

$\omega[n] \ \& \ \neg[n,P] \ \& \ P \sim Q$,! 8 (&I: 7)	i
$(\omega[n] \ \& \ \neg[n,P] \ \& \ P \sim Q \Rightarrow \neg[n,Q])$,! 9 (\forall E: C4.2)	i
$\omega[n] \ \& \ \neg[n,P] \ \& \ P \sim Q \Rightarrow \neg[n,Q]$,! 10 ($($)E: 9)	i
$\neg[n,Q]$,! 11 (\Rightarrow E: 8,10)	i
$\omega[n]$,! 12 (&E: 7)	i
$\omega[n] \ \& \ \neg[n,Q]$,! 13 (&I: 11,12)	i
$(\omega[n] \ \& \ \neg[n,Q] \Rightarrow f Q)$,! 14 (\forall E: P2)	i
$\omega[n] \ \& \ \neg[n,Q] \Rightarrow f Q$,! 15 ($($)E: 14)	i
$f Q$,! 16 (\Rightarrow E: 13,15)	i
$f P \ \& \ P \sim Q \Rightarrow f Q$,! 17 (\Rightarrow I: 2,16)	i
$(f P \ \& \ P \sim Q \Rightarrow f Q)$,! 18 ($($)I: 17)	i
$\forall P \forall Q (f P \ \& \ P \sim Q \Rightarrow f Q)$! 19 (\forall I: 1,18)	i

□

! 4.

$\vdash \forall P \forall Q (f P \ \& \ Q \sim P \Rightarrow f Q)$		i
P, Q	,! 1 (Prem)	i
$f P \ \& \ Q \sim P$,! 2 (Prem)	i
$Q \sim P$,! 3 (&E: 2)	i
$(Q \sim P \Rightarrow P \sim Q)$,! 4 (\forall E: III13.4)	i
$Q \sim P \Rightarrow P \sim Q$,! 5 ($($)E: 4)	i
$P \sim Q$,! 6 (\Rightarrow E: 3,5)	i
$f P$,! 7 (&E: 2)	i
$f P \ \& \ P \sim Q$,! 8 (&I: 6,7)	i
$(f P \ \& \ P \sim Q \Rightarrow f Q)$,! 9 (\forall E: P3)	i
$f P \ \& \ P \sim Q \Rightarrow f Q$,! 10 ($($)E: 9)	i
$f Q$,! 11 (\Rightarrow E: 8,10)	i
$f P \ \& \ Q \sim P \Rightarrow f Q$,! 12 (\Rightarrow I: 2,11)	i
$(f P \ \& \ Q \sim P \Rightarrow f Q)$,! 13 ($($)I: 12)	i

$\forall P \forall Q (f P \ \& \ Q \sim P \Rightarrow f Q)$! 14 ($\forall I$: 1,13) i

□

! P5 and P6 assert that predicates equivalent to finite predicates are themselves finite. i

! 5. i

$\vdash \forall P \forall Q (f P \ \& \ P \equiv Q \Rightarrow f Q)$ i

P, Q ,! 1 (Prem) i

$f P \ \& \ P \equiv Q$,! 2 (Prem) i

$P \equiv Q$,! 3 ($\&E$: 2) i

$(P \equiv Q \Rightarrow P \sim Q)$,! 4 ($\forall E$: III13.2) i

$P \equiv Q \Rightarrow P \sim Q$,! 5 ($()E$: 4) i

$P \sim Q$,! 6 ($\Rightarrow E$: 3,5) i

$f P$,! 7 ($\&E$: 2) i

$f P \ \& \ P \sim Q$,! 8 ($\&I$: 6,7) i

$(f P \ \& \ P \sim Q \Rightarrow f Q)$,! 9 ($\forall E$: P3) i

$f P \ \& \ P \sim Q \Rightarrow f Q$,! 10 ($()E$: 9) i

$f Q$,! 11 ($\Rightarrow E$: 8,10) i

$f P \ \& \ P \equiv Q \Rightarrow f Q$,! 12 ($\Rightarrow I$: 2,11) i

$(f P \ \& \ P \equiv Q \Rightarrow f Q)$,! 13 ($()I$: 12) i

$\forall P \forall Q (f P \ \& \ P \equiv Q \Rightarrow f Q)$! 14 ($\forall I$: 1,13) i

□

! 6. i

$\vdash \forall P \forall Q (f P \ \& \ Q \equiv P \Rightarrow f Q)$ i

P, Q ,! 1 (Prem) i

$f P \ \& \ Q \equiv P$,! 2 (Prem) i

$Q \equiv P$,! 3 ($\&E$: 2) i

$(Q \equiv P \Rightarrow P \equiv Q)$,! 4 ($\forall E$: II1.10) i

$Q \equiv P \Rightarrow P \equiv Q$,! 5 ($()E$: 4) i

$P \equiv Q$,! 6 (\Rightarrow E: 3,5)	i
$f P$,! 7 ($\&$ E: 2)	i
$f P \& P \equiv Q$,! 8 ($\&$ I: 6,7)	i
$(f P \& P \equiv Q \Rightarrow f Q)$,! 9 (\forall E: P5)	i
$f P \& P \equiv Q \Rightarrow f Q$,! 10 ($(())$ E: 9)	i
$f Q$,! 11 (\Rightarrow E: 8,10)	i
$f P \& Q \equiv P \Rightarrow f Q$,! 12 (\Rightarrow I: 2,11)	i
$(f P \& Q \equiv P \Rightarrow f Q)$,! 13 ($(())$ I: 12)	i
$\forall P \forall Q (f P \& Q \equiv P \Rightarrow f Q)$,! 14 (\forall I: 1,13)	i

□

! P7 asserts that our empty predicate is finite. P8 and P9 state two simple consequences. i

$\vdash f \phi$		i
$\omega[0] \& \mathcal{N}[0,\phi]$,! 1 ($\&$ I: $\omega 0, C3.14$)	i
$(\omega[0] \& \mathcal{N}[0,\phi])$,! 2 ($(())$ I: 1)	i
$\exists n (\omega[n] \& \mathcal{N}[n,\phi])$,! 3 (\exists I: 2)	i
$f \phi$,! 4 ($\$$ I: P1,3)	i

□

! 8. i

$\vdash \forall P (P \equiv \phi \Rightarrow f P)$		i
P	,! 1 (Prem)	i
$P \equiv \phi$,! 2 (Prem)	i
$f \phi \& P \equiv \phi$,! 3 ($\&$ I: P7,2)	i
$(f \phi \& P \equiv \phi \Rightarrow f P)$,! 4 (\forall E: P6)	i
$f \phi \& P \equiv \phi \Rightarrow f P$,! 5 ($(())$ E: 4)	i
$f P$,! 6 (\Rightarrow E: 3,5)	i
$P \equiv \phi \Rightarrow f P$,! 7 (\Rightarrow I: 2,6)	i
$(P \equiv \phi \Rightarrow f P)$,! 8 ($(())$ I: 7)	i

$\forall P (P \equiv \phi \Rightarrow f P)$,! 9 ($\forall I$: 1,8) i

□

! 9. i

$\vdash \forall P (\neg \exists x P[x] \Rightarrow f P)$ i

P ,! 1 (Prem) i

$\neg \exists x P[x]$,! 2 (Prem) i

$(\neg \exists x P[x] \Rightarrow P \equiv \phi)$,! 3 ($\forall E$: II5.15) i

$\neg \exists x P[x] \Rightarrow P \equiv \phi$,! 4 ($(\Rightarrow)E$: 3) i

$P \equiv \phi$,! 5 ($\Rightarrow E$: 2,4) i

$(P \equiv \phi \Rightarrow f P)$,! 6 ($\forall E$: P8) i

$P \equiv \phi \Rightarrow f P$,! 7 ($(\Rightarrow)E$: 6) i

$f P$,! 8 ($\Rightarrow E$: 5,7) i

$\neg \exists x P[x] \Rightarrow f P$,! 9 ($\Rightarrow I$: 2,8) i

$(\neg \exists x P[x] \Rightarrow f P)$,! 10 ($(\Rightarrow)I$: 9) i

$\forall P (\neg \exists x P[x] \Rightarrow f P)$,! 11 ($\forall I$: 1,10) i

□

! 10. P10 is a corollary to C4.17. i

$\vdash \forall P \forall Q \forall R \forall S (f R \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$ i

P, Q, R, S ,! 1 (Prem) i

$f R \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q$,! 2 (Prem) i

$f R$,! 3 ($\&E$: 2) i

$P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q$,! 4 ($\&E$: 2) i

$f R$,! 5 ($\&E$: 2) i

$\exists n (\omega[n] \ \& \ \mathfrak{N}[n, R])$,! 6 ($\exists E$: P1,3) i

$(\omega[n] \ \& \ \mathfrak{N}[n, R])$,! 7 ($\exists E$: 5) i

$\omega[n] \ \& \ \mathfrak{N}[n, R]$,! 8 ($(\Rightarrow)E$: 6) i

$\omega[n] \ \& \ \mathfrak{N}[n, R] \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q$
 ,! 9 ($\&I$: 4,8) i

$(\omega[n] \ \& \ \mathfrak{N}_l[n,R] \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q)$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$
 ,! 10 ($\forall E$: C4.17) ;

$\omega[n] \ \& \ \mathfrak{N}_l[n,R] \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S)$
 ,! 11 ($(\)E$: 10) ;

$(P \setminus R) \sim (Q \setminus S)$
 ,! 12 ($\Rightarrow E$: 9,11) ;

$f R \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S)$
 ,! 13 ($\Rightarrow I$: 2,12) ;

$(f R \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q)$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$
 ,! 14 ($(\)I$: 13) ;

$\forall P \forall Q \forall R \forall S (f R \ \& \ P \sim Q \ \& \ R \sim S \ \& \ R \subseteq P \ \& \ S \subseteq Q)$
 $\Rightarrow (P \setminus R) \sim (Q \setminus S))$
 ! 15 ($\forall I$: 1,14) ;

□

! 11. P11 is a lemma for P12. The proof appeals to Induction and P10. Remark that it does not rely on C2.9: that is, the existence of a finite number carries with it the implication that there are also finite numbers of any lesser size, without recourse to the ontological suppositions made in C2.9. ;

$\vdash \forall n \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}_l[n,P] \ \& \ Q \subseteq P \Rightarrow f Q)$;

! First we will prove

$\forall n (\omega[n] \Rightarrow \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}_l[n,P] \ \& \ Q \subseteq P \Rightarrow f Q))$

by induction, taking ϕ to be

$\forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}_l[n,P] \ \& \ Q \subseteq P \Rightarrow f Q)$

It must be shown that

$\forall P \forall Q (\omega[0] \ \& \ \mathfrak{N}_l[n,P] \ \& \ Q \subseteq P \Rightarrow f Q)$

and

$\forall n \forall m (\omega[n] \ \& \ \sigma[n,m]$
 $\ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}_l[n,P] \ \& \ Q \subseteq P \Rightarrow f Q)$
 $\Rightarrow \forall P \forall Q (\omega[m] \ \& \ \mathfrak{N}_l[m,P] \ \& \ Q \subseteq P \Rightarrow f Q))$. ;

! To prove:

$\forall P \forall Q (\omega[0] \ \& \ \mathfrak{N}_l[n,P] \ \& \ Q \subseteq P \Rightarrow f Q)$;

P, Q ,! 1 (Prem) ;

$\omega[0] \ \& \ \mathfrak{N}_l[0,P] \ \& \ Q \subseteq P$,! 2 (Prem) ;

$\mathfrak{N}_l[0,P]$,! 3 ($\&E$: 2) ;

$Q \subseteq P$,! 4 ($\&E$: 2) ;

$(\mathcal{N}_L[0, P] \Rightarrow P \equiv \phi)$,! 5 ($\forall E$: C3.1)	i
$\mathcal{N}_L[0, P] \Rightarrow P \equiv \phi$,! 6 ($(())E$: 5)	i
$P \equiv \phi$,! 7 ($\Rightarrow E$: 3,6)	i
$P \equiv \phi \ \& \ Q \subseteq P$,! 8 ($\&I$: 4,7)	i
$(P \equiv \phi \ \& \ Q \subseteq P \Rightarrow Q \equiv \phi)$,! 9 ($\forall E$ II5.11)	i
$P \equiv \phi \ \& \ Q \subseteq P \Rightarrow Q \equiv \phi$,! 10 ($(())E$: 9)	i
$Q \equiv \phi$,! 11 ($\Rightarrow E$: 8,10)	i
$(Q \equiv \phi \Rightarrow f Q)$,! 12 ($\forall E$: P8)	i
$Q \equiv \phi \Rightarrow f Q$,! 13 ($(())E$: 12)	i
$f Q$,! 14 ($\Rightarrow E$: 11,13)	i
$\omega[0] \ \& \ \mathcal{N}_L[0, P] \ \& \ Q \subseteq P \Rightarrow f Q$,! 15 ($\Rightarrow I$: 2,14)	i
$(\omega[0] \ \& \ \mathcal{N}_L[0, P] \ \& \ Q \subseteq P \Rightarrow f Q)$,! 16 ($(())I$: 15)	i
$\forall P \forall Q (\omega[0] \ \& \ \mathcal{N}_L[0, P] \ \& \ Q \subseteq P \Rightarrow f Q)$,! 17 ($\forall I$: 1,16)	i
! To prove:		
$\forall n \forall m (\omega[n] \ \& \ \sigma[n, m]$		
$\ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathcal{N}_L[n, P] \ \& \ Q \subseteq P \Rightarrow f Q)$		
$\ \Rightarrow \forall P \forall Q (\omega[m] \ \& \ \mathcal{N}_L[m, P] \ \& \ Q \subseteq P \Rightarrow f Q))$		i
n, m	,! 18 (Prem)	i
$\omega[n] \ \& \ \sigma[n, m] \ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathcal{N}_L[n, P] \ \& \ Q \subseteq P \Rightarrow f Q)$		
	,! 19 (Prem)	i
$\omega[n] \ \& \ \sigma[n, m]$,! 20 ($\&E$: 19)	i
$\omega[n]$,! 21 ($\&E$: 19)	i
$\forall P \forall Q (\omega[n] \ \& \ \mathcal{N}_L[n, P] \ \& \ Q \subseteq P \Rightarrow f Q)$,! 22 ($\&E$: 19)	i
P, Q	,! 23 (Prem)	i
$\omega[m] \ \& \ \mathcal{N}_L[m, P] \ \& \ Q \subseteq P$,! 24 (Prem)	i
$\omega[m]$,! 25 ($\&E$: 24)	i
$\mathcal{N}_L[m, P]$,! 26 ($\&E$: 24)	i
$Q \subseteq P$,! 27 ($\&E$: 24)	i
$(Q \subseteq P \Rightarrow Q \equiv P \vee Q \subset P)$,! 28 ($\forall E$: III1.53)	i

$Q \subseteq P \Rightarrow Q \equiv P \vee Q \subset P$,! 29 ((E: 28) i
 $Q \equiv P \vee Q \subset P$,! 30 (\Rightarrow E: 27,29) i
 $Q \equiv P$,! 31 (Prem) i
 $\omega[m] \ \& \ \mathfrak{N}_l[m,P]$,! 32 (&I: 25,26) i
 $\omega[m] \ \& \ \mathfrak{N}_l[m,P] \ \& \ Q \equiv P$,! 33 (&I: 31,32) i
 $(\ \omega[m] \ \& \ \mathfrak{N}_l[m,P] \ \& \ Q \equiv P \Rightarrow \mathfrak{N}_l[m,Q] \)$
, ! 34 (\forall E: C4.6) i
 $\omega[m] \ \& \ \mathfrak{N}_l[m,P] \ \& \ Q \equiv P \Rightarrow \mathfrak{N}_l[m,Q]$
, ! 35 ((E: 34) i
 $\mathfrak{N}_l[m,Q]$,! 36 (\Rightarrow E: 33,35) i
 $\omega[m] \ \& \ \mathfrak{N}_l[m,Q]$,! 37 (&I: 25,36) i
 $(\ \omega[m] \ \& \ \mathfrak{N}_l[m,Q] \Rightarrow f \ Q \)$,! 38 (\forall E: P2) i
 $\omega[m] \ \& \ \mathfrak{N}_l[m,Q] \Rightarrow f \ Q$,! 39 ((E: 38) i
 $f \ Q$,! 40 (\Rightarrow E: 37,39) i
 $Q \equiv P \Rightarrow f \ Q$,! 41 (\Rightarrow I: 31,40) i
 $Q \subset P$,! 42 (Prem) i
 $(\ Q \subset P \Rightarrow \exists a (\ P[a] \ \& \ Q \subseteq (P \setminus (a^\bullet)) \) \)$
, ! 43 (\forall E: II8.55) i
 $Q \subset P \Rightarrow \exists a (\ P[a] \ \& \ Q \subseteq (P \setminus (a^\bullet)) \)$
, ! 44 ((E: 43) i
 $\exists a (\ P[a] \ \& \ Q \subseteq (P \setminus (a^\bullet)) \)$,! 45 (\Rightarrow E: 42,44) i
 $(\ P[a] \ \& \ Q \subseteq (P \setminus (a^\bullet)) \)$,! 46 (\exists E: 45) i
 $P[a] \ \& \ Q \subseteq (P \setminus (a^\bullet))$,! 47 ((E: 46) i
 $P[a]$,! 48 (&E: 47) i
 $\omega[n] \ \& \ \sigma[n,m] \ \& \ P[a]$,! 49 (&I: 20,48) i
 $\omega[n] \ \& \ \sigma[n,m] \ \& \ P[a] \ \& \ \mathfrak{N}_l[m,P]$,! 50 (&I: 26,49) i
 $(\ \omega[n] \ \& \ \sigma[n,m] \ \& \ P[a] \ \& \ \mathfrak{N}_l[m,P] \Rightarrow \mathfrak{N}_l[n,(P \setminus (a^\bullet))] \)$
, ! 51 (\forall E: C1.11) i
 $\omega[n] \ \& \ \sigma[n,m] \ \& \ P[a] \ \& \ \mathfrak{N}_l[m,P] \Rightarrow \mathfrak{N}_l[n,(P \setminus (a^\bullet))]$
, ! 52 ((E: 51) i

$\mathcal{I}_l[n, (P \setminus (a^*))]$, ! 53 ($\Rightarrow E$: 50, 52)	i
$\omega[n] \ \& \ \mathcal{I}_l[n, (P \setminus (a^*))]$, ! 54 ($\& I$: 21, 53)	i
$Q \subseteq (P \setminus (a^*))$, ! 55 ($\& E$: 47)	i
$\omega[n] \ \& \ \mathcal{I}_l[n, (P \setminus (a^*))] \ \& \ Q \subseteq (P \setminus (a^*))$, ! 56 ($\& I$: 54, 55)	i
! Applying the induction hypothesis...		
$(\omega[n] \ \& \ \mathcal{I}_l[n, (P \setminus (a^*))] \ \& \ Q \subseteq (P \setminus (a^*)) \Rightarrow f \ Q)$, ! 57 ($\forall E$: 22)	i
$\omega[n] \ \& \ \mathcal{I}_l[n, (P \setminus (a^*))] \ \& \ Q \subseteq (P \setminus (a^*)) \Rightarrow f \ Q$, ! 58 ($() E$: 57)	i
$f \ Q$, ! 59 ($\Rightarrow E$: 56, 58)	i
$Q \subset P \Rightarrow f \ Q$, ! 60 ($\Rightarrow I$: 42, 59)	i
$f \ Q$, ! 61 ($\vee E$: 30, 41, 60)	i
$\omega[m] \ \& \ \mathcal{I}_l[m, P] \ \& \ Q \subseteq P \Rightarrow f \ Q$, ! 62 ($\Rightarrow I$: 24, 61)	i
$(\omega[m] \ \& \ \mathcal{I}_l[m, P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$, ! 63 ($() I$: 62)	i
$\forall P \forall Q (\omega[m] \ \& \ \mathcal{I}_l[m, P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$, ! 64 ($\forall I$: 23, 63)	i
$\omega[n] \ \& \ \sigma[n, m] \ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathcal{I}_l[n, P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$		
$\Rightarrow \forall P \forall Q (\omega[m] \ \& \ \mathcal{I}_l[m, P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$, ! 65 ($\Rightarrow I$: 19, 64)	i
$(\omega[n] \ \& \ \sigma[n, m] \ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathcal{I}_l[n, P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$		
$\Rightarrow \forall P \forall Q (\omega[m] \ \& \ \mathcal{I}_l[m, P] \ \& \ Q \subseteq P \Rightarrow f \ Q))$, ! 66 ($() I$: 65)	i
$\forall n \forall m (\omega[n] \ \& \ \sigma[n, m] \ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathcal{I}_l[n, P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$		
$\Rightarrow \forall P \forall Q (\omega[m] \ \& \ \mathcal{I}_l[m, P] \ \& \ Q \subseteq P \Rightarrow f \ Q))$, ! 67 ($\forall I$: 18, 66)	i
$\forall n (\omega[n] \Rightarrow \forall P \forall Q (\omega[n] \ \& \ \mathcal{I}_l[n, P] \ \& \ Q \subseteq P \Rightarrow f \ Q))$! 68 (Induct: 17, 67)	i
n, P, Q	, ! 69 (Prem)	i
$\omega[n] \ \& \ \mathcal{I}_l[n, P] \ \& \ Q \subseteq P$, ! 70 (Prem)	i
$\omega[n]$, ! 71 ($\& E$: 70)	i
$(\omega[n] \Rightarrow \forall P \forall Q (\omega[n] \ \& \ \mathcal{I}_l[n, P] \ \& \ Q \subseteq P \Rightarrow f \ Q))$, ! 72 ($\forall E$: 68)	i

$\omega[n] \Rightarrow \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$,! 73 ((E: 72)	i
$\forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$,! 74 (\Rightarrow E: 71,73)	i
$(\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$,! 75 (\forall E: 74)	i
$\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q$,! 76 ((E: 75)	i
$f \ Q$,! 77 (\Rightarrow E: 70,76)	i
$\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q$,! 78 (\Rightarrow I: 70,77)	i
$(\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$,! 79 ((I: 78)	i
$\forall n \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$! 80 (\forall I: 69,79)	i

□

! 12. Sub-predicates of finite predicates are themselves finite.
Remark that P12, like P11, does not rely on C2.9. i

$\vdash \forall P \forall Q (f \ P \ \& \ Q \subseteq P \Rightarrow f \ Q)$		i
P, Q	,! 1 (Prem)	i
$f \ P \ \& \ Q \subseteq P$,! 2 (Prem)	i
$f \ P$,! 3 ($\&$ E: 2)	i
$Q \subseteq P$,! 4 ($\&$ E: 2)	i
$\exists n (\omega[n] \ \& \ \mathfrak{N}[n,P])$,! 5 (\exists E: P1,3)	i
$(\omega[n] \ \& \ \mathfrak{N}[n,P])$,! 6 (\exists E: 5)	i
$\omega[n] \ \& \ \mathfrak{N}[n,P]$,! 7 ((E: 6)	i
$\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P$,! 8 ($\&$ I: 4,7)	i
$(\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q)$,! 9 (\forall E: P11)	i
$\omega[n] \ \& \ \mathfrak{N}[n,P] \ \& \ Q \subseteq P \Rightarrow f \ Q$,! 10 ((E: 9)	i
$f \ Q$,! 11 (\Rightarrow E: 8,10)	i
$f \ P \ \& \ Q \subseteq P \Rightarrow f \ Q$,! 12 (\Rightarrow I: 2,11)	i
$(f \ P \ \& \ Q \subseteq P \Rightarrow f \ Q)$,! 13 ((I: 12)	i
$\forall P \forall Q (f \ P \ \& \ Q \subseteq P \Rightarrow f \ Q)$! 14 (\forall I: 1,13)	i

□

! P13 through P15 assert some simple consequences of P12. i

! 13. i

$\vdash \forall P \forall Q (f (P \cup Q) \Rightarrow f P)$ i

P, Q ,! 1 (Prem) i

$f (P \cup Q)$,! 2 (Prem) i

$P \subseteq (P \cup Q)$,! 3 ($\forall E$ II2.12) i

$f (P \cup Q) \ \& \ P \subseteq (P \cup Q)$,! 4 ($\&I$: 2,3) i

$(f (P \cup Q) \ \& \ P \subseteq (P \cup Q) \Rightarrow f P)$,! 5 ($\forall E$: P12) i

$f (P \cup Q) \ \& \ P \subseteq (P \cup Q) \Rightarrow f P$,! 6 ($(\)E$: 5) i

$f P$,! 7 ($\Rightarrow E$: 4,6) i

$f (P \cup Q) \Rightarrow f P$,! 8 ($\Rightarrow I$: 2,7) i

$(f (P \cup Q) \Rightarrow f P)$,! 9 ($(\)I$: 8) i

$\forall P \forall Q (f (P \cup Q) \Rightarrow f P)$! 10 ($\forall I$: 1,9) i

□

! 14. i

$\vdash \forall P \forall Q (f (P \cup Q) \Rightarrow f Q)$ i

P, Q ,! 1 (Prem) i

$f (P \cup Q)$,! 2 (Prem) i

$Q \subseteq (P \cup Q)$,! 3 ($\forall E$ II2.13) i

$f (P \cup Q) \ \& \ Q \subseteq (P \cup Q)$,! 4 ($\&I$: 2,3) i

$(f (P \cup Q) \ \& \ Q \subseteq (P \cup Q) \Rightarrow f Q)$,! 5 ($\forall E$: P12) i

$f (P \cup Q) \ \& \ Q \subseteq (P \cup Q) \Rightarrow f Q$,! 6 ($(\)E$: 5) i

$f Q$,! 7 ($\Rightarrow E$: 4,6) i

$f (P \cup Q) \Rightarrow f Q$,! 8 ($\Rightarrow I$: 2,7) i

$(f (P \cup Q) \Rightarrow f Q)$,! 9 ($(\)I$: 8) i

$\forall P \forall Q (f (P \cup Q) \Rightarrow f Q)$! 10 ($\forall I$: 1,9) i

□

! 15. i

$\vdash \forall P \forall Q (f (P \cup Q) \Rightarrow f P \ \& \ f Q)$		i
P, Q	,! 1 (Prem)	i
$f (P \cup Q)$,! 2 (Prem)	i
$(f (P \cup Q) \Rightarrow f P)$,! 3 ($\forall E$: P13)	i
$f (P \cup Q) \Rightarrow f P$,! 4 ($(\)E$: 3)	i
$f P$,! 5 ($\Rightarrow E$: 2,4)	i
$(f (P \cup Q) \Rightarrow f Q)$,! 6 ($\forall E$: P14)	i
$f (P \cup Q) \Rightarrow f Q$,! 7 ($(\)E$: 6)	i
$f Q$,! 8 ($\Rightarrow E$: 2,7)	i
$f P \ \& \ f Q$,! 9 ($\&I$: 5,8)	i
$f (P \cup Q) \Rightarrow f P \ \& \ f Q$,! 10 ($\Rightarrow I$: 2,9)	i
$(f (P \cup Q) \Rightarrow f P \ \& \ f Q)$,! 11 ($(\)I$: 10)	i
$\forall P \forall Q (f (P \cup Q) \Rightarrow f P \ \& \ f Q)$! 12 ($\forall I$: 1,11)	i

□

! 16. P16 is important to the proof of P18, as it will be used in its Induction step.

$\vdash \forall P \forall a (f P \Rightarrow f (P \cup (a^\bullet)))$		i
P, a	,! 1 (Prem)	i
$f P$,! 2 (Prem)	i
$\exists n (\omega[n] \ \& \ \mathcal{I}_k[n, P])$,! 3 ($\exists E$: P1,2)	i
$(\omega[n] \ \& \ \mathcal{I}_k[n, P])$,! 4 ($\exists E$: 3)	i
$\omega[n] \ \& \ \mathcal{I}_k[n, P]$,! 5 ($(\)E$: 4)	i
$(\omega[n] \ \& \ \mathcal{I}_k[n, P] \Rightarrow \exists m (\omega[m] \ \& \ \mathcal{I}_k[m, (P \cup (a^\bullet))]))$,! 6 ($\forall E$: C4.16)	i
$\omega[n] \ \& \ \mathcal{I}_k[n, P] \Rightarrow \exists m (\omega[m] \ \& \ \mathcal{I}_k[m, (P \cup (a^\bullet))])$,! 7 ($(\)E$: 6)	i
$\exists m (\omega[m] \ \& \ \mathcal{I}_k[m, (P \cup (a^\bullet))])$,! 8 ($\Rightarrow E$: 5,7)	i
$(\omega[n] \ \& \ \mathcal{I}_k[n, (P \cup (a^\bullet))])$,! 9 ($\exists E$: 8)	i
$\exists n (\omega[n] \ \& \ \mathcal{I}_k[n, (P \cup (a^\bullet))])$,! 10 ($\exists I$: 9)	i

$f (P \cup (a^*))$,! 11 ($\$I$: P1,10) i
 $f P \Rightarrow f (P \cup (a^*))$,! 12 ($\Rightarrow I$: 2,11) i
 $(f P \Rightarrow f (P \cup (a^*)))$,! 13 ($(\)I$: 12) i
 $\forall P \forall a (f P \Rightarrow f (P \cup (a^*)))$! 14 ($\forall I$: 1,13) i

□

! 17. Our singleton predicates are finite. i

$\vdash \forall a f (a^*)$ i

a ,! 1 (Prem) i

$(f \phi \Rightarrow f (\phi \cup (a^*)))$,! 2 ($\forall E$: P16) i

$f \phi \Rightarrow f (\phi \cup (a^*))$,! 3 ($(\)E$: 2) i

$f (\phi \cup (a^*))$,! 4 ($\Rightarrow E$: P7,3) i

$(\phi \cup (a^*)) \equiv (a^*)$,! 5 ($\forall E$: II5.20) i

$f (\phi \cup (a^*)) \& (\phi \cup (a^*)) \equiv (a^*)$,! 6 ($\&I$: 4,5) i

$(f (\phi \cup (a^*)) \& (\phi \cup (a^*)) \equiv (a^*) \Rightarrow f (a^*))$
, ! 7 ($\forall E$: P5) i

$f (\phi \cup (a^*)) \& (\phi \cup (a^*)) \equiv (a^*) \Rightarrow f (a^*)$
, ! 8 ($(\)E$: 7) i

$f (a^*)$,! 9 ($\Rightarrow E$: 6,8) i

$\forall a f (a^*)$! 10 ($\forall I$: 1,9) i

□

! 18. P20 will rely on P18 for its proof. i

$\vdash \forall n \forall P \forall Q (\omega[n] \& \mathfrak{N}[n,Q] \& f P \Rightarrow f (P \cup Q))$ i

! We will first prove

$\forall n (\omega[n] \Rightarrow \forall P \forall Q (\omega[n] \& \mathfrak{N}[n,Q] \& f P \Rightarrow f (P \cup Q)))$

by induction, taking ϕ to be

$\forall P \forall Q (\omega[n] \& \mathfrak{N}[n,Q] \& f P \Rightarrow f (P \cup Q))$

It must be shown that

$\forall P \forall Q (\omega[n] \& \mathfrak{N}[n,Q] \& f P \Rightarrow f (P \cup Q))$

and

$\forall n \forall m (\omega[n] \& \sigma[n,m] \& \forall P \forall Q (\omega[n] \& \mathfrak{N}[n,Q] \& f P \Rightarrow f (P \cup Q)))$
 $\Rightarrow \forall P \forall Q (\omega[m] \& \mathfrak{N}[m,Q] \& f P \Rightarrow f (P \cup Q)))$. i

! To prove:

$\forall P \forall Q (\omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f P \Rightarrow f (P \cup Q))$		i
P, Q	,! 1 (Prem)	i
$\omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f P$,! 2 (Prem)	i
$\mathfrak{N}[0, Q]$,! 3 (&E: 2)	i
$f P$,! 4 (&E: 2)	i
$(\ \mathfrak{N}[0, Q] \Rightarrow Q \equiv \phi \)$,! 5 (\forall E: C3.1)	i
$\mathfrak{N}[0, Q] \Rightarrow Q \equiv \phi$,! 6 (()E: 5)	i
$Q \equiv \phi$,! 7 (\Rightarrow E: 3,6)	i
$(\ Q \equiv \phi \Rightarrow (P \cup Q) \equiv P \)$,! 8 (\forall E: II5.22)	i
$Q \equiv \phi \Rightarrow (P \cup Q) \equiv P$,! 9 (()E: 8)	i
$(P \cup Q) \equiv P$,! 10 (\Rightarrow E: 7,9)	i
$f P \ \& \ (P \cup Q) \equiv P$,! 11 (&I: 4,10)	i
$(\ f P \ \& \ (P \cup Q) \equiv P \Rightarrow f (P \cup Q) \)$,! 12 (\forall E: P6)	i
$f P \ \& \ (P \cup Q) \equiv P \Rightarrow f (P \cup Q)$,! 13 (()E: 12)	i
$f (P \cup Q)$,! 14 (\Rightarrow E: 11,13)	i
$\omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f P \Rightarrow f (P \cup Q)$,! 15 (\Rightarrow I: 2,14)	i
$(\omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f P \Rightarrow f (P \cup Q))$,! 16 (()I: 15)	i
$\forall P \forall Q (\omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f P \Rightarrow f (P \cup Q))$,! 17 (\forall I: 1,16)	i

! To prove:

$\forall n \forall m (\omega[n] \ \& \ \sigma[n, m]$		
$\ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n, Q] \ \& \ f P \Rightarrow f (P \cup Q))$		
$\ \Rightarrow \ \forall P \forall Q (\omega[m] \ \& \ \mathfrak{N}[m, Q] \ \& \ f P \Rightarrow f (P \cup Q)) \)$		i
n, m	,! 18 (Prem)	i
$\omega[n] \ \& \ \sigma[n, m] \ \& \ \forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n, Q] \ \& \ f P \Rightarrow f (P \cup Q))$,! 19 (Prem)	i
$\omega[n] \ \& \ \sigma[n, m]$,! 20 (&E: 19)	i
$\omega[n]$,! 21 (&E: 19)	i
$\forall P \forall Q (\omega[n] \ \& \ \mathfrak{N}[n, Q] \ \& \ f P \Rightarrow f (P \cup Q))$,! 22 (&E: 19)	i
P, Q	,! 23 (Prem)	i

$\omega[m] \ \& \ \mathfrak{N}[m, Q] \ \& \ f \ P$,! 24 (Prem)	i
$\omega[m] \ \& \ \mathfrak{N}[m, Q]$,! 25 (&E: 24)	i
$f \ P$,! 26 (&E: 24)	i
$(\ m = 0 \ \vee \ \neg \ m = 0 \)$,! 27 (\forall E: I3.4)	i
$m = 0 \ \vee \ \neg \ m = 0$,! 28 (()E: 27)	i
$m = 0$,! 29 (Prem)	i
$\omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f \ P$,! 30 (=E: 24,29)	i
$(\ \omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q) \)$,! 31 (\forall E: 17)	i
$\omega[0] \ \& \ \mathfrak{N}[0, Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)$,! 32 (()E: 31)	i
$f \ (P \cup Q)$,! 33 (\Rightarrow E: 30,32)	i
$m = 0 \Rightarrow f \ (P \cup Q)$,! 34 (\Rightarrow I: 29,33)	i
$\neg \ m = 0$,! 35 (Prem)	i
$\mathfrak{N}[m, Q]$,! 36 (&E: 24)	i
$\mathfrak{N}[m, Q] \ \& \ \neg \ m = 0$,! 37 (&I: 35,36)	i
$(\ \mathfrak{N}[m, Q] \ \& \ \neg \ m = 0 \Rightarrow \exists x \ Q[x] \)$,! 38 (\forall E: C3.17)	i
$\mathfrak{N}[m, Q] \ \& \ \neg \ m = 0 \Rightarrow \exists x \ Q[x]$,! 39 (()E: 38)	i
$\exists x \ Q[x]$,! 40 (\Rightarrow E: 37,39)	i
$Q[a]$,! 41 (\exists E: 40)	i
$\omega[n] \ \& \ \sigma[n, m] \ \& \ Q[a]$,! 42 (&I: 20,41)	i
$\omega[n] \ \& \ \sigma[n, m] \ \& \ Q[a] \ \& \ \mathfrak{N}[m, Q]$,! 43 (&I: 36,42)	i
$(\ \omega[n] \ \& \ \sigma[n, m] \ \& \ Q[a] \ \& \ \mathfrak{N}[m, Q] \Rightarrow \mathfrak{N}[n, (Q \setminus (a^\bullet))] \)$,! 44 (\forall E: C2.11)	i
$\omega[n] \ \& \ \sigma[n, m] \ \& \ Q[a] \ \& \ \mathfrak{N}[m, Q] \Rightarrow \mathfrak{N}[n, (Q \setminus (a^\bullet))]$,! 45 (()E: 44)	i
$\mathfrak{N}[n, (Q \setminus (a^\bullet))]$,! 46 (\Rightarrow E: 43,45)	i
$\omega[n] \ \& \ \mathfrak{N}[n, (Q \setminus (a^\bullet))]$,! 47 (&I: 21,46)	i
$\omega[n] \ \& \ \mathfrak{N}[n, (Q \setminus (a^\bullet))] \ \& \ f \ P$,! 48 (&I: 26,47)	i

! Applying the induction hypothesis...

i

$(\omega[n] \ \& \ \mathfrak{I}_l[n, (Q \setminus (a^\bullet))] \ \& \ f \ P \Rightarrow f (P \cup (Q \setminus (a^\bullet))))$
, ! 49 ($\forall E$: 22) i

$\omega[n] \ \& \ \mathfrak{I}_l[n, (Q \setminus (a^\bullet))] \ \& \ f \ P \Rightarrow f (P \cup (Q \setminus (a^\bullet)))$
, ! 50 ($()E$: 49) i

$f (P \cup (Q \setminus (a^\bullet)))$, ! 51 ($\Rightarrow E$: 48,50) i

$(f (P \cup (Q \setminus (a^\bullet))) \Rightarrow f ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet)))$
, ! 52 ($\forall E$: P16) i

$f (P \cup (Q \setminus (a^\bullet))) \Rightarrow f ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
, ! 53 ($()E$: 52) i

$f ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$, ! 54 ($\Rightarrow E$: 51,53) i

$(Q[a] \Rightarrow (P \cup Q) \equiv ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet)))$
, ! 55 ($\forall E$: II8.67) i

$Q[a] \Rightarrow (P \cup Q) \equiv ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
, ! 56 ($()E$: 55) i

$(P \cup Q) \equiv ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
, ! 57 ($\Rightarrow E$: 41,56) i

$f ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
 $\& (P \cup Q) \equiv ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
, ! 58 ($\&I$: 54,57) i

$(f ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
 $\& (P \cup Q) \equiv ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
 $\Rightarrow f (P \cup Q))$
, ! 59 ($\forall E$: P6) i

$f ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
 $\& (P \cup Q) \equiv ((P \cup (Q \setminus (a^\bullet))) \cup (a^\bullet))$
 $\Rightarrow f (P \cup Q)$
, ! 60 ($()E$: 59) i

$f (P \cup Q)$, ! 61 ($\Rightarrow E$: 58,60) i

$\neg m = 0 \Rightarrow f (P \cup Q)$, ! 62 ($\Rightarrow I$: 35,61) i

$f (P \cup Q)$, ! 63 ($\forall E$: 28,34,62) i

$\omega[m] \ \& \ \mathfrak{I}_l[m, Q] \ \& \ f \ P \Rightarrow f (P \cup Q)$, ! 64 ($\Rightarrow I$: 24,63) i

$(\omega[m] \ \& \ \mathfrak{I}_l[m, Q] \ \& \ f \ P \Rightarrow f (P \cup Q))$, ! 65 ($()I$: 64) i

$\forall P \forall Q (\omega[m] \ \& \ \mathfrak{I}_l[m, Q] \ \& \ f \ P \Rightarrow f (P \cup Q))$

,! 66 ($\forall I$: 23,65) i

$\omega[n] \ \& \ \sigma[n,m] \ \& \ \forall P \forall Q \ (\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q))$
 $\Rightarrow \forall P \forall Q \ (\omega[m] \ \& \ \mathfrak{N}[m,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q))$

,! 67 ($\Rightarrow I$: 19,66) i

$(\ \omega[n] \ \& \ \sigma[n,m] \ \& \ \forall P \forall Q \ (\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q))$
 $\Rightarrow \forall P \forall Q \ (\omega[m] \ \& \ \mathfrak{N}[m,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)) \)$

,! 68 ($(\) I$: 67) i

$\forall n \forall m \ (\omega[n] \ \& \ \sigma[n,m]$

$\ \& \ \forall P \forall Q \ (\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q))$

$\Rightarrow \forall P \forall Q \ (\omega[m] \ \& \ \mathfrak{N}[m,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)) \)$

,! 69 ($\forall I$: 18,68) i

$\forall n \ (\ \omega[n] \Rightarrow \forall P \forall Q \ (\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)) \)$

! 70 (Induct: 17,69)

i

n, P, Q

,! 71 (Prem) i

$\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P$

,! 72 (Prem) i

$\omega[n]$

,! 73 ($\& E$: 72) i

$(\ \omega[n] \Rightarrow \forall P \forall Q \ (\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)) \)$

,! 74 ($\forall E$: 70) i

$\omega[n] \Rightarrow \forall P \forall Q \ (\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q))$

,! 75 ($(\) E$: 74) i

$\forall P \forall Q \ (\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q))$

,! 76 ($\Rightarrow E$: 73,75) i

$(\ \omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q) \)$

,! 77 ($\forall E$: 76) i

$\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)$

,! 78 ($(\) E$: 77) i

$f \ (P \cup Q)$

,! 79 ($\Rightarrow E$: 72,78) i

$\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)$

,! 80 ($\Rightarrow I$: 72,79) i

$(\ \omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q) \)$

,! 81 ($(\) I$: 80) i

$\forall n \forall P \forall Q \ (\ \omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q) \)$

! 82 ($\forall I$: 71,81) i

□

! 19.

i

$\vdash \forall P \forall Q \forall m \forall k \ (\ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[m,P] \ \& \ \mathfrak{N}[k,Q]$

$\Rightarrow \exists n \ (\omega[n] \ \& \ \mathfrak{N}[n,(P \cup Q)]) \)$

i

P, Q, m, k	,! 1 (Prem)	i
$\omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[m, P] \ \& \ \mathcal{N}[k, Q]$,! 2 (Prem)	i
$\omega[m]$,! 3 (&E: 2)	i
$\omega[k]$,! 4 (&E: 2)	i
$\mathcal{N}[m, P]$,! 5 (&E: 2)	i
$\mathcal{N}[k, Q]$,! 6 (&E: 2)	i
$\omega[k] \ \& \ \mathcal{N}[k, Q]$,! 7 (&I: 4,6)	i
$\omega[m] \ \& \ \mathcal{N}[m, P]$,! 8 (&I: 3,5)	i
$(\ \omega[m] \ \& \ \mathcal{N}[m, P] \ \Rightarrow \ f \ P \)$,! 9 (\forall E: P2)	i
$\omega[m] \ \& \ \mathcal{N}[m, P] \ \Rightarrow \ f \ P$,! 10 ($(\)$ E: 9)	i
$f \ P$,! 11 (\Rightarrow E: 8,10)	i
$\omega[k] \ \& \ \mathcal{N}[k, Q] \ \& \ f \ P$,! 12 (&I: 7,11)	i
$(\ \omega[k] \ \& \ \mathcal{N}[k, Q] \ \& \ f \ P \ \Rightarrow \ f \ (P \cup Q) \)$,! 13 (\forall E: P18)	i
$\omega[k] \ \& \ \mathcal{N}[k, Q] \ \& \ f \ P \ \Rightarrow \ f \ (P \cup Q)$,! 14 ($(\)$ E: 13)	i
$f \ (P \cup Q)$,! 15 (\Rightarrow E: 12,14)	i
$\exists n \ (\omega[n] \ \& \ \mathcal{N}[n, (P \cup Q)])$,! 16 (\exists E: P1,15)	i
$\omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[m, P] \ \& \ \mathcal{N}[k, Q] \ \Rightarrow \ \exists n \ (\omega[n] \ \& \ \mathcal{N}[n, (P \cup Q)])$,! 17 (\Rightarrow I: 2,16)	i
$(\ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[m, P] \ \& \ \mathcal{N}[k, Q] \ \Rightarrow \ \exists n \ (\omega[n] \ \& \ \mathcal{N}[n, (P \cup Q)]) \)$,! 18 ($(\)$ I: 17)	i
$\forall P \forall Q \forall m \forall k \ (\ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[m, P] \ \& \ \mathcal{N}[k, Q] \ \Rightarrow \ \exists n \ (\omega[n] \ \& \ \mathcal{N}[n, (P \cup Q)]) \)$! 19 (\forall I: 1,18)	i

□

! 20. The union of finite predicates is itself finite. i

$\vdash \forall P \forall Q \ (\ f \ P \ \& \ f \ Q \ \Rightarrow \ f \ (P \cup Q) \)$ i

P, Q ,! 1 (Prem) i

$f \ P \ \& \ f \ Q$,! 2 (Prem) i

$f \ Q$,! 3 (&E: 2) i

$\exists n \ (\omega[n] \ \& \ \mathcal{N}[n, Q])$,! 4 (\exists E: P1,3) i

$(\omega[n] \ \& \ \mathfrak{N}[n,Q])$,! 5 ($\exists E$: 4)	i
$\omega[n] \ \& \ \mathfrak{N}[n,Q]$,! 6 ($()E$: 5)	i
$f \ P$,! 7 ($\&E$: 2)	i
$\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P$,! 8 ($\&I$: 6,7)	i
$(\ \omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q) \)$,! 9 ($\forall E$: P18)	i
$\omega[n] \ \& \ \mathfrak{N}[n,Q] \ \& \ f \ P \Rightarrow f \ (P \cup Q)$,! 10 ($()E$: 9)	i
$f \ (P \cup Q)$,! 11 ($\Rightarrow E$: 8,10)	i
$f \ P \ \& \ f \ Q \Rightarrow f \ (P \cup Q)$,! 12 ($\Rightarrow I$: 2,11)	i
$(\ f \ P \ \& \ f \ Q \Rightarrow f \ (P \cup Q) \)$,! 13 ($()I$: 12)	i
$\forall P \forall Q \ (\ f \ P \ \& \ f \ Q \Rightarrow f \ (P \cup Q) \)$! 14 ($\forall I$: 1,13)	i

□

! 21.

$\vdash \forall P \forall Q \forall R \ (\ f \ P \ \& \ f \ Q \ \& \ (P \cup Q) \equiv R \Rightarrow f \ R \)$		i
P, Q, R	,! 1 (Prem)	i
$f \ P \ \& \ f \ Q \ \& \ (P \cup Q) \equiv R$,! 2 (Prem)	i
$f \ P$,! 3 ($\&E$: 2)	i
$f \ Q$,! 4 ($\&E$: 2)	i
$(P \cup Q) \equiv R$,! 5 ($\&E$: 2)	i
$f \ P \ \& \ f \ Q$,! 6 ($\&I$: 3,4)	i
$(\ f \ P \ \& \ f \ Q \Rightarrow f \ (P \cup Q) \)$,! 7 ($\forall E$: P20)	i
$f \ P \ \& \ f \ Q \Rightarrow f \ (P \cup Q)$,! 8 ($()E$: 7)	i
$f \ (P \cup Q)$,! 9 ($\Rightarrow E$: 6,8)	i
$f \ (P \cup Q) \ \& \ (P \cup Q) \equiv R$,! 10 ($\&I$: 5,9)	i
$(\ f \ (P \cup Q) \ \& \ (P \cup Q) \equiv R \Rightarrow f \ R \)$,! 11 ($\forall E$: P5)	i
$f \ (P \cup Q) \ \& \ (P \cup Q) \equiv R \Rightarrow f \ R$,! 12 ($()E$: 11)	i
$f \ R$,! 13 ($\Rightarrow E$: 10,12)	i
$f \ P \ \& \ f \ Q \ \& \ (P \cup Q) \equiv R \Rightarrow f \ R$,! 14 ($\Rightarrow I$: 2,13)	i
$(\ f \ P \ \& \ f \ Q \ \& \ (P \cup Q) \equiv R \Rightarrow f \ R \)$,! 15 ($()I$: 14)	i

$\forall P \forall Q \forall R (f P \ \& \ f Q \ \& \ (P \cup Q) \equiv R \Rightarrow f R)$! 16 ($\forall I$: 1,15) i

□

! 22. i

$\vdash \forall P \forall Q \forall R (f P \ \& \ f Q \ \& \ R \equiv (P \cup Q) \Rightarrow f R)$ i

P, Q, R ,! 1 (Prem) i

$f P \ \& \ f Q \ \& \ R \equiv (P \cup Q)$,! 2 (Prem) i

$f P \ \& \ f Q$,! 3 ($\&E$: 2) i

$R \equiv (P \cup Q)$,! 4 ($\&E$: 2) i

$(R \equiv (P \cup Q) \Rightarrow (P \cup Q) \equiv R)$,! 5 ($\forall E$: II1.10) i

$R \equiv (P \cup Q) \Rightarrow (P \cup Q) \equiv R$,! 6 ($()E$: 5) i

$(P \cup Q) \equiv R$,! 7 ($\Rightarrow E$: 4,6) i

$f P \ \& \ f Q \ \& \ (P \cup Q) \equiv R$,! 8 ($\&E$: 3,7) i

$(f P \ \& \ f Q \ \& \ (P \cup Q) \equiv R \Rightarrow f R)$,! 9 ($\forall E$: P21) i

$f P \ \& \ f Q \ \& \ (P \cup Q) \equiv R \Rightarrow f R$,! 10 ($()E$: 9) i

$f R$,! 11 ($\Rightarrow E$: 8,10) i

$f P \ \& \ f Q \ \& \ R \equiv (P \cup Q) \Rightarrow f R$,! 12 ($\Rightarrow I$: 2,11) i

$(f P \ \& \ f Q \ \& \ R \equiv (P \cup Q) \Rightarrow f R)$,! 13 ($()I$: 12) i

$\forall P \forall Q \forall R (f P \ \& \ f Q \ \& \ R \equiv (P \cup Q) \Rightarrow f R)$! 14 ($\forall I$: 1,13) i

□

! P23 through P25 establish that our doublet, triplet, and quadruplet predicates are finite. i

! 23. i

$\vdash \forall a \forall b f (a \ b \ \ddagger)$ i

a, b ,! 1 (Prem) i

$(a \ b \ \ddagger) \equiv ((a^\bullet) \cup (b^\bullet))$,! 2 ($\forall E$: II9.3) i

$f (a^\bullet)$,! 3 ($\forall E$: P17) i

$f (b^\bullet)$,! 4 ($\forall E$: P17) i

$f (a^\bullet) \ \& \ f (b^\bullet)$,! 5 ($\&I$: 3,4) i

$$f(a^\bullet) \& f(b^\bullet) \& (a \ b \ \dagger) \equiv ((a^\bullet) \cup (b^\bullet))$$

, ! 6 (&I: 2,5) i

$$(f(a^\bullet) \& f(b^\bullet) \& (a \ b \ \dagger) \equiv ((a^\bullet) \cup (b^\bullet)) \Rightarrow f(a \ b \ \dagger))$$

, ! 7 (\forall E: P22) i

$$f(a^\bullet) \& f(b^\bullet) \& (a \ b \ \dagger) \equiv ((a^\bullet) \cup (b^\bullet)) \Rightarrow f(a \ b \ \dagger)$$

, ! 8 ($()$ E: 7) i

$$f(a \ b \ \dagger)$$

, ! 9 (\Rightarrow E: 6,8) i

$$\forall a \forall b \ f(a \ b \ \dagger)$$

! 10 (\forall I: 1,9) i

□

! 24.

$$\vdash \forall a \forall b \forall c \ f(a \ b \ c \ \forall)$$

i

$$a, b, c$$

, ! 1 (Prem) i

$$(a \ b \ c \ \forall) \equiv ((a \ b \ \dagger) \cup (c^\bullet))$$

, ! 2 (\forall E: II9.7) i

$$f(a \ b \ \dagger)$$

, ! 3 (\forall E: P23) i

$$f(c^\bullet)$$

, ! 4 (\forall E: P17) i

$$f(a \ b \ \dagger) \& f(c^\bullet)$$

, ! 5 (&I: 3,4) i

$$f(a \ b \ \dagger) \& f(c^\bullet) \& (a \ b \ c \ \forall) \equiv ((a \ b \ \dagger) \cup (c^\bullet))$$

, ! 6 (&I: 2,5) i

$$(f(a \ b \ \dagger) \& f(c^\bullet) \& (a \ b \ c \ \forall) \equiv ((a \ b \ \dagger) \cup (c^\bullet)) \Rightarrow f(a \ b \ c \ \forall))$$

, ! 7 (\forall E: P22) i

$$f(a \ b \ \dagger) \& f(c^\bullet) \& (a \ b \ c \ \forall) \equiv ((a \ b \ \dagger) \cup (c^\bullet)) \Rightarrow f(a \ b \ c \ \forall)$$

, ! 8 ($()$ E: 7) i

$$f(a \ b \ c \ \forall)$$

, ! 9 (\Rightarrow E: 6,8) i

$$\forall a \forall b \forall c \ f(a \ b \ c \ \forall)$$

! 10 (\forall I: 1,9) i

□

! 25.

$$\vdash \forall a \forall b \forall c \forall d \ f(a \ b \ c \ d \ \forall)$$

i

$$a, b, c, d$$

, ! 1 (Prem) i

$$(a \ b \ c \ d \ \forall) \equiv ((a \ b \ c \ \forall) \cup (d^\bullet))$$

, ! 2 (\forall E: II9.11) i

$f (a b c \vee)$,! 3 ($\forall E$: P24)	i
$f (d^{\bullet})$,! 4 ($\forall E$: P17)	i
$f (a b c \vee) \& f (d^{\bullet})$,! 5 ($\&I$: 3,4)	i
$f (a b c \vee) \& f (d^{\bullet}) \& (a b c d \forall)$	$\equiv ((a b c \vee) \cup (d^{\bullet}))$	
	,! 6 ($\&I$: 2,5)	i
$(f (a b c \vee) \& f (d^{\bullet}) \& (a b c d \forall) \equiv ((a b c \vee) \cup (d^{\bullet}))$		
$\Rightarrow f (a b c d \forall))$,! 7 ($\forall E$: P22)	i
$f (a b c \vee) \& f (d^{\bullet}) \& (a b c d \forall) \equiv ((a b c \vee) \cup (d^{\bullet}))$		
$\Rightarrow f (a b c d \forall)$,! 8 ($(\)E$: 7)	i
$f (a b c d \forall)$,! 9 ($\Rightarrow E$: 6,8)	i
$\forall a \forall b \forall c \forall d f (a b c d \forall)$! 10 ($\forall I$: 1,9)	i

□

! 26. P26 establishes conditions when a predicate has a maximum natural number. i

$\vdash \forall P (P \subseteq \omega \& \neg P \equiv \phi \& f P$		
$\Rightarrow \neg (\omega \cap P) \equiv \phi \& f (\omega \cap P))$		i
P	,! 1 (Prem)	i
P \subseteq ω & $\neg P \equiv \phi$ & $f P$,! 2 (Prem)	i
P \subseteq ω	,! 3 ($\&E$: 2)	i
$\neg P \equiv \phi$,! 4 ($\&E$: 2)	i
$f P$,! 5 ($\&E$: 2)	i
(P \subseteq $\omega \Rightarrow (\omega \cap P) \equiv P$)	,! 6 ($\forall E$: II3.26)	i
P \subseteq $\omega \Rightarrow (\omega \cap P) \equiv P$,! 7 ($(\)E$: 6)	i
($\omega \cap P) \equiv P$,! 8 ($\Rightarrow E$: 3,7)	i
($\omega \cap P) \equiv P$ & $\neg P \equiv \phi$,! 9 ($\&I$: 4,8)	i
(($\omega \cap P) \equiv P$ & $\neg P \equiv \phi \Rightarrow \neg (\omega \cap P) \equiv \phi)$		
	,! 10 ($\forall E$: III1.40)	i
($\omega \cap P) \equiv P$ & $\neg P \equiv \phi \Rightarrow \neg (\omega \cap P) \equiv \phi$		
	,! 11 ($(\)E$: 10)	i
$\neg (\omega \cap P) \equiv \phi$,! 12 ($\Rightarrow E$: 9,11)	i

$f P \ \& \ (\omega \cap P) \equiv P$,! 13 (&E: 5,8) i
 $(f P \ \& \ (\omega \cap P) \equiv P \Rightarrow f (\omega \cap P))$,! 14 (\forall E: P6) i
 $f P \ \& \ (\omega \cap P) \equiv P \Rightarrow f (\omega \cap P)$,! 15 ((\Rightarrow)E: 14) i
 $f (\omega \cap P)$,! 16 (\Rightarrow E: 13,15) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$,! 17 (&I: 12,16) i
 $P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$,! 18 (\Rightarrow I: 2,17) i
 $(P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P))$,! 19 ((\Rightarrow)I: 18) i
 $\forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P))$! 20 (\forall I: 1,19) i

□