&E: 2) ;
 $\mathbf{f} R \ \& \ (R^D)[\mathbf{x}]$,! 5 (\mathbb{T} E: C8.20,4) ;
 $\mathbf{f} R \ \& \ (R^D)[\mathbf{y}]$,! 6 (\mathbb{T} E: C8.20,4) ;
 $R[\mathbf{x}, (R \backslash \mathbf{x})]$,! 7 (\mathbb{T} I: C8.20,5) ;
 $R[\mathbf{y}, (R \backslash \mathbf{y})]$,! 8 (\mathbb{T} I: C8.20,6) ;
 $\mathbf{1} R \ \& \ R[\mathbf{x}, (R \backslash \mathbf{x})]$,! 9 (&I: 3,7) ;
 $R[\mathbf{y}, (R \backslash \mathbf{x})]$,! 10 (=E: 4,8) ;
 $\mathbf{1} R \ \& \ R[\mathbf{x}, (R \backslash \mathbf{x})] \ \& \ R[\mathbf{y}, (R \backslash \mathbf{x})]$,! 11 (&I: 9,10) ;
 $(\mathbf{1} R \ \& \ R[\mathbf{x}, (R \backslash \mathbf{x})] \ \& \ R[\mathbf{y}, (R \backslash \mathbf{x})] \Rightarrow \mathbf{x} = \mathbf{y})$
,! 12 (\forall E: P2;
 $(R \backslash \mathbf{x})$: C8.20,5) ;
 $\mathbf{1} R \ \& \ R[\mathbf{x}, (R \backslash \mathbf{x})] \ \& \ R[\mathbf{y}, (R \backslash \mathbf{x})] \Rightarrow \mathbf{x} = \mathbf{y}$
,! 13 ((E: 12) ;
 $\mathbf{x} = \mathbf{y}$,! 14 (\Rightarrow E: 11,13) ;
 $\mathbf{1} R \ \& \ (R \backslash \mathbf{x}) = (R \backslash \mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y}$,! 15 (\Rightarrow I: 2,14) ;

$(\mathbf{1} R \ \& \ (R'x) = (R'y) \Rightarrow x = y)$,! 16 (()I: 15) ;
 $\forall R \forall x \forall y (\mathbf{1} R \ \& \ (R'x) = (R'y) \Rightarrow x = y)$! 17 (\forall I: 1,16) ;
 \square

! 45. A proof along the lines of P44's would save two steps (17 versus the actual 19).

$\vdash \forall R \forall x \forall y (\mathbf{f} R \ \& \ (R'x) = (R'y) \Rightarrow x = y)$;
 R, x, y ,! 1 (Prem) ;
 $\mathbf{f} R \ \& \ (R'x) = (R'y)$,! 2 (Prem) ;
 $\mathbf{f} R$,! 3 (&E: 2) ;
 $(R'x) = (R'y)$,! 4 (&E: 2) ;
 $\mathbf{1} R \ \& \ (R^I)[x]$,! 5 (\mathbb{T} E: P35,4) ;
 $\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[x]$,! 6 (&I: 3,5) ;
 $(\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[x] \Rightarrow (R'(R'x)) = x)$,! 7 (\forall E: P43) ;
 $\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[x] \Rightarrow (R'(R'x)) = x$,! 8 (()E: 7) ;
 $(R'(R'x)) = x$,! 9 (\Rightarrow E: 6,8) ;
 $\mathbf{1} R \ \& \ (R^I)[y]$,! 10 (\mathbb{T} E: P35,4) ;
 $\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^I)[y]$,! 11 (&I: 3,10) ;
 $(\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^D)[y] \Rightarrow (R'(R'y)) = y)$,! 12 (\forall E: P43) ;
 $\mathbf{f} R \ \& \ \mathbf{1} R \ \& \ (R^D)[y] \Rightarrow (R'(R'y)) = y$,! 13 (()E: 12) ;
 $(R'(R'y)) = y$,! 14 (\Rightarrow E: 11,13) ;
 $(R'(R'x)) = y$,! 15 (=E: 4,14) ;
 $x = y$,! 16 (=E: 9,15) ;
 $\mathbf{f} R \ \& \ (R'x) = (R'y) \Rightarrow x = y$,! 17 (\Rightarrow I: 2,16) ;
 $(\mathbf{f} R \ \& \ (R'x) = (R'y) \Rightarrow x = y)$,! 18 (()I: 17) ;
 $\forall R \forall x \forall y (\mathbf{f} R \ \& \ (R'x) = (R'y) \Rightarrow x = y)$! 19 (\forall I: 1,18) ;
 \square

! P46 and P47 concern duality. An alternative development of the last third of this chapter would have established these

propositions first and the others subsequently.

! 46.

$\vdash \forall R \forall x (f R \ \& \ (R^D)[x] \Rightarrow ((R^*) \setminus x) = (R'x))$

R, x	,! 1 (Prem)	i
$f R \ \& \ (R^D)[x]$,! 2 (Prem)	i
$R[x, (R'x)]$,! 3 ($\mathbb{T}I$: C8.20,2)	i
$(R[x, (R'x)] \Rightarrow (R^*)[(R'x), x])$,! 4 ($\forall E$: C3.4; ($R'x$): C8.20,2)	i
$R[x, (R'x)] \Rightarrow (R^*)[(R'x), x]$,! 5 ($(\Rightarrow)E$: 4)	i
$(R^*)[(R'x), x]$,! 6 ($\Rightarrow E$: 3,5)	i
$f R$,! 7 ($\&E$: 2)	i
$(f R \Rightarrow \mathbf{1} (R^*))$,! 8 ($\forall E$: P7)	i
$f R \Rightarrow \mathbf{1} (R^*)$,! 9 ($(\Rightarrow)E$: 8)	i
$\mathbf{1} (R^*)$,! 10 ($\Rightarrow E$: 7,9)	i
$\mathbf{1} (R^*) \ \& \ (R^*)[(R'x), x]$,! 11 ($\&I$: 6,10)	i
$(\mathbf{1} (R^*) \ \& \ (R^*)[(R'x), x] \Rightarrow ((R^*) \setminus x) = (R'x))$,! 12 ($\forall E$: P37; ($R'x$): C8.20,2)	i
$\mathbf{1} (R^*) \ \& \ (R^*)[(R'x), x] \Rightarrow ((R^*) \setminus x) = (R'x)$,! 13 ($(\Rightarrow)E$: 12)	i
$((R^*) \setminus x) = (R'x)$,! 14 ($\Rightarrow E$: 11,13)	i
$f R \ \& \ (R^D)[x] \Rightarrow ((R^*) \setminus x) = (R'x)$,! 15 ($\Rightarrow I$: 2,14)	i
$(f R \ \& \ (R^D)[x] \Rightarrow ((R^*) \setminus x) = (R'x))$,! 16 ($(\Rightarrow)I$: 15)	i
$\forall R \forall x (f R \ \& \ (R^D)[x] \Rightarrow ((R^*) \setminus x) = (R'x))$! 17 ($\forall I$: 1,16)	i

□

! 47.

$\vdash \forall R \forall y (\mathbf{1} R \ \& \ (R^I)[y] \Rightarrow ((R^*) \setminus y) = (R'y))$

R, y	,! 1 (Prem)	i
$\mathbf{1} R \ \& \ (R^I)[y]$,! 2 (Prem)	i

$R[(R \setminus y), y]$,!	3	($\mathbb{T}I$: P35,2)	i
$(R[(R \setminus y), y] \Rightarrow (R^*)[y, (R \setminus y)])$,!	4	($\forall E$: C3.4; ($R \setminus y$): P35,2)	i
$R[(R \setminus y), y] \Rightarrow (R^*)[y, (R \setminus y)]$,!	5	($()E$: 4)	i
$(R^*)[y, (R \setminus y)]$,!	6	($\Rightarrow E$: 3,5)	i
$\mathbf{1} R$,!	7	($\&E$: 2)	i
$(\mathbf{1} R \Rightarrow \mathbf{f} (R^*))$,!	8	($\forall E$: P3)	i
$\mathbf{1} R \Rightarrow \mathbf{f} (R^*)$,!	9	($()E$: 8)	i
$\mathbf{f} (R^*)$,!	10	($\Rightarrow E$: 7,9)	i
$\mathbf{f} (R^*) \ \& \ (R^*)[y, (R \setminus y)]$,!	11	($\&I$: 6,10)	i
$(\mathbf{f} (R^*) \ \& \ (R^*)[y, (R \setminus y)] \Rightarrow ((R^*) \setminus y) = (R \setminus y))$,!	12	($\forall E$: C8.22; ($R \setminus y$): P35,2)	i
$\mathbf{f} (R^*) \ \& \ (R^*)[y, (R \setminus y)] \Rightarrow ((R^*) \setminus y) = (R \setminus y)$,!	13	($()E$: 12)	i
$((R^*) \setminus y) = (R \setminus y)$,!	14	($\Rightarrow E$: 11,13)	i
$\mathbf{1} R \ \& \ (R^I)[y] \Rightarrow ((R^*) \setminus y) = (R \setminus y)$,!	15	($\Rightarrow I$: 2,14)	i
$(\mathbf{1} R \ \& \ (R^I)[y] \Rightarrow ((R^*) \setminus y) = (R \setminus y))$,!	16	($()I$: 15)	i
$\forall R \forall y (\mathbf{1} R \ \& \ (R^I)[y] \Rightarrow ((R^*) \setminus y) = (R \setminus y))$!	17	($\forall I$: 1,16)	i

□