

! CHAPTER 11

k-UNIFORM PREDICATES;

! This chapter introduces the notion of k-uniform predicates, which are useful in the development of multiplication. **R** is k-uniform with respect to **P** if the image of **R**, when restricted to any element of **P**, is of size **k**. i

! 1. \cup represents uniformity. i

$\$ \cup ; R \cup_k P; \omega[k] \ \& \ \forall x (P[x] \Rightarrow \mathcal{N}[k, ((R \upharpoonright (x^\bullet))^I)])$ i

! 2. i

$\vdash \forall R \forall P \forall k (R \cup_k P \Rightarrow \omega[k])$ i

R, P, k , ! 1 (Prem) i

R \cup_k P , ! 2 (Prem) i

$\omega[k] \ \& \ \forall x (P[x] \Rightarrow \mathcal{N}[k, ((R \upharpoonright (x^\bullet))^I)])$, ! 3 ($\E: P1,2) i

$\omega[k]$, ! 4 ($\&E$: 3) i

$R \cup_k P \Rightarrow \omega[k]$, ! 5 ($\Rightarrow I$: 2,4) i

$(R \cup_k P \Rightarrow \omega[k])$, ! 6 ($()I$: 5) i

$\forall R \forall P \forall k (R \cup_k P \Rightarrow \omega[k])$! 7 ($\forall I$: 1,6) i

□

! P3 through P7 show that inclusion and equivalence maintain uniformity. P4 through P7 rely on P3 for their proofs. i

! 3. i

$\vdash \forall R \forall S \forall P \forall Q \forall k (R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv S \Rightarrow S \cup_k Q)$ i

R, S, P, Q, k , ! 1 (Prem) i

$R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv S$, ! 2 (Prem) i

$R \cup_k P$, ! 3 ($\&E$: 2) i

$Q \subseteq P$, ! 4 ($\&E$: 2) i

$R \equiv S$, ! 5 ($\&E$: 2) i

$\omega[k] \ \& \ \forall x (P[x] \Rightarrow \mathcal{N}[k, ((R \upharpoonright (x^\bullet))^I)])$, ! 6 ($\E: P1,3) i

$\omega[k]$, ! 7 ($\&E$: 6) i

$\forall x (P[x] \Rightarrow \mathcal{N}[k, ((R \lceil (x^\bullet))^I)])$,! 8 (&E: 6)	i
x	,! 9 (Prem)	i
$(P[x] \Rightarrow \mathcal{N}[k, ((R \lceil (x^\bullet))^I)])$,! 10 (\forall E: 8)	i
$P[x] \Rightarrow \mathcal{N}[k, ((R \lceil (x^\bullet))^I)]$,! 11 ($(\)$ E: 10)	i
$Q[x]$,! 12 (Prem)	i
$Q[x] \ \& \ Q \subseteq P$,! 13 (&I: 4,12)	i
$(Q[x] \ \& \ Q \subseteq P \Rightarrow P[x])$,! 14 (\forall E: III1.2)	i
$Q[x] \ \& \ Q \subseteq P \Rightarrow P[x]$,! 15 ($(\)$ E: 14)	i
$P[x]$,! 16 (\Rightarrow E: 13,15)	i
$\mathcal{N}[k, ((R \lceil (x^\bullet))^I)]$,! 17 (\Rightarrow E: 11,16)	i
$\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))^I)]$,! 18 (&I: 7,17)	i
$(R \equiv S \Rightarrow (R \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet)))$,! 19 (\forall E: III7.12)	i
$R \equiv S \Rightarrow (R \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$,! 20 ($(\)$ E: 19)	i
$(R \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$,! 21 (\Rightarrow E: 5,20)	i
$\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))^I)] \ \& \ (R \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$,! 22 (&I: 18,21)	i
$(\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))^I)] \ \& \ (R \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$ $\Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))^I)])$,! 23 (\forall E: C4.10)	i
$\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))^I)] \ \& \ (R \lceil (x^\bullet)) \equiv (S \lceil (x^\bullet))$ $\Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))^I)]$,! 24 ($(\)$ E: 23)	i
$\mathcal{N}[k, ((S \lceil (x^\bullet))^I)]$,! 25 (\Rightarrow E: 22,24)	i
$Q[x] \Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))^I)]$,! 26 (\Rightarrow I: 12,25)	i
$(Q[x] \Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))^I)])$,! 27 ($(\)$ I: 26)	i
$\forall x (Q[x] \Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))^I)])$,! 28 (\forall I: 9,27)	i
$\omega[k] \ \& \ \forall x (Q[x] \Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))^I)])$,! 29 (&I: 7,28)	i
$S \cup_k Q$,! 30 (\mathbb{S} I: P1,29)	i

$R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv S \Rightarrow S \cup_k Q$,! 31 (\Rightarrow I: 2,30) i
 $(R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv S \Rightarrow S \cup_k Q)$,! 32 ($(\)$ I: 31) i
 $\forall R \forall S \forall P \forall Q \forall k (R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv S \Rightarrow S \cup_k Q)$
 ! 33 (\forall I: 1,32) i

□

! 4. i

$\vdash \forall R \forall P \forall Q \forall k (R \cup_k P \ \& \ Q \subseteq P \Rightarrow R \cup_k Q)$ i

R, P, Q, k ,! 1 (Prem) i

$R \cup_k P \ \& \ Q \subseteq P$,! 2 (Prem) i

$R \equiv R$,! 3 (\forall E: III1.7) i

$R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv R$,! 4 ($\&$ I: 2,3) i

$(R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv R \Rightarrow R \cup_k Q)$,! 5 (\forall E: P3) i

$R \cup_k P \ \& \ Q \subseteq P \ \& \ R \equiv R \Rightarrow R \cup_k Q$,! 6 ($(\)$ E: 5) i

$R \cup_k Q$,! 7 (\Rightarrow E: 4,6) i

$R \cup_k P \ \& \ Q \subseteq P \Rightarrow R \cup_k Q$,! 8 (\Rightarrow I: 2,7) i

$(R \cup_k P \ \& \ Q \subseteq P \Rightarrow R \cup_k Q)$,! 9 ($(\)$ I: 8) i

$\forall R \forall P \forall Q \forall k (R \cup_k P \ \& \ Q \subseteq P \Rightarrow R \cup_k Q)$! 10 (\forall I: 1,9) i

□

! 5. i

$\vdash \forall R \forall P \forall Q \forall k (R \cup_k P \ \& \ Q \equiv P \Rightarrow R \cup_k Q)$ i

R, P, Q, k ,! 1 (Prem) i

$R \cup_k P \ \& \ Q \equiv P$,! 2 (Prem) i

$R \cup_k P$,! 3 ($\&$ E: 2) i

$Q \equiv P$,! 4 ($\&$ E: 2) i

$(Q \equiv P \Rightarrow Q \subseteq P)$,! 5 (\forall E: III1.11) i

$Q \equiv P \Rightarrow Q \subseteq P$,! 6 ($(\)$ E: 5) i

$Q \subseteq P$,! 7 (\Rightarrow E: 4,6) i

$R \cup_k P \ \& \ Q \subseteq P$,! 8 (&I: 3,7)	i
$(R \cup_k P \ \& \ Q \subseteq P \Rightarrow R \cup_k Q)$,! 9 (\forall E: P4)	i
$R \cup_k P \ \& \ Q \subseteq P \Rightarrow R \cup_k Q$,! 10 (()E: 9)	i
$R \cup_k Q$,! 11 (\Rightarrow E: 8,10)	i
$R \cup_k P \ \& \ Q \equiv P \Rightarrow R \cup_k Q$,! 12 (\Rightarrow I: 2,11)	i
$(R \cup_k P \ \& \ Q \equiv P \Rightarrow R \cup_k Q)$,! 13 (()I: 12)	i
$\forall R \forall P \forall Q \forall k (R \cup_k P \ \& \ Q \equiv P \Rightarrow R \cup_k Q)$! 14 (\forall I: 1,13)	i

□

! 6.

$\vdash \forall R \forall P \forall Q \forall k (R \cup_k P \ \& \ P \equiv Q \Rightarrow R \cup_k Q)$		i
R, P, Q, k	,! 1 (Prem)	i
$R \cup_k P \ \& \ P \equiv Q$,! 2 (Prem)	i
$R \cup_k P$,! 3 (&E: 2)	i
$P \equiv Q$,! 4 (&E: 2)	i
$(P \equiv Q \Rightarrow Q \equiv P)$,! 5 (\forall E: III.1.10)	i
$P \equiv Q \Rightarrow Q \equiv P$,! 6 (()E: 5)	i
$Q \equiv P$,! 7 (\Rightarrow E: 4,6)	i
$R \cup_k P \ \& \ Q \equiv P$,! 8 (&I: 3,7)	i
$(R \cup_k P \ \& \ Q \equiv P \Rightarrow R \cup_k Q)$,! 9 (\forall E: P5)	i
$R \cup_k P \ \& \ Q \equiv P \Rightarrow R \cup_k Q$,! 10 (()E: 9)	i
$R \cup_k Q$,! 11 (\Rightarrow E: 8,10)	i
$R \cup_k P \ \& \ P \equiv Q \Rightarrow R \cup_k Q$,! 12 (\Rightarrow I: 2,11)	i
$(R \cup_k P \ \& \ P \equiv Q \Rightarrow R \cup_k Q)$,! 13 (()I: 12)	i
$\forall R \forall P \forall Q \forall k (R \cup_k P \ \& \ P \equiv Q \Rightarrow R \cup_k Q)$! 14 (\forall I: 1,13)	i

□

! 7.

$\vdash \forall R \forall S \forall P \forall k (R \cup_k P \ \& \ R \equiv S \Rightarrow S \cup_k P)$		i
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R, P, Q, k	,! 1 (Prem)	i
$R \cup_k P \ \& \ R \equiv S$,! 2 (Prem)	i
$P \subseteq P$,! 3 ($\forall E$: III.1.4)	i
$R \cup_k P \ \& \ P \subseteq P \ \& \ R \equiv S$,! 4 ($\&I$: 2,3)	i
$(R \cup_k P \ \& \ P \subseteq P \ \& \ R \equiv S \Rightarrow S \cup_k P)$,! 5 ($\forall E$: P3)	i
$R \cup_k P \ \& \ P \subseteq P \ \& \ R \equiv S \Rightarrow S \cup_k P$,! 6 ($(\)E$: 5)	i
$S \cup_k P$,! 7 ($\Rightarrow E$: 4,6)	i
$R \cup_k P \ \& \ R \equiv S \Rightarrow S \cup_k P$,! 8 ($\Rightarrow I$: 2,7)	i
$(R \cup_k P \ \& \ R \equiv S \Rightarrow R \cup_k Q)$,! 9 ($(\)I$: 8)	i
$\forall R \forall S \forall P \forall k (R \cup_k P \ \& \ R \equiv S \Rightarrow S \cup_k P)$! 10 ($\forall I$: 1,9)	i
\square		
! 8.		i
$\vdash \forall R \forall P \forall A \forall k (R \cup_k P \ \& \ A \subseteq P \Rightarrow (R \uparrow A) \cup_k A)$		i
R, P, A, k	,! 1 (Prem)	i
$R \cup_k P \ \& \ A \subseteq P$,! 2 (Prem)	i
$(R \cup_k P \ \& \ A \subseteq P \Rightarrow R \cup_k A)$,! 3 ($\forall E$: P4)	i
$R \cup_k P \ \& \ A \subseteq P \Rightarrow R \cup_k A$,! 4 ($(\)E$: 3)	i
$R \cup_k A$,! 5 ($\Rightarrow E$: 2,4)	i
$\omega[k] \ \& \ \forall x (A[x] \Rightarrow \mathcal{N}[k, ((R \uparrow (x^\bullet))^I)])$,! 6 ($\mathcal{S}E$: P1,5)	i
$\omega[k]$,! 7 ($\&E$: 6)	i
$\forall x (A[x] \Rightarrow \mathcal{N}[k, ((R \uparrow (x^\bullet))^I)])$,! 8 ($\&E$: 6)	i
x	,! 9 (Prem)	i
$A[x]$,! 10 (Prem)	i
$(A[x] \Rightarrow \mathcal{N}[k, ((R \uparrow (x^\bullet))^I)])$,! 11 ($\forall E$: 8)	i
$A[x] \Rightarrow \mathcal{N}[k, ((R \uparrow (x^\bullet))^I)]$,! 12 ($(\)E$: 11)	i
$\mathcal{N}[k, ((R \uparrow (x^\bullet))^I)]$,! 13 ($\Rightarrow E$: 10,12)	i

$\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))^\perp)]$,! 14 (&I: 7,13) ;
 $(\mathbf{A}[\mathbf{x}] \Rightarrow ((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet)) \equiv (\mathbf{R} \lceil (\mathbf{x}^\bullet)))$
, ! 15 (\forall E: III7.16) ;
 $\mathbf{A}[\mathbf{x}] \Rightarrow ((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet)) \equiv (\mathbf{R} \lceil (\mathbf{x}^\bullet))$
, ! 16 (()E: 15) ;
 $((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet)) \equiv (\mathbf{R} \lceil (\mathbf{x}^\bullet))$,! 17 (\Rightarrow E: 10,16) ;
 $\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))^\perp)] \ \& \ ((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet)) \equiv (\mathbf{R} \lceil (\mathbf{x}^\bullet))$
, ! 18 (&I: 14,17) ;
 $(\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))^\perp)]$
 $\ \& \ ((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet)) \equiv (\mathbf{R} \lceil (\mathbf{x}^\bullet))$
 $\Rightarrow \mathfrak{N}[\mathbf{k}, (((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet))^\perp)])$
, ! 19 (\forall E: C4.11) ;
 $\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))^\perp)] \ \& \ ((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet)) \equiv (\mathbf{R} \lceil (\mathbf{x}^\bullet))$
 $\Rightarrow \mathfrak{N}[\mathbf{k}, (((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet))^\perp)]$
, ! 20 (()E: 19) ;
 $\mathfrak{N}[\mathbf{k}, (((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet))^\perp)]$,! 21 (\Rightarrow E: 18,20) ;
 $\mathbf{A}[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, (((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet))^\perp)]$,! 22 (\Rightarrow I: 10,21) ;
 $(\mathbf{A}[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, (((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet))^\perp)])$
, ! 23 (()I: 22) ;
 $\forall \mathbf{x} (\mathbf{A}[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, (((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet))^\perp)])$
, ! 24 (\forall I: 9,23) ;
 $\omega[\mathbf{k}] \ \& \ \forall \mathbf{x} (\mathbf{A}[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, (((\mathbf{R} \lceil \mathbf{A}) \lceil (\mathbf{x}^\bullet))^\perp)])$
, ! 25 (&I: 7,24) ;
 $(\mathbf{R} \lceil \mathbf{A}) \cup_{\mathbf{k}} \mathbf{A}$,! 26 (\mathfrak{S} I: P1,25) ;
 $\mathbf{R} \cup_{\mathbf{k}} \mathbf{P} \ \& \ \mathbf{A} \subseteq \mathbf{P} \Rightarrow (\mathbf{R} \lceil \mathbf{A}) \cup_{\mathbf{k}} \mathbf{A}$,! 27 (\Rightarrow I: 2,26) ;
 $(\mathbf{R} \cup_{\mathbf{k}} \mathbf{P} \ \& \ \mathbf{A} \subseteq \mathbf{P} \Rightarrow (\mathbf{R} \lceil \mathbf{A}) \cup_{\mathbf{k}} \mathbf{A})$,! 28 (()I: 27) ;
 $\forall \mathbf{R} \forall \mathbf{P} \forall \mathbf{A} \forall \mathbf{k} (\mathbf{R} \cup_{\mathbf{k}} \mathbf{P} \ \& \ \mathbf{A} \subseteq \mathbf{P} \Rightarrow (\mathbf{R} \lceil \mathbf{A}) \cup_{\mathbf{k}} \mathbf{A})$
! 29 (\forall I: 1,28) ;
 \square
! P9 through P11 speak of uniformity and empty predicates. ;
! 9. ;
 $\vdash \forall \mathbf{R} \forall \mathbf{k} (\omega[\mathbf{k}] \Rightarrow \mathbf{R} \cup_{\mathbf{k}} \phi)$;
 \mathbf{R}, \mathbf{k} ,! 1 (Prem) ;

$\omega[\mathbf{k}]$,! 2 (Prem)	i
\mathbf{x}	,! 3 (Prem)	i
$\phi[\mathbf{x}]$,! 4 (Prem)	i
$\neg \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$,! 5 (Prem)	i
$\neg \phi[\mathbf{x}]$,! 6 ($\forall E$: II5.3)	i
\mathfrak{F}	,! 7 ($\mathfrak{F}I$: 4,6)	i
$\neg \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})] \Rightarrow \mathfrak{F}$,! 8 ($\Rightarrow I$: 5,7)	i
$\neg\neg \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$,! 9 ($\neg I$: 8)	i
$\mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$,! 10 ($\neg E$: 9)	i
$\phi[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$,! 11 ($\Rightarrow I$: 4,10)	i
$(\phi[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})])$,! 12 ($(\)I$: 11)	i
$\forall \mathbf{x} (\phi[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})])$,! 13 ($\forall I$: 3,12)	i
$\omega[\mathbf{k}] \ \& \ \forall \mathbf{x} (\phi[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \lceil (\mathbf{x}^\bullet))\mathbb{I})])$,! 14 ($\&I$: 2,13)	i
$\mathbf{R} \cup_{\mathbf{k}} \phi$,! 15 ($\mathfrak{S}I$: P1,14)	i
$\omega[\mathbf{k}] \Rightarrow \mathbf{R} \cup_{\mathbf{k}} \phi$,! 16 ($\Rightarrow I$: 2,15)	i
$(\omega[\mathbf{k}] \Rightarrow \mathbf{R} \cup_{\mathbf{k}} \phi)$,! 17 ($(\)I$: 16)	i
$\forall \mathbf{R} \forall \mathbf{k} (\omega[\mathbf{k}] \Rightarrow \mathbf{R} \cup_{\mathbf{k}} \phi)$! 18 ($\forall I$: 1,17)	i

□

! 10. i

$\vdash \forall \mathbf{P} \Phi \cup_0 \mathbf{P}$ i

\mathbf{P} ,! 1 (Prem) i

\mathbf{x} ,! 2 (Prem) i

$\mathbf{P}[\mathbf{x}]$,! 3 (Prem) i

$((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I}) \equiv \phi$,! 4 ($\forall E$: III7.38) i

$(((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I}) \equiv \phi \Rightarrow \mathfrak{N}[0, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})])$
, ! 5 ($\forall E$: C3.2) i

$((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I}) \equiv \phi \Rightarrow \mathfrak{N}[0, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]$
, ! 6 ($(\)E$: 5) i

$\mathfrak{N}_{[0, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]}$, ! 7 ($\Rightarrow\text{E}$: 4,6)	i
$\mathbf{P}[\mathbf{x}] \Rightarrow \mathfrak{N}_{[0, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]}$, ! 8 ($\Rightarrow\text{I}$: 3,7)	i
$(\mathbf{P}[\mathbf{x}] \Rightarrow \mathfrak{N}_{[0, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]})$, ! 9 ($(\)\text{I}$: 8)	i
$\forall \mathbf{x} (\mathbf{P}[\mathbf{x}] \Rightarrow \mathfrak{N}_{[0, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]})$, ! 10 ($\forall\text{I}$: 2,9)	i
$\omega[0] \ \& \ \forall \mathbf{x} (\mathbf{P}[\mathbf{x}] \Rightarrow \mathfrak{N}_{[0, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]})$, ! 11 ($\&\text{I}$: $\omega 0, 10$)	i
$\Phi \cup_0 \mathbf{P}$, ! 12 ($\mathfrak{S}\text{I}$: P1,11)	i
$\forall \mathbf{P} \Phi \cup_0 \mathbf{P}$! 13 ($\forall\text{I}$: 1,12)	i
\square		
! 11.		i
$\vdash \forall \mathbf{R} \forall \mathbf{P} \forall \mathbf{k} (\mathbf{R} \equiv \Phi \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ \mathbf{R} \cup_{\mathbf{k}} \mathbf{P} \Rightarrow \mathbf{k} = 0)$		i
$\mathbf{R}, \mathbf{P}, \mathbf{k}$, ! 1 (Prem)	i
$\mathbf{R} \equiv \Phi \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ \mathbf{R} \cup_{\mathbf{k}} \mathbf{P}$, ! 2 (Prem)	i
$\mathbf{R} \equiv \Phi$, ! 3 ($\&\text{E}$: 2)	i
$\neg \mathbf{P} \equiv \phi$, ! 4 ($\&\text{E}$: 2)	i
$\mathbf{R} \cup_{\mathbf{k}} \mathbf{P}$, ! 5 ($\&\text{E}$: 2)	i
$\mathbf{R} \cup_{\mathbf{k}} \mathbf{P} \ \& \ \mathbf{R} \equiv \Phi$, ! 6 ($\&\text{I}$: 3,5)	i
$(\mathbf{R} \cup_{\mathbf{k}} \mathbf{P} \ \& \ \mathbf{R} \equiv \Phi \Rightarrow \Phi \cup_{\mathbf{k}} \mathbf{P})$, ! 7 ($\forall\text{E}$: P7)	i
$\mathbf{R} \cup_{\mathbf{k}} \mathbf{P} \ \& \ \mathbf{R} \equiv \Phi \Rightarrow \Phi \cup_{\mathbf{k}} \mathbf{P}$, ! 8 ($(\)\text{E}$: 7)	i
$\Phi \cup_{\mathbf{k}} \mathbf{P}$, ! 9 ($\Rightarrow\text{E}$: 6,8)	i
$\omega[\mathbf{k}] \ \& \ \forall \mathbf{x} (\mathbf{P}[\mathbf{x}] \Rightarrow \mathfrak{N}_{[\mathbf{k}, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]})$, ! 10 ($\mathfrak{S}\text{E}$: P1,9)	i
$\forall \mathbf{x} (\mathbf{P}[\mathbf{x}] \Rightarrow \mathfrak{N}_{[\mathbf{k}, ((\Phi \lceil (\mathbf{x}^\bullet))\mathbb{I})]})$, ! 11 ($\&\text{E}$: 10)	i
$(\neg \mathbf{P} \equiv \phi \Rightarrow \exists \mathbf{x} \mathbf{P}[\mathbf{x}])$, ! 12 ($\forall\text{E}$: II5.16)	i
$\neg \mathbf{P} \equiv \phi \Rightarrow \exists \mathbf{x} \mathbf{P}[\mathbf{x}]$, ! 13 ($(\)\text{E}$: 12)	i
$\exists \mathbf{x} \mathbf{P}[\mathbf{x}]$, ! 14 ($\Rightarrow\text{E}$: 4,13)	i
$\mathbf{P}[\mathbf{x}]$, ! 15 ($\exists\text{E}$: 14)	i

$(P[x] \Rightarrow \mathcal{N}[k, ((\Phi \lceil (x^\bullet))^I)])$,! 16 ($\forall E$: 11) ;
 $P[x] \Rightarrow \mathcal{N}[k, ((\Phi \lceil (x^\bullet))^I)]$,! 17 ($(\)E$: 16) ;
 $\mathcal{N}[k, ((\Phi \lceil (x^\bullet))^I)]$,! 18 ($\Rightarrow E$: 15,17) ;
 $(\Phi \lceil (x^\bullet)) \equiv \Phi$,! 19 ($\forall E$: III7.20) ;
 $((\Phi \lceil (x^\bullet)) \equiv \Phi \Rightarrow ((\Phi \lceil (x^\bullet))^I) \equiv \phi)$
,! 20 ($\forall E$: III6.29) ;
 $(\Phi \lceil (x^\bullet)) \equiv \Phi \Rightarrow ((\Phi \lceil (x^\bullet))^I) \equiv \phi$,! 21 ($(\)E$: 20) ;
 $((\Phi \lceil (x^\bullet))^I) \equiv \phi$,! 22 ($\Rightarrow E$: 19,21) ;
 $(((\Phi \lceil (x^\bullet))^I) \equiv \phi \Rightarrow \forall x \neg ((\Phi \lceil (x^\bullet))^I)[x])$
,! 23 ($\forall E$: II5.5) ;
 $((\Phi \lceil (x^\bullet))^I) \equiv \phi \Rightarrow \forall x \neg ((\Phi \lceil (x^\bullet))^I)[x]$
,! 24 ($(\)E$: 23) ;
 $\forall x \neg ((\Phi \lceil (x^\bullet))^I)[x]$,! 25 ($\Rightarrow E$: 22,24) ;
 $\mathcal{N}[k, ((\Phi \lceil (x^\bullet))^I)] \& \forall x \neg ((\Phi \lceil (x^\bullet))^I)[x]$
,! 26 ($\&I$: 18,25) ;
 $(\mathcal{N}[k, ((\Phi \lceil (x^\bullet))^I)] \& \forall x \neg ((\Phi \lceil (x^\bullet))^I)[x] \Rightarrow k = 0)$
,! 27 ($\forall E$: C3.16) ;
 $\mathcal{N}[k, ((\Phi \lceil (x^\bullet))^I)] \& \forall x \neg ((\Phi \lceil (x^\bullet))^I)[x] \Rightarrow k = 0$
,! 28 ($(\)E$: 27) ;
 $k = 0$,! 29 ($\Rightarrow E$: 26,28) ;
 $R \equiv \Phi \& \neg P \equiv \phi \& R \vee_k P \Rightarrow k = 0$,! 30 ($\Rightarrow I$: 2,29) ;
 $(R \equiv \Phi \& \neg P \equiv \phi \& R \vee_k P \Rightarrow k = 0)$
,! 31 ($(\)I$: 30) ;
 $\forall R \forall P \forall k (R \equiv \Phi \& \neg P \equiv \phi \& R \vee_k P \Rightarrow k = 0)$
! 32 ($\forall I$: 1,31) ;

□

! 12. ;

$\vdash \forall B \forall a \forall k (\omega[k] \& \mathcal{N}[k, B] \Rightarrow (a \times B) \vee_k (a^\bullet))$;
 B, a, k ,! 1 (Prem) ;
 $\omega[k] \& \mathcal{N}[k, B]$,! 2 (Prem) ;
 $\omega[k]$,! 3 ($\&E$: 2) ;

\mathbf{x}	,! 4 (Prem)	i
$(\mathbf{a}^\bullet)[\mathbf{x}]$,! 5 (Prem)	i
$((\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a})$,! 6 ($\forall\text{E}$: II8.3)	i
$(\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathbf{x} = \mathbf{a}$,! 7 ($(\)\text{E}$: 6)	i
$\mathbf{x} = \mathbf{a}$,! 8 ($\Rightarrow\text{E}$: 5,7)	i
$((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \text{ X } \mathbf{B})$,! 9 ($\forall\text{E}$: III16.24)	i
$(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \text{ X } \mathbf{B})$ $\Rightarrow (((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}}) \equiv \mathbf{B})$,! 10 ($\forall\text{E}$: III16.22)	i
$((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet)) \equiv (\mathbf{a} \text{ X } \mathbf{B}) \Rightarrow (((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}}) \equiv \mathbf{B}$,! 11 ($(\)\text{E}$: 10)	i
$(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}}) \equiv \mathbf{B}$,! 12 ($\Rightarrow\text{E}$: 9,11)	i
$\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k},\mathbf{B}] \ \& \ (((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}}) \equiv \mathbf{B}$,! 13 ($\&\text{I}$: 2,12)	i
$(\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k},\mathbf{B}] \ \& \ (((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}}) \equiv \mathbf{B}$ $\Rightarrow \mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}})])$,! 14 ($\forall\text{E}$: C4.6)	i
$\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k},\mathbf{B}] \ \& \ (((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}}) \equiv \mathbf{B}$ $\Rightarrow \mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}})]$,! 15 ($(\)\text{E}$: 14)	i
$\mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{a}^\bullet))^{\text{I}})]$,! 16 ($\Rightarrow\text{E}$: 13,15)	i
$\mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{x}^\bullet))^{\text{I}})]$,! 17 ($=\text{E}$: 8,16)	i
$(\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{x}^\bullet))^{\text{I}})]$,! 18 ($\Rightarrow\text{I}$: 5,17)	i
$((\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{x}^\bullet))^{\text{I}})])$,! 19 ($(\)\text{I}$: 18)	i
$\forall \mathbf{x} ((\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{x}^\bullet))^{\text{I}})])$,! 20 ($\forall\text{I}$: 4,19)	i
$\omega[\mathbf{k}] \ \& \ \forall \mathbf{x} ((\mathbf{a}^\bullet)[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k},(((\mathbf{a} \text{ X } \mathbf{B}) \lceil (\mathbf{x}^\bullet))^{\text{I}})])$,! 21 ($\&\text{I}$: 3,20)	i
$(\mathbf{a} \text{ X } \mathbf{B}) \cup_{\mathbf{k}} (\mathbf{a}^\bullet)$,! 22 ($\mathfrak{S}\text{I}$: P1,21)	i

$(P[x] \Rightarrow \mathcal{N}[k, ((R \lceil (x^\bullet))\mathbb{I})])$,! 18 ($\forall E$: 10) ;
 $P[x] \Rightarrow \mathcal{N}[k, ((R \lceil (x^\bullet))\mathbb{I})]$,! 19 ($(\)E$: 18) ;
 $\mathcal{N}[k, ((R \lceil (x^\bullet))\mathbb{I})]$,! 20 ($\Rightarrow E$: 17,19) ;
 $\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))\mathbb{I})]$,! 21 ($\&I$: 9,20) ;
 $P[x] \ \& \ ((S^D) \cap P) \equiv \phi$,! 22 ($\&I$: 5,17) ;
 $(P[x] \ \& \ ((S^D) \cap P) \equiv \phi \Rightarrow \neg (S^D)[x])$
,! 23 ($\forall E$: II5.24) ;
 $P[x] \ \& \ ((S^D) \cap P) \equiv \phi \Rightarrow \neg (S^D)[x]$
,! 24 ($(\)E$: 23) ;
 $\neg (S^D)[x]$,! 25 ($\Rightarrow E$: 22,24) ;
 $(\neg (S^D)[x] \Rightarrow ((R \sqcup S) \lceil (x^\bullet))\mathbb{I}) \equiv ((R \lceil (x^\bullet))\mathbb{I})$
,! 26 ($\forall E$: III7.45) ;
 $\neg (S^D)[x] \Rightarrow ((R \sqcup S) \lceil (x^\bullet))\mathbb{I} \equiv ((R \lceil (x^\bullet))\mathbb{I})$
,! 27 ($(\)E$: 26) ;
 $((R \sqcup S) \lceil (x^\bullet))\mathbb{I} \equiv ((R \lceil (x^\bullet))\mathbb{I})$
,! 28 ($\Rightarrow E$: 25,27) ;
 $\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))\mathbb{I})]$
 $\ \& \ ((R \sqcup S) \lceil (x^\bullet))\mathbb{I} \equiv ((R \lceil (x^\bullet))\mathbb{I})$
,! 29 ($\&I$: 21,28) ;
 $(\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))\mathbb{I})]$
 $\ \& \ ((R \sqcup S) \lceil (x^\bullet))\mathbb{I} \equiv ((R \lceil (x^\bullet))\mathbb{I})$
 $\Rightarrow \mathcal{N}[k, ((R \sqcup S) \lceil (x^\bullet))\mathbb{I}])$
,! 30 ($\forall E$: C4.6) ;
 $\omega[k] \ \& \ \mathcal{N}[k, ((R \lceil (x^\bullet))\mathbb{I})]$
 $\ \& \ ((R \sqcup S) \lceil (x^\bullet))\mathbb{I} \equiv ((R \lceil (x^\bullet))\mathbb{I})$
 $\Rightarrow \mathcal{N}[k, ((R \sqcup S) \lceil (x^\bullet))\mathbb{I}]$
,! 31 ($(\)E$: 30) ;
 $\mathcal{N}[k, ((R \sqcup S) \lceil (x^\bullet))\mathbb{I}]$,! 32 ($\Rightarrow E$: 29,31) ;
 $P[x] \Rightarrow \mathcal{N}[k, ((R \sqcup S) \lceil (x^\bullet))\mathbb{I}]$,! 33 ($\Rightarrow I$: 17,32) ;
 $Q[x]$,! 34 (Prem) ;
 $(Q[x] \Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))\mathbb{I})])$,! 35 ($\forall E$: 11) ;
 $Q[x] \Rightarrow \mathcal{N}[k, ((S \lceil (x^\bullet))\mathbb{I})]$,! 36 ($(\)E$: 35) ;
 $\mathcal{N}[k, ((S \lceil (x^\bullet))\mathbb{I})]$,! 37 ($\Rightarrow E$: 34,36) ;

$\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$,! 38 (&I: 9,37) ;

$\mathbf{Q}[\mathbf{x}] \ \& \ ((\mathbf{R}^D) \cap \mathbf{Q}) \equiv \phi$,! 39 (&I: 6,34) ;

$(\mathbf{Q}[\mathbf{x}] \ \& \ ((\mathbf{R}^D) \cap \mathbf{Q}) \equiv \phi \Rightarrow \neg (\mathbf{R}^D)[\mathbf{x}])$
 ,! 40 (\forall E: II5.24) ;

$\mathbf{Q}[\mathbf{x}] \ \& \ ((\mathbf{R}^D) \cap \mathbf{Q}) \equiv \phi \Rightarrow \neg (\mathbf{R}^D)[\mathbf{x}]$
 ,! 41 (()E: 40) ;

$\neg (\mathbf{R}^D)[\mathbf{x}]$,! 42 (\Rightarrow E: 39,41) ;

$(\neg (\mathbf{R}^D)[\mathbf{x}] \Rightarrow ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}) \equiv ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})$
 ,! 43 (\forall E: III7.44) ;

$\neg (\mathbf{R}^D)[\mathbf{x}] \Rightarrow ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I} \equiv ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})$
 ,! 44 (()E: 43) ;

$((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I} \equiv ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})$
 ,! 45 (\Rightarrow E: 42,44) ;

$\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$
 $\ \& \ ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I} \equiv ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})$
 ,! 46 (&I: 38,45) ;

$(\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$
 $\ \& \ ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I} \equiv ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})$
 $\Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}])$
 ,! 47 (\forall E: C4.6) ;

$\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k}, ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})]$
 $\ \& \ ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I} \equiv ((\mathbf{S} \lceil (\mathbf{x}^\bullet))\mathbb{I})$
 $\Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}]$
 ,! 48 (()E: 47) ;

$\mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}]$,! 49 (\Rightarrow E: 46,48) ;

$\mathbf{Q}[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}]$
 ,! 50 (\Rightarrow I: 34,49) ;

$\mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}]$,! 51 (\forall E: 16,33,50) ;

$(\mathbf{P} \cup \mathbf{Q})[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}]$
 ,! 52 (\Rightarrow I: 13,51) ;

$((\mathbf{P} \cup \mathbf{Q})[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}])$
 ,! 53 (()I: 52) ;

$\forall \mathbf{x} ((\mathbf{P} \cup \mathbf{Q})[\mathbf{x}] \Rightarrow \mathfrak{N}[\mathbf{k}, ((\mathbf{R} \sqcup \mathbf{S}) \lceil (\mathbf{x}^\bullet))\mathbb{I}])$

,! 54 ($\forall I$: 12,53) i

$\omega[k] \ \& \ \forall x \ ((P \cup Q)[x] \Rightarrow \mathfrak{N}[k, (((R \sqcup S) \lceil (x^\bullet))\bar{I})])$
,! 55 ($\&I$: 9,54) i

$(R \sqcup S) \cup_k (P \cup Q)$,! 56 ($\mathfrak{S}I$: P1,55) i

$R \cup_k P \ \& \ S \cup_k Q \ \& \ ((S^D) \cap P) \equiv \phi \ \& \ ((R^D) \cap Q) \equiv \phi$
 $\Rightarrow (R \sqcup S) \cup_k (P \cup Q)$
,! 57 ($\Rightarrow I$: 2,56) i

$(R \cup_k P \ \& \ S \cup_k Q \ \& \ ((S^D) \cap P) \equiv \phi \ \& \ ((R^D) \cap Q) \equiv \phi$
 $\Rightarrow (R \sqcup S) \cup_k (P \cup Q))$
,! 58 ($(\)I$: 57) i

$\forall R \forall S \forall P \forall Q \forall k \ (R \cup_k P \ \& \ S \cup_k Q \ \& \ ((S^D) \cap P) \equiv \phi \ \& \ ((R^D) \cap Q) \equiv \phi$
 $\Rightarrow (R \sqcup S) \cup_k (P \cup Q))$
! 59 ($\forall I$: 1,58) i

□

! 14. i

$\vdash \forall R \forall P \forall B \forall a \forall k \ (R \cup_k P \ \& \ (R^D) \subseteq P \ \& \ \neg P[a] \ \& \ \mathfrak{N}[k, B]$
 $\Rightarrow (R \sqcup (a \times B)) \cup_k (P \cup (a^\bullet)))$ i

R, P, B, a, k ,! 1 (Prem) i

$R \cup_k P \ \& \ (R^D) \subseteq P \ \& \ \neg P[a] \ \& \ \mathfrak{N}[k, B]$,! 2 (Prem) i

$R \cup_k P$,! 3 ($\&E$: 2) i

$(R^D) \subseteq P$,! 4 ($\&E$: 2) i

$\neg P[a]$,! 5 ($\&E$: 2) i

$\mathfrak{N}[k, B]$,! 6 ($\&E$: 2) i

$(R \cup_k P \Rightarrow \omega[k])$,! 7 ($\forall E$: P2) i

$R \cup_k P \Rightarrow \omega[k]$,! 8 ($(\)E$: 7) i

$\omega[k]$,! 9 ($\Rightarrow E$: 3,8) i

$(R \cup_k P \ \& \ (a \times B) \cup_k (a^\bullet) \ \& \ (((a \times B)^D) \cap P) \equiv \phi$
 $\ \& \ ((R^D) \cap (a^\bullet)) \equiv \phi$
 $\Rightarrow (R \sqcup (a \times B)) \cup_k (P \cup (a^\bullet)))$
,! 10 ($\forall E$: P13) i

$R \cup_k P \ \& \ (a \times B) \cup_k (a^\bullet) \ \& \ (((a \times B)^D) \cap P) \equiv \phi$

$\& ((R^D) \cap (a^\bullet)) \equiv \phi$
 $\Rightarrow (R \sqcup (a \times B)) \cup_k (P \cup (a^\bullet))$,! 11 ((E: 10) i
 $\omega[k] \& \eta[k, B]$,! 12 (&I: 6,9) i
 $(\omega[k] \& \eta[k, B] \Rightarrow (a \times B) \cup_k (a^\bullet))$,! 13 ($\forall E$: P12) i
 $\omega[k] \& \eta[k, B] \Rightarrow (a \times B) \cup_k (a^\bullet)$,! 14 ((E: 13) i
 $(a \times B) \cup_k (a^\bullet)$,! 15 ($\Rightarrow E$: 12,14) i
 $R \cup_k P \& (a \times B) \cup_k (a^\bullet)$,! 16 (&I: 3,15) i
 $((a \times B)^D) \subseteq (a^\bullet)$,! 17 ($\forall E$: III16.17) i
 $\neg P[a] \& ((a \times B)^D) \subseteq (a^\bullet)$,! 18 (&I: 5,17) i
 $(\neg P[a] \& ((a \times B)^D) \subseteq (a^\bullet) \Rightarrow ((a \times B)^D) \cap P \equiv \phi)$,! 19 ($\forall E$: II8.42) i
 $\neg P[a] \& ((a \times B)^D) \subseteq (a^\bullet) \Rightarrow (((a \times B)^D) \cap P) \equiv \phi$,! 20 ((E: 19) i
 $((a \times B)^D) \cap P \equiv \phi$,! 21 ($\Rightarrow E$: 18,20) i
 $R \cup_k P \& (a \times B) \cup_k (a^\bullet) \& (((a \times B)^D) \cap P) \equiv \phi$,! 22 (&I: 16,21) i
 $\neg P[a] \& (R^D) \subseteq P$,! 23 (&I: 4,5) i
 $(\neg P[a] \& (R^D) \subseteq P \Rightarrow ((R^D) \cap (a^\bullet)) \equiv \phi)$,! 24 ($\forall E$: II8.41) i
 $\neg P[a] \& (R^D) \subseteq P \Rightarrow ((R^D) \cap (a^\bullet)) \equiv \phi$,! 25 ((E: 24) i
 $((R^D) \cap (a^\bullet)) \equiv \phi$,! 26 ($\Rightarrow E$: 23,25) i
 $R \cup_k P \& (a \times B) \cup_k (a^\bullet) \& (((a \times B)^D) \cap P) \equiv \phi$
 $\& ((R^D) \cap (a^\bullet)) \equiv \phi$,! 27 (&I: 22,26) i
 $(R \sqcup (a \times B)) \cup_k (P \cup (a^\bullet))$,! 28 ($\Rightarrow E$: 11,27) i
 $R \cup_k P \& (R^D) \subseteq P \& \neg P[a] \& \eta[k, B]$
 $\Rightarrow (R \sqcup (a \times B)) \cup_k (P \cup (a^\bullet))$,! 29 ($\Rightarrow I$: 2,28) i
 $(R \cup_k P \& (R^D) \subseteq P \& \neg P[a] \& \eta[k, B]$

$\Rightarrow (R \sqcup (a \times B)) \cup_k (P \cup (a^\bullet))$,! 30 ((I: 29) i

$\forall R \forall P \forall B \forall a \forall k (R \cup_k P \ \& \ (R^D) \subseteq P \ \& \ \neg P[a] \ \& \ \mathcal{N}_k[k, B]$
 $\Rightarrow (R \sqcup (a \times B)) \cup_k (P \cup (a^\bullet))$)
 ! 31 ($\forall I$: 1,30) i

□

! 15. P15 will help to show that two finite numbers have a finite product. i

$\vdash \forall n \forall k \forall P \forall R (\omega[n] \ \& \ \mathcal{N}_{[n,P]} \ \& \ R \cup_k P \Rightarrow f ((R \sqsupset P)^I))$ i

! First, we prove that

$\forall n (\omega[n] \Rightarrow \forall k \forall P \forall R (\mathcal{N}_{[n,P]} \ \& \ R \cup_k P \Rightarrow f ((R \sqsupset P)^I)))$

by Induction, taking ϕ to be

$\forall k \forall P \forall R (\mathcal{N}_{[n,P]} \ \& \ R \cup_k P \Rightarrow f ((R \sqsupset P)^I))$

It must be shown that

$\forall k \forall P \forall R (\mathcal{N}_{[0,P]} \ \& \ R \cup_k P \Rightarrow f ((R \sqsupset P)^I))$

and

$\forall n \forall m (\omega[n] \ \& \ \sigma[n,m]$
 $\ \& \ \forall k \forall P \forall R (\mathcal{N}_{[n,P]} \ \& \ R \cup_k P \Rightarrow f ((R \sqsupset P)^I))$
 $\Rightarrow \forall k \forall P \forall R (\mathcal{N}_{[m,P]} \ \& \ R \cup_k P \Rightarrow f ((R \sqsupset P)^I)))$ i

! To prove:

$\forall k \forall P \forall R (\mathcal{N}_{[0,P]} \ \& \ R \cup_k P \Rightarrow f ((R \sqsupset P)^I))$ i

k, P, R ,! 1 (Prem) i

$\mathcal{N}_{[0,P]} \ \& \ R \cup_k P$,! 2 (Prem) i

$\mathcal{N}_{[0,P]}$,! 3 ($\&E$: 2) i

$(\mathcal{N}_{[0,P]} \Rightarrow P \equiv \phi)$,! 4 ($\forall E$: C3.1) i

$\mathcal{N}_{[0,P]} \Rightarrow P \equiv \phi$,! 5 ((E): 4) i

$P \equiv \phi$,! 6 ($\Rightarrow E$: 3,5) i

$(P \equiv \phi \Rightarrow ((R \sqsupset P)^I) \equiv \phi)$,! 7 ($\forall E$: III7.40) i

$P \equiv \phi \Rightarrow ((R \sqsupset P)^I) \equiv \phi$,! 8 ((E): 7) i

$((R \sqsupset P)^I) \equiv \phi$,! 9 ($\Rightarrow E$: 6,8) i

$(((R \sqsupset P)^I) \equiv \phi \Rightarrow f ((R \sqsupset P)^I))$,! 10 ($\forall E$: C5.8) i

$((R \sqsupset P)^I) \equiv \phi \Rightarrow f ((R \sqsupset P)^I)$,! 11 ((E): 10) i

$f ((R \sqsupset P)^I)$,! 12 ($\Rightarrow E$: 9,11) i

$\mathcal{N}_{[0,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})$,! 13 ($\Rightarrow\mathbf{I}$: 2,12) ;
 $(\ \mathcal{N}_{[0,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}) \)$,! 14 ($(\)\mathbf{I}$: 13) ;
 $\forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\ \mathcal{N}_{[0,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}) \)$
,! 15 ($\forall\mathbf{I}$: 1,14) ;
! To prove:
 $\forall \mathbf{n} \forall \mathbf{m} \ (\ \omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}]$
 $\ \& \ \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\ \mathcal{N}_{[\mathbf{n},\mathbf{P}]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}) \)$
 $\Rightarrow \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\ \mathcal{N}_{[\mathbf{m},\mathbf{P}]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}) \) \)$;
 \mathbf{n},\mathbf{m} ,! 16 (Prem) ;
 $\omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}] \ \& \ \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\ \mathcal{N}_{[\mathbf{n},\mathbf{P}]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}) \)$
,! 17 (Prem) ;
 $\omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}]$,! 18 ($\&\mathbf{E}$: 17) ;
 $\forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\ \mathcal{N}_{[\mathbf{n},\mathbf{P}]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}) \)$
,! 19 ($\&\mathbf{E}$: 17) ;
 $\mathbf{k},\mathbf{P},\mathbf{R}$,! 20 (Prem) ;
 $\mathcal{N}_{[\mathbf{m},\mathbf{P}]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P}$,! 21 (Prem) ;
 $\mathcal{N}_{[\mathbf{m},\mathbf{P}]}$,! 22 ($\&\mathbf{E}$: 21) ;
 $\mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P}$,! 23 ($\&\mathbf{E}$: 21) ;
 $(\ \omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}] \Rightarrow (\mathbf{n}') = \mathbf{m} \)$,! 24 ($\forall\mathbf{E}$: C8.18) ;
 $\omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}] \Rightarrow (\mathbf{n}') = \mathbf{m}$,! 25 ($(\)\mathbf{E}$: 25) ;
 $(\mathbf{n}') = \mathbf{m}$,! 26 ($\Rightarrow\mathbf{E}$: 18,25) ;
 $\neg (\mathbf{n}') = 0$,! 27 ($\forall\mathbf{E}$: C8.32) ;
 $\neg \mathbf{m} = 0$,! 28 ($=\mathbf{E}$: 26,27) ;
 $\mathcal{N}_{[\mathbf{m},\mathbf{P}]} \ \& \ \neg \mathbf{m} = 0$,! 29 ($\&\mathbf{I}$: 22,28) ;
 $(\ \mathcal{N}_{[\mathbf{m},\mathbf{P}]} \ \& \ \neg \mathbf{m} = 0 \Rightarrow \exists \mathbf{x} \ \mathbf{P}[\mathbf{x}] \)$,! 30 ($\forall\mathbf{E}$: C3.17) ;
 $\mathcal{N}_{[\mathbf{m},\mathbf{P}]} \ \& \ \neg \mathbf{m} = 0 \Rightarrow \exists \mathbf{x} \ \mathbf{P}[\mathbf{x}]$,! 31 ($(\)\mathbf{E}$: 30) ;
 $\exists \mathbf{x} \ \mathbf{P}[\mathbf{x}]$,! 32 ($\Rightarrow\mathbf{E}$: 29,31) ;
 $\mathbf{P}[\mathbf{a}]$,! 33 ($\exists\mathbf{E}$: 32) ;
 $(\ \mathbf{P}[\mathbf{a}] \Rightarrow (\ \mathbf{P} \setminus (\mathbf{a}^{\bullet}) \) \cup \ (\mathbf{a}^{\bullet}) \) \equiv \mathbf{P} \)$
,! 34 ($\forall\mathbf{E}$: II8.56) ;

$$P[a] \Rightarrow ((P \setminus (a^\bullet)) \cup (a^\bullet)) \equiv P, ! 35 ((E: 34) \quad ;$$

$$((P \setminus (a^\bullet)) \cup (a^\bullet)) \equiv P, ! 36 (\Rightarrow E: 33,35) \quad ;$$

$$\begin{aligned} & (((P \setminus (a^\bullet)) \cup (a^\bullet)) \equiv P \\ & \Rightarrow (((R \lceil (P \setminus (a^\bullet)))^I) \cup ((R \lceil (a^\bullet))^I)) \\ & \equiv ((R \lceil P)^I)) \end{aligned} ,! 37 (\forall E: III7.43) \quad ;$$

$$\begin{aligned} & ((P \setminus (a^\bullet)) \cup (a^\bullet)) \equiv P \\ & \Rightarrow (((R \lceil (P \setminus (a^\bullet)))^I) \cup ((R \lceil (a^\bullet))^I)) \equiv ((R \lceil P)^I) \end{aligned} ,! 38 ((E: 37) \quad ;$$

$$(((R \lceil (P \setminus (a^\bullet)))^I) \cup ((R \lceil (a^\bullet))^I)) \equiv ((R \lceil P)^I) ,! 39 (\Rightarrow E: 36,38) \quad ;$$

$$\omega[n] \ \& \ \sigma[n,m] \ \& \ P[a], ! 40 (\&I: 18,33) \quad ;$$

$$\omega[n] \ \& \ \sigma[n,m] \ \& \ P[a] \ \& \ \eta_l[m,P], ! 41 (\&I: 22,40) \quad ;$$

$$(\omega[n] \ \& \ \sigma[n,m] \ \& \ P[a] \ \& \ \eta_l[m,P] \Rightarrow \eta_l[n, (P \setminus (a^\bullet))]) ,! 42 (\forall E: C2.11) \quad ;$$

$$\omega[n] \ \& \ \sigma[n,m] \ \& \ P[a] \ \& \ \eta_l[m,P] \Rightarrow \eta_l[n, (P \setminus (a^\bullet))] ,! 43 ((E: 42) \quad ;$$

$$\eta_l[n, (P \setminus (a^\bullet))] ,! 44 (\Rightarrow E: 41,43) \quad ;$$

$$(P \setminus (a^\bullet)) \subseteq P, ! 45 (\forall E: II7.13) \quad ;$$

$$R \cup_k P \ \& \ (P \setminus (a^\bullet)) \subseteq P, ! 46 (\&I: 23,45) \quad ;$$

$$(R \cup_k P \ \& \ (P \setminus (a^\bullet)) \subseteq P \Rightarrow R \cup_k (P \setminus (a^\bullet))) ,! 47 (\forall E: P4) \quad ;$$

$$R \cup_k P \ \& \ (P \setminus (a^\bullet)) \subseteq P \Rightarrow R \cup_k (P \setminus (a^\bullet)) ,! 48 ((E: 47) \quad ;$$

$$R \cup_k (P \setminus (a^\bullet)) ,! 49 (\Rightarrow E: 46,48) \quad ;$$

$$\eta_l[n, (P \setminus (a^\bullet))] \ \& \ R \cup_k (P \setminus (a^\bullet)) ,! 50 (\&I: 44,49) \quad ;$$

! Applying the Induction hypothesis... i

$$\begin{aligned} & (\eta_l[n, (P \setminus (a^\bullet))] \ \& \ R \cup_k (P \setminus (a^\bullet)) \\ & \Rightarrow f ((R \lceil (P \setminus (a^\bullet)))^I)) \end{aligned} ,! 51 (\forall E: 19) \quad ;$$

$$\begin{aligned} & (\#(P \setminus (a^\bullet))) = n \ \& \ R \cup_k (P \setminus (a^\bullet)) \\ & \Rightarrow f ((R \lceil (P \setminus (a^\bullet)))^I) \end{aligned}$$

,! 52 ((E: 51) ;

$$f ((R \lceil (P \setminus (a^\bullet)))^I)$$

,! 53 (\Rightarrow E: 50,52) ;

$$f ((R \lceil (P \setminus (a^\bullet)))^I)$$

$$\& (((R \lceil (P \setminus (a^\bullet)))^I) \cup ((R \lceil (a^\bullet))^I)) \equiv ((R \lceil P)^I)$$

,! 54 (&I: 39,53) ;

$$\omega[k] \& \forall x (P[x] \Rightarrow (\#((R \lceil (x^\bullet))^I)) = k)$$

,! 55 ($\$$ E: P1,23) ;

$$\omega[k]$$

,! 56 (&E: 55) ;

$$\forall x (P[x] \Rightarrow \mathfrak{N}[k, ((R \lceil (x^\bullet))^I)])$$

,! 57 (&E: 55) ;

$$(P[a] \Rightarrow \mathfrak{N}[k, ((R \lceil (a^\bullet))^I)])$$

,! 58 (\forall E: 57) ;

$$P[a] \Rightarrow \mathfrak{N}[k, ((R \lceil (a^\bullet))^I)]$$

,! 59 ((E: 58) ;

$$\mathfrak{N}[k, ((R \lceil (a^\bullet))^I)]$$

,! 60 (\Rightarrow E: 33,59) ;

$$\omega[k] \& \mathfrak{N}[k, ((R \lceil (a^\bullet))^I)]$$

,! 61 (&I: 56,60) ;

$$(\omega[k] \& \mathfrak{N}[k, ((R \lceil (a^\bullet))^I)] \Rightarrow f ((R \lceil (a^\bullet))^I))$$

,! 62 (\forall E: C5.2) ;

$$\omega[k] \& \mathfrak{N}[k, ((R \lceil (a^\bullet))^I)] \Rightarrow f ((R \lceil (a^\bullet))^I)$$

,! 63 ((E: 62) ;

$$f ((R \lceil (a^\bullet))^I)$$

,! 64 (\Rightarrow E: 61,63) ;

$$f ((R \lceil (P \setminus (a^\bullet)))^I)$$

$$\& f ((R \lceil (a^\bullet))^I)$$

$$\& (((R \lceil (P \setminus (a^\bullet)))^I) \cup ((R \lceil (a^\bullet))^I)) \equiv ((R \lceil P)^I)$$

,! 65 (&I: 54,64) ;

$$(f ((R \lceil (P \setminus (a^\bullet)))^I)$$

$$\& f ((R \lceil (a^\bullet))^I)$$

$$\& (((R \lceil (P \setminus (a^\bullet)))^I) \cup ((R \lceil (a^\bullet))^I)) \equiv ((R \lceil P)^I)$$

$$\Rightarrow f ((R \lceil P)^I))$$

,! 66 (\forall E: C5.21) ;

$$f ((R \lceil (P \setminus (a^\bullet)))^I)$$

$$\& f ((R \lceil (a^\bullet))^I)$$

$$\& (((R \lceil (P \setminus (a^\bullet)))^I) \cup ((R \lceil (a^\bullet))^I)) \equiv ((R \lceil P)^I)$$

$$\Rightarrow f ((R \lceil P)^I)$$

,! 67 ((E: 66) ;

$$f ((R \lceil P)^I)$$

,! 68 (\Rightarrow E: 65,67) ;

$$\mathfrak{N}[m, P] \& R \cup_k P \Rightarrow f ((R \lceil P)^I)$$

,! 69 (\Rightarrow I: 21,68) ;

$(\mathcal{N}_{[m,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$, ! 70 ((I: 69) i
$\forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[m,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$, ! 71 (\forall I: 20,70) i
$\omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}] \ \& \ \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$	
$\Rightarrow \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[m,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$, ! 72 (\Rightarrow I: 17,71) i
$(\omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}] \ \& \ \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$	
$\Rightarrow \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[m,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})))$, ! 73 ((I: 72) i
$\forall \mathbf{n} \forall \mathbf{m} \ (\omega[\mathbf{n}] \ \& \ \sigma[\mathbf{n},\mathbf{m}] \ \& \ \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$	
$\Rightarrow \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[m,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})))$, ! 74 (\forall I: 16,73) i
$\forall \mathbf{n} \ (\omega[\mathbf{n}] \Rightarrow \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})))$, ! 75 (Induct: 15,74) i
! Now to prove the proposition proper...	i
n, k, P, R	, ! 76 (Prem) i
$\omega[\mathbf{n}] \ \& \ \mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P}$, ! 77 (Prem) i
$\omega[\mathbf{n}]$, ! 78 (&E: 77) i
$\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P}$, ! 79 (&E: 77) i
$(\omega[\mathbf{n}] \Rightarrow \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})))$, ! 80 (\forall E: 75) i
$\omega[\mathbf{n}] \Rightarrow \forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$, ! 81 ((E: 80) i
$\forall \mathbf{k} \forall \mathbf{P} \forall \mathbf{R} \ (\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$, ! 82 (\Rightarrow E: 78,81) i
$(\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}}))$, ! 83 (\forall E: 82) i
$\mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})$, ! 84 ((E: 83) i
$f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})$, ! 85 (\Rightarrow E: 79,84) i
$\omega[\mathbf{n}] \ \& \ \mathcal{N}_{[n,P]} \ \& \ \mathbf{R} \ \cup_{\mathbf{k}} \ \mathbf{P} \Rightarrow f \ ((\mathbf{R} \ \lceil \ \mathbf{P})^{\mathbf{I}})$, ! 86 (\Rightarrow I: 77,85) i

($\omega[n] \ \& \ \mathfrak{N}_{[n,P]} \ \& \ R \ \cup_k \ P \Rightarrow f \left((R \uparrow P)^I \right)$)
, ! 87 ((I: 86) i

$\forall n \forall k \forall P \forall R \left(\omega[n] \ \& \ \mathfrak{N}_{[n,P]} \ \& \ R \ \cup_k \ P \Rightarrow f \left((R \uparrow P)^I \right) \right)$
! 88 ($\forall I$: 76,87) i

□

! 16. P16 is important because it shows that, given two finite numbers, it is possible to find a relationship and a predicate, which permit the Mult axiom to be used. i

$\vdash \forall n \forall j \left(\omega[n] \ \& \ \omega[j] \right.$
 $\Rightarrow \exists P \exists R \left(\mathfrak{N}_{[n,P]} \ \& \ R \ \cup_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f \left(R^I \right) \right))$
i

! First, we use induction to prove that

$\forall n \left(\omega[n] \right.$
 $\Rightarrow \forall j \left(\omega[j] \right.$
 $\Rightarrow \exists P \exists R \left(\mathfrak{N}_{[n,P]} \ \& \ R \ \cup_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f \left(R^I \right) \right)))$

taking ϕ to be

$\forall j \left(\omega[j] \right.$
 $\Rightarrow \exists P \exists R \left(\mathfrak{N}_{[n,P]} \ \& \ R \ \cup_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f \left(R^I \right) \right))$

It must be shown that

$\forall j \left(\omega[j] \right.$
 $\Rightarrow \exists P \exists R \left(\mathfrak{N}_{[0,P]} \ \& \ R \ \cup_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f \left(R^I \right) \right))$

and

$\forall n \forall m \left(\omega[n] \ \& \ \sigma[n,m] \right.$
 $\ \& \ \forall j \left(\omega[j] \Rightarrow \exists P \exists R \left(\mathfrak{N}_{[n,P]} \ \& \ R \ \cup_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f \left(R^I \right) \right) \right))$
 $\Rightarrow \forall j \left(\omega[j] \Rightarrow \exists P \exists R \left(\mathfrak{N}_{[m,P]} \ \& \ R \ \cup_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f \left(R^I \right) \right) \right))$
i

! To prove:

$\forall j \left(\omega[j] \right.$
 $\Rightarrow \exists P \exists R \left(\mathfrak{N}_{[0,P]} \ \& \ R \ \cup_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f \left(R^I \right) \right))$

ϕ will be used for P and Φ for R. i

j , ! 1 (Prem) i

$\omega[j]$, ! 2 (Prem) i

$\Phi \cup_j \phi$, ! 3 ($\forall E$: P10) i

$\mathfrak{N}_{[0,\phi]} \ \& \ \Phi \cup_j \phi$, ! 4 ($\&I$: C3.14,3) i

$\mathfrak{N}_{[0,\phi]} \ \& \ \Phi \cup_j \phi \ \& \ \mathbf{1} \ \Phi$, ! 5 ($\&I$: III9.18,4) i

$\mathfrak{N}_{[0,\phi]} \ \& \ \Phi \cup_j \phi \ \& \ \mathbf{1} \ \Phi \ \& \ (\Phi^D) \subseteq \phi$, ! 6 ($\&I$: III5.21,5)

i

$f \phi \ \& \ (\Phi^I) \equiv \phi$,! 7 (&I: III6.28,C5.7) i

$(f \phi \ \& \ (\Phi^I) \equiv \phi \Rightarrow f (\Phi^I))$,! 8 (\forall E: C5.6) i

$f \phi \ \& \ (\Phi^I) \equiv \phi \Rightarrow f (\Phi^I)$,! 9 ($(\)$ E: 8) i

$f (\Phi^I)$,! 10 (\Rightarrow E: 7,9) i

$\mathfrak{N}_{[0,\phi]} \ \& \ \Phi \ \vee \ j \ \phi \ \& \ \mathbf{1} \ \Phi \ \& \ (\Phi^D) \subseteq \phi \ \& \ f (\Phi^I)$
,! 11 (&I: 6,10) i

$(\mathfrak{N}_{[0,\phi]} \ \& \ \Phi \ \vee \ j \ \phi \ \& \ \mathbf{1} \ \Phi \ \& \ (\Phi^D) \subseteq \phi \ \& \ f (\Phi^I))$
,! 12 ($(\)$ I: 11) i

$\exists R (\mathfrak{N}_{[0,\phi]} \ \& \ R \ \vee \ j \ \phi \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq \phi \ \& \ f (R^I))$
,! 13 (\exists I: 12) i

$\exists P \exists R (\mathfrak{N}_{[0,P]} \ \& \ R \ \vee \ j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I))$
,! 14 (\exists I: 13) i

$\omega[j]$
 $\Rightarrow \exists P \exists R (\mathfrak{N}_{[0,P]} \ \& \ R \ \vee \ j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I))$
,! 15 (\Rightarrow I: 2,14) i

$(\omega[j]$
 $\Rightarrow \exists P \exists R (\mathfrak{N}_{[0,P]} \ \& \ R \ \vee \ j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)))$
,! 16 ($(\)$ I: 15) i

$\forall j (\omega[j] \Rightarrow \exists P \exists R (\mathfrak{N}_{[0,P]} \ \& \ R \ \vee \ j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)))$
,! 17 (\forall I: 1,16) i

! To prove:
 $\forall n \forall m (\omega[n] \ \& \ \sigma[n,m]$
 $\ \& \ \forall j (\omega[j] \Rightarrow \exists P \exists R (\mathfrak{N}_{[n,P]} \ \& \ R \ \vee \ j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P$
 $\ \& \ f (R^I)))$
 $\Rightarrow \forall j (\omega[j] \Rightarrow \exists P \exists R (\mathfrak{N}_{[m,P]} \ \& \ R \ \vee \ j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P$
 $\ \& \ f (R^I))))$ i

n,m ,! 18 (Prem) i

$\omega[n] \ \& \ \sigma[n,m]$
 $\ \& \ \forall j (\omega[j]$
 $\Rightarrow \exists P \exists R (\mathfrak{N}_{[n,P]} \ \& \ R \ \vee \ j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)))$
,! 19 (Prem) i

$\omega[n] \ \& \ \sigma[n,m]$,! 20 (&E: 19) i

$$\begin{array}{l}
\forall j (\omega[j] \\
\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I))) \\
, ! \ 21 \ (\&E: 19) \quad ; \\
j \\
, ! \ 22 \ (\text{Prem}) \quad ; \\
\omega[j] \\
, ! \ 23 \ (\text{Prem}) \quad ; \\
(\omega[j] \\
\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I))) \\
, ! \ 24 \ (\forall E: 21) \quad ; \\
\omega[j] \\
\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)) \\
, ! \ 25 \ (()E: 24) \quad ; \\
\exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)) \\
, ! \ 26 \ (\Rightarrow E: 23, 25) \quad ; \\
\exists R (\mathcal{N}[n, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)) \\
, ! \ 27 \ (\exists E: 26) \quad ; \\
(\mathcal{N}[n, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I)) \\
, ! \ 28 \ (\exists E: 27) \quad ; \\
\mathcal{N}[n, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I) \\
, ! \ 29 \ (()E: 28) \quad ; \\
\mathcal{N}[n, P] \\
, ! \ 30 \ (\&E: 29) \quad ; \\
R \ \cup_j P \\
, ! \ 31 \ (\&E: 29) \quad ; \\
\mathbf{1} \ R \\
, ! \ 32 \ (\&E: 29) \quad ; \\
(R^D) \subseteq P \\
, ! \ 33 \ (\&E: 29) \quad ; \\
f (R^I) \\
, ! \ 34 \ (\&E: 29) \quad ;
\end{array}$$

! To prove:

$$\exists P \exists R (\mathcal{N}[m, P] \ \& \ R \ \cup_j P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f (R^I))$$

It will be shown that there exists an \mathbf{a} s.t. $\neg P[\mathbf{a}]$ and \mathbf{B} s.t. $((R^I) \cap \mathbf{B}) \equiv \phi$ and $\mathcal{N}[j, \mathbf{B}]$. $(P \cup (\mathbf{a}^\bullet))$ will be used for P and $(R \sqcup (\mathbf{a} \times \mathbf{B}))$ for R . i

$$\omega[n] \ \& \ \mathcal{N}[n, P] \ \& \ \sigma[n, m] \quad , ! \ 35 \ (\&I: 20, 30) \quad ;$$

$$\begin{array}{l}
(\omega[n] \ \& \ \mathcal{N}[n, P] \ \& \ \sigma[n, m] \\
\Rightarrow \exists \mathbf{a} (\neg P[\mathbf{a}] \ \& \ \mathcal{N}[m, (P \cup (\mathbf{a}^\bullet))])) \\
, ! \ 36 \ (\forall E: C7.16) \quad ;
\end{array}$$

$\mathbf{1} \mathbf{R} \ \& \ ((\mathbf{R}^{\mathbf{I}}) \cap \mathbf{B}) \equiv \phi$,! 58 (&I: 32,50) i

$(\mathbf{1} \mathbf{R} \ \& \ ((\mathbf{R}^{\mathbf{I}}) \cap \mathbf{B}) \equiv \phi \Rightarrow \mathbf{1} (\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})))$
,! 59 (\forall E: III16.31) i

$\mathbf{1} \mathbf{R} \ \& \ ((\mathbf{R}^{\mathbf{I}}) \cap \mathbf{B}) \equiv \phi \Rightarrow \mathbf{1} (\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))$
,! 60 ((E): 59) i

$\mathbf{1} (\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))$,! 61 (\Rightarrow E: 58,60) i

$\mathfrak{N}[\mathbf{m}, (\mathbf{P} \cup (\mathbf{a}^{\bullet}))]$
& $(\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \ \vee_j (\mathbf{P} \cup (\mathbf{a}^{\bullet}))$
& $\mathbf{1} (\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))$
,! 62 (&I: 57,61) i

$((\mathbf{R}^{\mathbf{D}}) \subseteq \mathbf{P} \Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{D}}) \subseteq (\mathbf{P} \cup (\mathbf{a}^{\bullet})))$
,! 63 (\forall E: III16.20) i

$(\mathbf{R}^{\mathbf{D}}) \subseteq \mathbf{P} \Rightarrow ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{D}}) \subseteq (\mathbf{P} \cup (\mathbf{a}^{\bullet}))$
,! 64 ((E): 63) i

$((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{D}}) \subseteq (\mathbf{P} \cup (\mathbf{a}^{\bullet}))$,! 65 (\Rightarrow E: 33,64) i

$\mathfrak{N}[\mathbf{m}, (\mathbf{P} \cup (\mathbf{a}^{\bullet}))]$
& $(\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B})) \ \vee_j (\mathbf{P} \cup (\mathbf{a}^{\bullet}))$
& $\mathbf{1} (\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))$
& $((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{D}}) \subseteq (\mathbf{P} \cup (\mathbf{a}^{\bullet}))$
,! 66 (&I: 62,65) i

$\omega[\mathbf{j}] \ \& \ \mathfrak{N}[\mathbf{j}, \mathbf{B}] \ \& \ f (\mathbf{R}^{\mathbf{I}})$,! 67 (&I: 43,49) i

$(\omega[\mathbf{j}] \ \& \ \mathfrak{N}[\mathbf{j}, \mathbf{B}] \ \& \ f (\mathbf{R}^{\mathbf{I}}) \Rightarrow f ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B}))$
,! 68 (\forall E: C5.18) i

$\omega[\mathbf{j}] \ \& \ \mathfrak{N}[\mathbf{j}, \mathbf{B}] \ \& \ f (\mathbf{R}^{\mathbf{I}}) \Rightarrow f ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B})$
,! 69 ((E): 68) i

$f ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B})$,! 70 (\Rightarrow E: 67,69) i

$((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{I}}) \equiv ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B})$,! 71 (\forall E: III16.23) i

$f ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B}) \ \& \ ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{I}}) \equiv ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B})$
,! 72 (&I: 70,71) i

$(f ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B}) \ \& \ ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{I}}) \equiv ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B})$
 $\Rightarrow f ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{I}}))$
,! 73 (\forall E: C5.6) i

$f ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B}) \ \& \ ((\mathbf{R} \sqcup (\mathbf{a} \times \mathbf{B}))^{\mathbf{I}}) \equiv ((\mathbf{R}^{\mathbf{I}}) \cup \mathbf{B})$

$$\Rightarrow f ((R \sqcup (a \times B))^I) \quad ,! 74 (())E: 73) \quad ;$$

$$f ((R \sqcup (a \times B))^I) \quad ,! 75 (\Rightarrow E: 72,74) \quad ;$$

$$\begin{aligned} & \mathcal{N}_k[m, (P \cup (a^\bullet))] \\ & \& (R \sqcup (a \times B)) \cup_j (P \cup (a^\bullet)) \\ & \& \mathbf{1} (R \sqcup (a \times B)) \\ & \& ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet)) \\ & \& f ((R \sqcup (a \times B))^I) \end{aligned} \quad ,! 76 (\&I: 66,75) \quad ;$$

$$\begin{aligned} & (\mathcal{N}_k[m, (P \cup (a^\bullet))] \\ & \& (R \sqcup (a \times B)) \cup_j (P \cup (a^\bullet)) \\ & \& \mathbf{1} (R \sqcup (a \times B)) \\ & \& ((R \sqcup (a \times B))^D) \subseteq (P \cup (a^\bullet)) \\ & \& f ((R \sqcup (a \times B))^I)) \end{aligned} \quad ,! 77 (())I: 76) \quad ;$$

$$\begin{aligned} \exists R (\mathcal{N}_k[m, (P \cup (a^\bullet))] \\ & \& R \cup_j (P \cup (a^\bullet)) \\ & \& \mathbf{1} R \\ & \& (R^D) \subseteq (P \cup (a^\bullet)) \\ & \& f (R^I)) \end{aligned} \quad ,! 78 (\exists I: 77) \quad ;$$

$$\begin{aligned} \exists P \exists R (\mathcal{N}_k[m, P] \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f (R^I)) \\ & ,! 79 (\exists I: 78) \quad ; \end{aligned}$$

$$\begin{aligned} \omega[j] \Rightarrow \exists P \exists R (\mathcal{N}_k[m, P] \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f (R^I)) \\ & ,! 80 (\Rightarrow I: 23,79) \quad ; \end{aligned}$$

$$\begin{aligned} (\omega[j] \\ \Rightarrow \exists P \exists R (\mathcal{N}_k[m, P] \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f (R^I))) \\ & ,! 81 (())I: 80) \quad ; \end{aligned}$$

$$\begin{aligned} \forall j (\omega[j] \\ \Rightarrow \exists P \exists R (\mathcal{N}_k[m, P] \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f (R^I))) \\ & ,! 82 (\forall I: 22,81) \quad ; \end{aligned}$$

$$\begin{aligned} \omega[n] \& \sigma[n, m] \\ \& \forall j (\omega[j] \\ \Rightarrow \exists P \exists R (\mathcal{N}_k[n, P] \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f (R^I))) \end{aligned}$$

$$\begin{aligned} \Rightarrow \forall j (\omega[j] \\ \Rightarrow \exists P \exists R (\mathcal{N}_k[m, P] \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f (R^I))) \\ & ,! 83 (\Rightarrow I: 19,82) \quad ; \end{aligned}$$

$$(\omega[n] \& \sigma[n, m])$$

$\omega[j] \Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$
, ! 95 (()E: 94) i

$\exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$
, ! 96 (\Rightarrow E: 90, 95) i

$\omega[n] \ \& \ \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$
, ! 97 (\Rightarrow I: 88, 96) i

($\omega[n] \ \& \ \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$)
, ! 98 (()I: 97) i

$\forall n \forall j (\omega[n] \ \& \ \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$)
, ! 99 (\forall I: 87, 98) i

□

! 17. i

$\vdash \forall n \forall j (\omega[n] \ \& \ \omega[j] \Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R))$ i

n, j , ! 1 (Prem) i

$\omega[n] \ \& \ \omega[j]$, ! 2 (Prem) i

($\omega[n] \ \& \ \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$)
, ! 3 (\forall E: P16) i

$\omega[n] \ \& \ \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$
, ! 4 (()E: 3) i

$\exists P \exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$
, ! 5 (\Rightarrow E: 2, 4) i

$\exists R (\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I))$
, ! 6 (\exists E: 5) i

($\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I)$)
, ! 7 (\exists E: 6) i

$\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R \ \& \ (R^D) \subseteq P \ \& \ f(R^I)$
, ! 8 (()E: 7) i

$\mathcal{N}[n, P] \ \& \ R \ \vee_j \ P \ \& \ \mathbf{1} \ R$, ! 9 ($\&$ E: 8) i

$(\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R)$,! 10 ((I: 9) i
 $\exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R)$,! 11 (\exists I: 10) i
 $\exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R)$,! 12 (\exists I: 11) i
 $\omega[n] \& \omega[j] \Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R)$
,! 13 (\Rightarrow I: 2,12) i
 $(\omega[n] \& \omega[j] \Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R))$
,! 14 ((I: 13) i
 $\forall n \forall j (\omega[n] \& \omega[j] \Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R))$
! 15 (\forall I: 1,14) i
 \square
! 18. i
 $\vdash \forall n \forall j (\omega[n] \& \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P)$) i
 n, j ,! 1 (Prem) i
 $\omega[n] \& \omega[j]$,! 2 (Prem) i
 $(\omega[n] \& \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f(R^I))$)
,! 3 (\forall E: P16) i
 $\omega[n] \& \omega[j]$
 $\Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f(R^I))$
,! 4 ((E: 3) i
 $\exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f(R^I))$
,! 5 (\Rightarrow E: 2,4) i
 $\exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f(R^I))$
,! 6 (\exists E: 5) i
 $(\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f(R^I))$
,! 7 (\exists E: 6) i
 $\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P \& f(R^I)$
,! 8 ((E: 7) i
 $\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P$,! 9 ($\&$ E: 8) i
 $(\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P)$,! 10 ((I: 9) i
 $\exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P)$

,! 11 ($\exists I: 10$) i

$\exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P)$

,! 12 ($\exists I: 11$) i

$\omega[n] \& \omega[j] \Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P)$

,! 13 ($\Rightarrow I: 2,12$) i

$(\omega[n] \& \omega[j] \Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P))$

,! 14 ($(\) I: 13$) i

$\forall n \forall j (\omega[n] \& \omega[j]$

$\Rightarrow \exists P \exists R (\mathcal{N}_{[n,P]} \& R \cup_j P \& \mathbf{1} R \& (R^D) \subseteq P))$

! 15 ($\forall I: 1,14$) i

\square