

! CHAPTER 4

STRICT INEQUALITY;

! This chapter introduces and develops the concept of strict inequality. Given that strict inequality is defined in terms of inequality, many proofs in this chapter appeal to propositions from C3.

Trichotomy appears as P44. Another version of the Well-Ordering Principle appears as P49.

The chapter finishes with a proposition (P50) which shows how to conclude that P is infinite. It is used immediately (P51) to prove that the finite numbers are infinite. i

! 1. < represents strict inequality. i

$\mathbb{D} < ; < ; ; \{n,m : \leq[n,m] \ \& \ \neg n = m\}$ i

! P2 through P8 establish connections between, on the one side, strict inequality and, on the other, inequality and equality. i

! 2. i

$\vdash \forall x \forall y (<[x,y] \Leftrightarrow \leq[x,y] \ \& \ \neg x = y)$ i

$\forall x \forall y (\{n,m : \leq[n,m] \ \& \ \neg n = m\}[x,y] \Leftrightarrow \leq[x,y] \ \& \ \neg x = y)$
, ! 1 (Pred) i

$\forall x \forall y (<[x,y] \Leftrightarrow \leq[x,y] \ \& \ \neg x = y)$! 2 ($\mathbb{D}E$: P1,1) i

□

! 3. i

$\vdash \forall x \forall y (<[x,y] \Rightarrow \leq[x,y] \ \& \ \neg x = y)$ i

$\mathbf{x,y}$, ! 1 (Prem) i

$<[\mathbf{x,y}]$, ! 2 (Prem) i

$(<[\mathbf{x,y}] \Leftrightarrow \leq[\mathbf{x,y}] \ \& \ \neg \mathbf{x} = \mathbf{y})$, ! 3 ($\forall E$: P2) i

$<[\mathbf{x,y}] \Leftrightarrow \leq[\mathbf{x,y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 4 ($(())E$: 3) i

$<[\mathbf{x,y}] \Rightarrow \leq[\mathbf{x,y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 5 ($\Leftrightarrow E$: 4) i

$\leq[\mathbf{x,y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 6 ($\Rightarrow E$: 2,5) i

$<[\mathbf{x,y}] \Rightarrow \leq[\mathbf{x,y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 7 ($\Rightarrow I$: 2,6) i

$(<[\mathbf{x,y}] \Rightarrow \leq[\mathbf{x,y}] \ \& \ \neg \mathbf{x} = \mathbf{y})$, ! 8 ($(())I$: 7) i

$\forall x \forall y (<[x,y] \Rightarrow \leq[x,y] \ \& \ \neg x = y)$, ! 9 ($\forall I$: 1,8) i

□

! 4. i

$\vdash \forall x \forall y (\leq[x,y] \ \& \ \neg x = y \Rightarrow <[x,y])$ i

\mathbf{x}, \mathbf{y}	, ! 1 (Prem)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 2 (Prem)	i
$(\ <[\mathbf{x}, \mathbf{y}] \Leftrightarrow \leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \)$, ! 3 ($\forall E$: P2)	i
$\ <[\mathbf{x}, \mathbf{y}] \Leftrightarrow \leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 4 ($(())E$: 3)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \Rightarrow \ <[\mathbf{x}, \mathbf{y}]$, ! 5 ($\Leftrightarrow E$: 4)	i
$\ <[\mathbf{x}, \mathbf{y}]$, ! 6 ($\Rightarrow E$: 2,5)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \Rightarrow \ <[\mathbf{x}, \mathbf{y}]$, ! 7 ($\Rightarrow I$: 2,6)	i
$(\ \leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \Rightarrow \ <[\mathbf{x}, \mathbf{y}] \)$, ! 8 ($(())I$: 7)	i
$\forall x \forall y (\ \leq[x, y] \ \& \ \neg x = y \Rightarrow \ <[x, y] \)$, ! 9 ($\forall I$: 1,8)	i

□

! 5. i

$\vdash \forall x \forall y (\ <[x, y] \Rightarrow \leq[x, y] \)$ i

\mathbf{x}, \mathbf{y}	, ! 1 (Prem)	i
$\ <[\mathbf{x}, \mathbf{y}]$, ! 2 (Prem)	i
$(\ <[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \)$, ! 3 ($\forall E$: P3)	i
$\ <[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 4 ($(())E$: 3)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 5 ($\Rightarrow E$: 2,4)	i
$\leq[\mathbf{x}, \mathbf{y}]$, ! 6 ($\&E$: 5)	i
$\ <[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}]$, ! 7 ($\Rightarrow I$: 2,6)	i
$(\ <[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \)$, ! 8 ($(())I$: 7)	i
$\forall x \forall y (\ <[x, y] \Rightarrow \leq[x, y] \)$, ! 9 ($\forall I$: 1,8)	i

□

! 6. i

$\vdash \forall x \forall y (\ <[x, y] \Rightarrow \neg x = y \)$ i

\mathbf{x}, \mathbf{y}	, ! 1 (Prem)	i
$\ <[\mathbf{x}, \mathbf{y}]$, ! 2 (Prem)	i
$(\ <[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \)$, ! 3 ($\forall E$: P3)	i
$\ <[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 4 ($(())E$: 3)	i

$\leq[x,y] \ \& \ \neg x = y$,! 5 ($\Rightarrow E$: 2,4)	i
$\neg x = y$,! 6 ($\&E$: 5)	i
$\langle[x,y] \Rightarrow \neg x = y$,! 7 ($\Rightarrow I$: 2,6)	i
$(\langle[x,y] \Rightarrow \neg x = y)$,! 8 ($(\)I$: 7)	i
$\forall x \forall y (\langle[x,y] \Rightarrow \neg x = y)$! 9 ($\forall I$: 1,8)	i
\square		

! 7. i

$\vdash \forall x \forall y (\neg \leq[x,y] \Rightarrow \neg \langle[x,y])$		i
x, y	,! 1 (Prem)	i
$\neg \leq[x,y]$,! 2 (Prem)	i
$\langle[x,y]$,! 3 (Prem)	i
$(\langle[x,y] \Rightarrow \leq[x,y])$,! 4 ($\forall E$: P5)	i
$\langle[x,y] \Rightarrow \leq[x,y]$,! 5 ($(\)E$: 4)	i
$\leq[x,y]$,! 6 ($\Rightarrow E$: 3,5)	i
\exists	,! 7 ($\exists I$: 2,6)	i
$\langle[x,y] \Rightarrow \exists$,! 8 ($\Rightarrow I$: 3,7)	i
$\neg \langle[x,y]$,! 9 ($\neg I$: 8)	i
$\neg \leq[x,y] \Rightarrow \neg \langle[x,y]$,! 10 ($\Rightarrow I$: 2,9)	i
$(\neg \leq[x,y] \Rightarrow \neg \langle[x,y])$,! 11 ($(\)I$: 10)	i
$\forall x \forall y (\neg \leq[x,y] \Rightarrow \neg \langle[x,y])$! 12 ($\forall I$: 1,11)	i

\square

! 8. i

$\vdash \forall x \forall y (\leq[x,y] \Rightarrow \langle[x,y] \vee x = y)$		i
x, y	,! 1 (Prem)	i
$\leq[x,y]$,! 2 (Prem)	i
$(x = y \vee \neg x = y)$,! 3 ($\forall E$: I3.4)	i
$x = y \vee \neg x = y$,! 4 ($(\)E$: 3)	i
$x = y$,! 5 (Prem)	i
$\langle[x,y] \vee x = y$,! 6 ($\vee I$: 5)	i

$\mathbf{x} = \mathbf{y} \Rightarrow \langle [\mathbf{x}, \mathbf{y}] \vee \mathbf{x} = \mathbf{y}$, ! 7 (\Rightarrow I: 5,6)	i
$\neg \mathbf{x} = \mathbf{y}$, ! 8 (Prem)	i
$\leq [\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$, ! 9 ($\&$ I: 2,8)	i
$(\leq [\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \Rightarrow \langle [\mathbf{x}, \mathbf{y}])$, ! 10 (\forall E: P4)	i
$\leq [\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \Rightarrow \langle [\mathbf{x}, \mathbf{y}]$, ! 11 ($(\)$ E: 10)	i
$\langle [\mathbf{x}, \mathbf{y}]$, ! 12 (\Rightarrow E: 9,11)	i
$\langle [\mathbf{x}, \mathbf{y}] \vee \mathbf{x} = \mathbf{y}$, ! 13 (\vee I: 12)	i
$\neg \mathbf{x} = \mathbf{y} \Rightarrow \langle [\mathbf{x}, \mathbf{y}] \vee \mathbf{x} = \mathbf{y}$, ! 14 (\Rightarrow I: 8,13)	i
$\langle [\mathbf{x}, \mathbf{y}] \vee \mathbf{x} = \mathbf{y}$, ! 15 (\vee E: 4,7,14)	i
$\leq [\mathbf{x}, \mathbf{y}] \Rightarrow \langle [\mathbf{x}, \mathbf{y}] \vee \mathbf{x} = \mathbf{y}$, ! 16 (\Rightarrow I: 2,15)	i
$(\leq [\mathbf{x}, \mathbf{y}] \Rightarrow \langle [\mathbf{x}, \mathbf{y}] \vee \mathbf{x} = \mathbf{y})$, ! 17 ($(\)$ I: 16)	i
$\forall \mathbf{x} \forall \mathbf{y} (\leq [\mathbf{x}, \mathbf{y}] \Rightarrow \langle [\mathbf{x}, \mathbf{y}] \vee \mathbf{x} = \mathbf{y})$! 18 (\forall I: 1,17)	i

□

! 9.

$\vdash \forall \mathbf{x} \forall \mathbf{y} (\langle [\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$		i
\mathbf{x}, \mathbf{y}	, ! 1 (Prem)	i
$\langle [\mathbf{x}, \mathbf{y}]$, ! 2 (Prem)	i
$(\langle [\mathbf{x}, \mathbf{y}] \Rightarrow \leq [\mathbf{x}, \mathbf{y}])$, ! 3 (\forall E: P5)	i
$\langle [\mathbf{x}, \mathbf{y}] \Rightarrow \leq [\mathbf{x}, \mathbf{y}]$, ! 4 ($(\)$ E: 3)	i
$\leq [\mathbf{x}, \mathbf{y}]$, ! 5 (\Rightarrow E: 2,4)	i
$(\leq [\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$, ! 6 (\forall E: C3.5)	i
$\leq [\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$, ! 7 ($(\)$ E: 6)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$, ! 8 (\Rightarrow E: 5,7)	i
$\langle [\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$, ! 9 (\Rightarrow I: 2,8)	i
$(\langle [\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$, ! 10 ($(\)$ I: 9)	i
$\forall \mathbf{x} \forall \mathbf{y} (\langle [\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$! 11 (\forall I: 1,10)	i

□

! 10.

i

$\vdash \forall x \forall y (<[x,y] \Rightarrow \omega[x])$		i
$\mathbf{x, y}$,! 1 (Prem)	i
$<[\mathbf{x}, \mathbf{y}]$,! 2 (Prem)	i
$(<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$,! 3 ($\forall E$: P9)	i
$<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 4 ($(())E$: 3)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 5 ($\Rightarrow E$: 2,4)	i
$\omega[\mathbf{x}]$,! 6 ($\&E$: 5)	i
$<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}]$,! 7 ($\Rightarrow I$: 2,6)	i
$(<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}])$,! 8 ($(())I$: 7)	i
$\forall x \forall y (<[x,y] \Rightarrow \omega[x])$! 9 ($\forall I$: 1,8)	i

□

! 11. i

$\vdash \forall x \forall y (<[x,y] \Rightarrow \omega[y])$		i
$\mathbf{x, y}$,! 1 (Prem)	i
$<[\mathbf{x}, \mathbf{y}]$,! 2 (Prem)	i
$(<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$,! 3 ($\forall E$: P9)	i
$<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 4 ($(())E$: 3)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 5 ($\Rightarrow E$: 2,4)	i
$\omega[\mathbf{y}]$,! 6 ($\&E$: 5)	i
$<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{y}]$,! 7 ($\Rightarrow I$: 2,6)	i
$(<[\mathbf{x}, \mathbf{y}] \Rightarrow \omega[\mathbf{y}])$,! 8 ($(())I$: 7)	i
$\forall x \forall y (<[x,y] \Rightarrow \omega[y])$! 9 ($\forall I$: 1,8)	i

□

! 12. Strict Inequality is Irreflexive. i

$\vdash \forall x \neg <[x,x]$		i
\mathbf{x}	,! 1 (Prem)	i
$<[\mathbf{x}, \mathbf{x}]$,! 2 (Prem)	i
$(<[\mathbf{x}, \mathbf{x}] \Rightarrow \neg \mathbf{x} = \mathbf{x})$,! 3 ($\forall E$: P6)	i

$\langle [x, x] \Rightarrow \neg x = x$, ! 4 (()E: 3)	i
$\neg x = x$, ! 5 (\Rightarrow E: 2,4)	i
$x = x$, ! 6 (=I)	i
\mathcal{F}	, ! 7 (\mathcal{F} I: 5,6)	i
$\langle [x, x] \Rightarrow \mathcal{F}$, ! 8 (\Rightarrow I: 2,7)	i
$\neg \langle [x, x]$, ! 9 (\neg I: 8)	i
$\forall x \neg \langle [x, x]$! 10 (\forall I: 1,9)	i

□

! 13.

$\vdash \forall x \neg \langle [x, 0]$		i
x	, ! 1 (Prem)	i
$\langle [x, 0]$, ! 2 (Prem)	i
$(\langle [x, 0] \Rightarrow \leq [x, 0] \ \& \ \neg x = 0)$, ! 3 (\forall E: P3)	i
$\langle [x, 0] \Rightarrow \leq [x, 0] \ \& \ \neg x = 0$, ! 4 (()E: 3)	i
$\leq [x, 0] \ \& \ \neg x = 0$, ! 5 (\Rightarrow E: 2,4)	i
$\leq [x, 0]$, ! 6 (&E: 5)	i
$\neg x = 0$, ! 7 (&E: 5)	i
$(\leq [x, 0] \Rightarrow x = 0)$, ! 8 (\forall E: C3.25)	i
$\leq [x, 0] \Rightarrow x = 0$, ! 9 (()E: 8)	i
$x = 0$, ! 10 (\Rightarrow E: 6,9)	i
\mathcal{F}	, ! 11 (\mathcal{F} I: 7,10)	i
$\langle [x, 0] \Rightarrow \mathcal{F}$, ! 12 (\Rightarrow I: 2,11)	i
$\neg \langle [x, 0]$, ! 13 (\neg I: 12)	i
$\forall x \neg \langle [x, 0]$! 14 (\forall I: 1,13)	i

□

! 14.

$\vdash \forall x (\langle [0, x] \Rightarrow \neg x = 0)$		i
x	, ! 1 (Prem)	i
$\langle [0, x]$, ! 2 (Prem)	i

$x = 0$,! 3 (Prem)	i
$<[x, x]$,! 4 (=E: 2,3)	i
$\neg <[x, x]$,! 5 (\forall E: P12)	i
\exists	,! 6 (\exists I: 4,5)	i
$x = 0 \Rightarrow \exists$,! 7 (\Rightarrow I: 3,6)	i
$\neg x = 0$,! 8 (\neg I: 7)	i
$<[0, x] \Rightarrow \neg x = 0$,! 9 (\Rightarrow I: 2,8)	i
$(<[0, x] \Rightarrow \neg x = 0)$,! 10 (()I: 9)	i
$\forall x (<[0, x] \Rightarrow \neg x = 0)$! 11 (\forall I: 1,10)	i

□

! 15.

$\vdash \forall x (\omega[x] \ \& \ \neg x = 0 \Rightarrow <[0, x])$		i
x	,! 1 (Prem)	i
$\omega[x] \ \& \ \neg x = 0$,! 2 (Prem)	i
$\omega[x]$,! 3 ($\&$ E: 2)	i
$\neg x = 0$,! 4 ($\&$ E: 2)	i
$(\omega[x] \Rightarrow \leq[0, x])$,! 5 (\forall E: C3.24)	i
$\omega[x] \Rightarrow \leq[0, x]$,! 6 (()E: 5)	i
$\leq[0, x]$,! 7 (\Rightarrow E: 3,6)	i
$(\neg x = 0 \Rightarrow \neg 0 = x)$,! 8 (\forall E: I3.3)	i
$\neg x = 0 \Rightarrow \neg 0 = x$,! 9 (()E: 8)	i
$\neg 0 = x$,! 10 (\Rightarrow E: 4,9)	i
$\leq[0, x] \ \& \ \neg 0 = x$,! 11 ($\&$ I: 7,10)	i
$(\leq[0, x] \ \& \ \neg 0 = x \Rightarrow <[0, x])$,! 12 (\forall E: P4)	i
$\leq[0, x] \ \& \ \neg 0 = x \Rightarrow <[0, x]$,! 13 (()E: 12)	i
$<[0, x]$,! 14 (\Rightarrow E: 11,13)	i
$\omega[x] \ \& \ \neg x = 0 \Rightarrow <[0, x]$,! 15 (\Rightarrow I: 2,14)	i
$(\omega[x] \ \& \ \neg x = 0 \Rightarrow <[0, x])$,! 16 (()I: 15)	i

$\forall x (\omega[x] \ \& \ \neg x = 0 \Rightarrow <[0,x])$! 17 (\forall I: 1,16) ;

□

! P18 expresses the transitivity of strict inequality. P16 and P17 express stronger forms, namely one inequality of the antecedent can be normal rather than strict. ;

! 16. ;

$\vdash \forall x \forall y \forall z (<[x,y] \ \& \ \leq[y,z] \Rightarrow <[x,z])$;

x, y, z ,! 1 (Prem) ;

$<[x,y] \ \& \ \leq[y,z]$,! 2 (Prem) ;

$<[x,y]$,! 3 ($\&$ E: 2) ;

$\leq[y,z]$,! 4 ($\&$ E: 2) ;

$(<[x,y] \Rightarrow \leq[x,y] \ \& \ \neg x = y)$,! 5 (\forall E: P3) ;

$<[x,y] \Rightarrow \leq[x,y] \ \& \ \neg x = y$,! 6 ($($)E: 5) ;

$\leq[x,y] \ \& \ \neg x = y$,! 7 (\Rightarrow E: 3,6) ;

$\leq[x,y]$,! 8 ($\&$ E: 7) ;

$\neg x = y$,! 9 ($\&$ E: 7) ;

$\leq[x,y] \ \& \ \leq[y,z]$,! 10 ($\&$ I: 4,8) ;

$(\leq[x,y] \ \& \ \leq[y,z] \Rightarrow \leq[x,z])$,! 11 (\forall E: C3.20) ;

$\leq[x,y] \ \& \ \leq[y,z] \Rightarrow \leq[x,z]$,! 12 ($($)E: 11) ;

$\leq[x,z]$,! 13 (\Rightarrow E: 10,12) ;

$x = z$,! 14 (Prem) ;

$\leq[x,y] \ \& \ \leq[y,x]$,! 15 ($=$ E: 10,14) ;

$(\leq[x,y] \ \& \ \leq[y,x] \Rightarrow x = y)$,! 16 (\forall E: C3.22) ;

$\leq[x,y] \ \& \ \leq[y,x] \Rightarrow x = y$,! 17 ($($)E: 16) ;

$x = y$,! 18 (\Rightarrow E: 15,17) ;

\mathfrak{F} ,! 19 (\mathfrak{F} I: 9,18) ;

$x = z \Rightarrow \mathfrak{F}$,! 20 (\Rightarrow I: 14,19) ;

$\neg x = z$,! 21 (\neg I: 20) ;

$\leq[x,z] \ \& \ \neg x = z$,! 22 ($\&$ I: 13,21) ;

$(\leq[x,z] \ \& \ \neg x = z \Rightarrow <[x,z])$,! 23 (\forall E: P4) ;

$\leq[x, z] \ \& \ \neg x = z \Rightarrow <[x, z]$, ! 24 (()E: 23)	i
$<[x, z]$, ! 25 (\Rightarrow E: 22, 24)	i
$<[x, y] \ \& \ \leq[y, z] \Rightarrow <[x, z]$, ! 26 (\Rightarrow I: 2, 25)	i
$(<[x, y] \ \& \ \leq[y, z] \Rightarrow <[x, z])$, ! 27 (()I: 26)	i
$\forall x \forall y \forall z (<[x, y] \ \& \ \leq[y, z] \Rightarrow <[x, z])$! 28 (\forall I: 1, 27)	i
\square		
! 17.		i
$\vdash \forall x \forall y \forall z (\leq[x, y] \ \& \ <[y, z] \Rightarrow <[x, z])$		i
x, y, z	, ! 1 (Prem)	i
$\leq[x, y] \ \& \ <[y, z]$, ! 2 (Prem)	i
$\leq[x, y]$, ! 3 ($\&$ E: 2)	i
$<[y, z]$, ! 4 ($\&$ E: 2)	i
$(<[y, z] \Rightarrow \leq[y, z] \ \& \ \neg y = z)$, ! 5 (\forall E: P3)	i
$<[y, z] \Rightarrow \leq[y, z] \ \& \ \neg y = z$, ! 6 (()E: 5)	i
$\leq[y, z] \ \& \ \neg y = z$, ! 7 (\Rightarrow E: 4, 6)	i
$\leq[y, z]$, ! 8 ($\&$ E: 7)	i
$\neg y = z$, ! 9 ($\&$ E: 7)	i
$\leq[x, y] \ \& \ \leq[y, z]$, ! 10 ($\&$ I: 3, 8)	i
$(\leq[x, y] \ \& \ \leq[y, z] \Rightarrow \leq[x, z])$, ! 11 (\forall E: C3.20)	i
$\leq[x, y] \ \& \ \leq[y, z] \Rightarrow \leq[x, z]$, ! 12 (()E: 11)	i
$\leq[x, z]$, ! 13 (\Rightarrow E: 10, 12)	i
$x = z$, ! 14 (Prem)	i
$\leq[y, z] \ \& \ \leq[x, y]$, ! 15 ($\&$ I: 3, 8)	i
$\leq[y, z] \ \& \ \leq[z, y]$, ! 16 ($=$ E: 14, 15)	i
$(\leq[y, z] \ \& \ \leq[z, y] \Rightarrow y = z)$, ! 17 (\forall E: C3.22)	i
$\leq[y, z] \ \& \ \leq[z, y] \Rightarrow y = z$, ! 18 (()E: 17)	i
$y = z$, ! 19 (\Rightarrow E: 16, 18)	i
\exists	, ! 20 (\exists I: 9, 19)	i
$x = z \Rightarrow \exists$, ! 21 (\Rightarrow I: 14, 20)	i

$\neg \mathbf{x} = \mathbf{z}$,! 22 (\neg I: 21)	i
$\leq[\mathbf{x}, \mathbf{z}] \ \& \ \neg \mathbf{x} = \mathbf{z}$,! 23 ($\&$ I: 13,22)	i
$(\leq[\mathbf{x}, \mathbf{z}] \ \& \ \neg \mathbf{x} = \mathbf{z} \Rightarrow <[\mathbf{x}, \mathbf{z}])$,! 24 (\forall E: P4)	i
$\leq[\mathbf{x}, \mathbf{z}] \ \& \ \neg \mathbf{x} = \mathbf{z} \Rightarrow <[\mathbf{x}, \mathbf{z}]$,! 25 ($()$ E: 24)	i
$<[\mathbf{x}, \mathbf{z}]$,! 26 (\Rightarrow E: 23,25)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow <[\mathbf{x}, \mathbf{z}]$,! 27 (\Rightarrow I: 2,26)	i
$(\leq[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow <[\mathbf{x}, \mathbf{z}])$,! 28 ($()$ I: 27)	i
$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} (\leq[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow <[\mathbf{x}, \mathbf{z}])$! 29 (\forall I: 1,28)	i
\square		

! 18. i

$\vdash \forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} (<[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow <[\mathbf{x}, \mathbf{z}])$		i
$\mathbf{x}, \mathbf{y}, \mathbf{z}$,! 1 (Prem)	i
$<[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}]$,! 2 (Prem)	i
$<[\mathbf{x}, \mathbf{y}]$,! 3 ($\&$ E: 2)	i
$<[\mathbf{y}, \mathbf{z}]$,! 4 ($\&$ E: 2)	i
$(<[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}])$,! 5 (\forall E: P5)	i
$<[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}]$,! 6 ($()$ E: 5)	i
$\leq[\mathbf{x}, \mathbf{y}]$,! 7 (\Rightarrow E: 3,6)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}]$,! 8 ($\&$ I: 4,7)	i
$(\leq[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow \mathbf{x} < \mathbf{z})$,! 9 (\forall E: P17)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow \mathbf{x} < \mathbf{z}$,! 10 ($()$ E: 9)	i
$\mathbf{x} < \mathbf{z}$,! 11 (\Rightarrow E: 8,10)	i
$<[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow <[\mathbf{x}, \mathbf{z}]$,! 12 (\Rightarrow I: 2,11)	i
$(<[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow <[\mathbf{x}, \mathbf{z}])$,! 13 ($()$ I: 12)	i
$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} (<[\mathbf{x}, \mathbf{y}] \ \& \ <[\mathbf{y}, \mathbf{z}] \Rightarrow <[\mathbf{x}, \mathbf{z}])$! 14 (\forall I: 1,13)	i
\square		

! 19. i

$\vdash \forall \mathbf{x} \forall \mathbf{y} (<[\mathbf{x}, \mathbf{y}] \ \& \ \leq[\mathbf{y}, \mathbf{x}] \Rightarrow \mathfrak{F})$		i
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x, y	,! 1 (Prem)	i
$\lt[x, y] \ \& \ \leq[y, x]$,! 2 (Prem)	i
$\lt[x, y]$,! 3 (&E: 2)	i
$\leq[y, x]$,! 4 (&E: 2)	i
$(\ \lt[x, y] \Rightarrow \leq[x, y] \ \& \ \neg \ x = y \)$,! 5 (\forall E: P3)	i
$\lt[x, y] \Rightarrow \leq[x, y] \ \& \ \neg \ x = y$,! 6 ($(\)$ E: 5)	i
$\leq[x, y] \ \& \ \neg \ x = y$,! 7 (\Rightarrow E: 3,6)	i
$\leq[x, y]$,! 8 (&E: 7)	i
$\neg \ x = y$,! 9 (&E: 7)	i
$\leq[x, y] \ \& \ \leq[y, x]$,! 10 (&I: 4,8)	i
$(\ \leq[x, y] \ \& \ \leq[y, x] \Rightarrow x = y \)$,! 11 (\forall E: C3.22)	i
$\leq[x, y] \ \& \ \leq[y, x] \Rightarrow x = y$,! 12 ($(\)$ E: 11)	i
$x = y$,! 13 (\Rightarrow E: 10,12)	i
\mathfrak{F}	,! 14 (\mathfrak{F} I: 9,13)	i
$\lt[x, y] \ \& \ \leq[y, x] \Rightarrow \mathfrak{F}$,! 15 (\Rightarrow I: 2,14)	i
$(\ \lt[x, y] \ \& \ \leq[y, x] \Rightarrow \mathfrak{F} \)$,! 16 ($(\)$ I: 15)	i
$\forall x \forall y (\ \lt[x, y] \ \& \ \leq[y, x] \Rightarrow \mathfrak{F} \)$! 17 (\forall I: 1,16)	i

□

! 20. i

$\vdash \forall x \forall y (\ \lt[x, y] \Rightarrow \neg \ \leq[y, x] \)$ i

x, y	,! 1 (Prem)	i
$\lt[x, y]$,! 2 (Prem)	i
$\leq[y, x]$,! 3 (Prem)	i
$\lt[x, y] \ \& \ \leq[y, x]$,! 4 (&I: 2,3)	i
$(\ \lt[x, y] \ \& \ \leq[y, x] \Rightarrow \mathfrak{F} \)$,! 5 (\forall E: P19)	i
$\lt[x, y] \ \& \ \leq[y, x] \Rightarrow \mathfrak{F}$,! 6 ($(\)$ E: 5)	i
\mathfrak{F}	,! 7 (\Rightarrow E: 4,6)	i
$\leq[y, x] \Rightarrow \mathfrak{F}$,! 8 (\Rightarrow I: 3,7)	i
$\neg \ \leq[y, x]$,! 9 (\neg I: 8)	i

$\langle [x, y] \Rightarrow \neg \leq [y, x]$, ! 10 (\Rightarrow I: 2,9)	i
$(\langle [x, y] \Rightarrow \neg \leq [y, x])$, ! 11 ($(())$ I: 10)	i
$\forall x \forall y (\langle [x, y] \Rightarrow \neg \leq [y, x])$! 12 (\forall I: 1,11)	i
\square		

! 21.

$\vdash \forall x \forall y (\langle [x, y] \Rightarrow \neg \langle [y, x])$		i
x, y	, ! 1 (Prem)	i
$\langle [x, y]$, ! 2 (Prem)	i
$(\langle [x, y] \Rightarrow \neg \leq [y, x])$, ! 3 (\forall E: P20)	i
$\langle [x, y] \Rightarrow \neg \leq [y, x]$, ! 4 ($(())$ E: 3)	i
$\neg \leq [y, x]$, ! 5 (\Rightarrow E: 2,4)	i
$(\neg \leq [y, x] \Rightarrow \neg \langle [y, x])$, ! 6 (\forall E: P7)	i
$\neg \leq [y, x] \Rightarrow \neg \langle [y, x]$, ! 7 ($(())$ E: 6)	i
$\neg \langle [y, x]$, ! 8 (\Rightarrow E: 5,7)	i
$\langle [x, y] \Rightarrow \neg \langle [y, x]$, ! 9 (\Rightarrow I: 2,8)	i
$(\langle [x, y] \Rightarrow \neg \langle [y, x])$, ! 10 ($(())$ I: 9)	i
$\forall x \forall y (\langle [x, y] \Rightarrow \neg \langle [y, x])$! 11 (\forall I: 1,10)	i
\square		

! 22.

$\vdash \forall x \forall y (\leq [x, y] \Rightarrow \neg \langle [y, x])$		i
x, y	, ! 1 (Prem)	i
$\leq [x, y]$, ! 2 (Prem)	i
$\langle [y, x]$, ! 3 (Prem)	i
$\langle [y, x] \ \& \ \leq [x, y]$, ! 4 ($\&$ I: 2,3)	i
$(\langle [y, x] \ \& \ \leq [x, y] \Rightarrow \mathfrak{F})$, ! 5 (\forall E: P19)	i
$\langle [y, x] \ \& \ \leq [x, y] \Rightarrow \mathfrak{F}$, ! 6 ($(())$ E: 5)	i
\mathfrak{F}	, ! 7 (\Rightarrow E: 4,6)	i
$\langle [y, x] \Rightarrow \mathfrak{F}$, ! 8 (\Rightarrow I: 3,7)	i

$\neg <[y, x]$, ! 9 ($\neg I$: 8)	i
$\leq[x, y] \Rightarrow \neg <[y, x]$, ! 10 ($\Rightarrow I$: 2,9)	i
$(\leq[x, y] \Rightarrow \neg <[y, x])$, ! 11 ($(())I$: 10)	i
$\forall x \forall y (\leq[x, y] \Rightarrow \neg <[y, x])$! 12 ($\forall I$: 1,11)	i

□

! 23.

$\vdash \forall x \forall y (<[x, y] \ \& \ \omega[z] \Rightarrow <[x, (y+z)])$		i
x, y	, ! 1 (Prem)	i
$<[x, y] \ \& \ \omega[z]$, ! 2 (Prem)	i
$<[x, y]$, ! 3 ($\&E$: 2)	i
$\omega[z]$, ! 4 ($\&E$: 2)	i
$(<[x, y] \Rightarrow \omega[y])$, ! 5 ($\forall E$: P11)	i
$<[x, y] \Rightarrow \omega[y]$, ! 6 ($(())E$: 5)	i
$\omega[y]$, ! 7 ($\Rightarrow E$: 3,6)	i
$\omega[y] \ \& \ \omega[z]$, ! 8 ($\&I$: 4,7)	i
$(\omega[y] \ \& \ \omega[z] \Rightarrow \leq[y, (y+z)])$, ! 9 ($\forall E$: C3.33)	i
$\omega[y] \ \& \ \omega[z] \Rightarrow \leq[y, (y+z)]$, ! 10 ($(())E$: 9)	i
$\leq[y, (y+z)]$, ! 11 ($\Rightarrow E$: 8,10)	i
$<[x, y] \ \& \ \leq[y, (y+z)]$, ! 12 ($\&I$: 3,11)	i
$(<[x, y] \ \& \ \leq[y, (y+z)] \Rightarrow <[x, (y+z)])$, ! 13 ($\forall E$: P16; ($y+z$): C1.7,8)	i
$<[x, y] \ \& \ \leq[y, (y+z)] \Rightarrow <[x, (y+z)]$, ! 14 ($(())E$: 13)	i
$<[x, (y+z)]$, ! 15 ($\Rightarrow E$: 12,14)	i
$<[x, y] \ \& \ \omega[z] \Rightarrow <[x, (y+z)]$, ! 16 ($\Rightarrow I$: 2,15)	i
$(<[x, y] \ \& \ \omega[z] \Rightarrow <[x, (y+z)])$, ! 17 ($(())I$: 16)	i
$\forall x \forall y (<[x, y] \ \& \ \omega[z] \Rightarrow <[x, (y+z)])$! 18 ($\forall I$: 1,17)	i

□

! 24.

$\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg y = 0 \Rightarrow <[x, (x+y)])$		i
---	--	---

\mathbf{x}, \mathbf{y}	,! 1 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0$,! 2 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 3 (&E: 2)	i
$\neg \mathbf{y} = 0$,! 4 (&E: 2)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \Rightarrow \ \leq[\mathbf{x}, (\mathbf{x}+\mathbf{y})] \)$,! 5 (\forall E: C3.33)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \Rightarrow \ \leq[\mathbf{x}, (\mathbf{x}+\mathbf{y})]$,! 6 (()E: 5)	i
$\leq[\mathbf{x}, (\mathbf{x}+\mathbf{y})]$,! 7 (\Rightarrow E: 3,6)	i
$\mathbf{x} = (\mathbf{x}+\mathbf{y})$,! 8 (Prem)	i
$(\ \mathbf{x} = (\mathbf{x}+\mathbf{y}) \ \Rightarrow \ \mathbf{y} = 0 \)$,! 9 (\forall E: C2.36)	i
$\mathbf{x} = (\mathbf{x}+\mathbf{y}) \ \Rightarrow \ \mathbf{y} = 0$,! 10 (()E: 9)	i
$\mathbf{y} = 0$,! 11 (\Rightarrow E: 8,10)	i
\mathfrak{F}	,! 12 (\mathfrak{F} I: 4,11)	i
$\mathbf{x} = (\mathbf{x}+\mathbf{y}) \ \Rightarrow \ \mathfrak{F}$,! 13 (\Rightarrow I: 8,12)	i
$\neg \mathbf{x} = (\mathbf{x}+\mathbf{y})$,! 14 (\neg I: 13)	i
$\leq[\mathbf{x}, (\mathbf{x}+\mathbf{y})] \ \& \ \neg \mathbf{x} = (\mathbf{x}+\mathbf{y})$,! 15 (&I: 7,14)	i
$(\ \leq[\mathbf{x}, (\mathbf{x}+\mathbf{y})] \ \& \ \neg \mathbf{x} = (\mathbf{x}+\mathbf{y}) \ \Rightarrow \ <[\mathbf{x}, (\mathbf{x}+\mathbf{y})] \)$,! 16 (\forall E: P4; ($\mathbf{x}+\mathbf{y}$): C1.7,3)	i
$\leq[\mathbf{x}, (\mathbf{x}+\mathbf{y})] \ \& \ \neg \mathbf{x} = (\mathbf{x}+\mathbf{y}) \ \Rightarrow \ <[\mathbf{x}, (\mathbf{x}+\mathbf{y})]$,! 17 (()E: 16)	i
$<[\mathbf{x}, (\mathbf{x}+\mathbf{y})]$,! 18 (\Rightarrow E: 15,17)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \ \Rightarrow \ <[\mathbf{x}, (\mathbf{x}+\mathbf{y})]$,! 19 (\Rightarrow I: 2,18)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \ \Rightarrow \ <[\mathbf{x}, (\mathbf{x}+\mathbf{y})] \)$,! 20 (()I: 19)	i
$\forall \mathbf{x} \forall \mathbf{y} (\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \ \Rightarrow \ <[\mathbf{x}, (\mathbf{x}+\mathbf{y})] \)$! 21 (\forall I: 1,20)	i

□

! 25. i

$\vdash \forall \mathbf{x} \forall \mathbf{y} (\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \ \Rightarrow \ <[\mathbf{x}, (\mathbf{y}+\mathbf{x})] \)$ i

\mathbf{x}, \mathbf{y} ,! 1 (Prem) i

$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0$,! 2 (Prem) i

$(\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \rangle)$, ! 3 ($\forall E$: P24)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \rangle$, ! 4 ($()E$: 3)	i
$\langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \rangle$, ! 5 ($\Rightarrow E$: 2,4)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$, ! 6 ($\mathbb{T}E$: C1.7,5)	i
$(\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow (\mathbf{x}+\mathbf{y}) = (\mathbf{y}+\mathbf{x}))$, ! 7 ($\forall E$: C2.5)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow (\mathbf{x}+\mathbf{y}) = (\mathbf{y}+\mathbf{x})$, ! 8 ($()E$: 7)	i
$(\mathbf{x}+\mathbf{y}) = (\mathbf{y}+\mathbf{x})$, ! 9 ($\Rightarrow E$: 6,8)	i
$\langle [\mathbf{x}, (\mathbf{y}+\mathbf{x})] \rangle$, ! 10 ($=E$: 5,9)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{y}+\mathbf{x})] \rangle$, ! 11 ($\Rightarrow I$: 2,10)	i
$(\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{y}+\mathbf{x})] \rangle)$, ! 12 ($()I$: 11)	i
$\forall \mathbf{x} \forall \mathbf{y} (\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{y}+\mathbf{x})] \rangle)$! 13 ($\forall I$: 1,12)	i

□

! 26.

$\vdash \forall \mathbf{x} (\omega[\mathbf{x}] \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+1)] \rangle)$		
\mathbf{x}	, ! 1 (Prem)	i
$\omega[\mathbf{x}]$, ! 2 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[1]$, ! 3 ($\&I$: IV9.2,2)	i
$\omega[\mathbf{x}] \ \& \ \omega[1] \ \& \ \neg 1 = 0$, ! 4 ($\&I$: IV9.6,3)	i
$(\omega[\mathbf{x}] \ \& \ \omega[1] \ \& \ \neg 1 = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+1)] \rangle)$, ! 5 ($\forall E$: P24)	i
$\omega[\mathbf{x}] \ \& \ \omega[1] \ \& \ \neg 1 = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+1)] \rangle$, ! 6 ($()E$: 5)	i
$\langle [\mathbf{x}, (\mathbf{x}+1)] \rangle$, ! 7 ($\Rightarrow E$: 4,6)	i
$\omega[\mathbf{x}] \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+1)] \rangle$, ! 8 ($\Rightarrow I$: 2,7)	i
$(\omega[\mathbf{x}] \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+1)] \rangle)$, ! 9 ($()I$: 8)	i
$\forall \mathbf{x} (\omega[\mathbf{x}] \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+1)] \rangle)$! 10 ($\forall I$: 1,9)	i

□

! 27. i

$\vdash <[0,1]$ i

$(\omega[0] \Rightarrow <[0,(0+1)])$,! 1 ($\forall E$: P26) i

$\omega[0] \Rightarrow <[0,(0+1)]$,! 2 ($()E$: 1) i

$<[0,(0+1)]$,! 3 ($\Rightarrow E$: $\omega 0,2$) i

$<[0,1]$! 4 ($=E$: C2.67,3) i

□

! P28 through P37 provide strong ways of connecting strict inequality and normal inequality, without recourse to equality. i

! 28. i

$\vdash \forall x \forall y (<[x,(y+1)] \Rightarrow \leq[x,y])$ i

\mathbf{x}, \mathbf{y} ,! 1 (Prem) i

$<[\mathbf{x},(\mathbf{y}+1)]$,! 2 (Prem) i

$\omega[\mathbf{y}] \ \& \ \omega[1]$,! 3 ($\mathbb{T}E$: C1.7,2) i

$(<[\mathbf{x},(\mathbf{y}+1)] \Rightarrow \leq[\mathbf{x},(\mathbf{y}+1)] \ \& \ \neg \mathbf{x} = (\mathbf{y}+1))$
,! 4 ($\forall E$: P3;
 $(\mathbf{y}+1)$: C1.7,3) i

$<[\mathbf{x},(\mathbf{y}+1)] \Rightarrow \leq[\mathbf{x},(\mathbf{y}+1)] \ \& \ \neg \mathbf{x} = (\mathbf{y}+1)$
,! 5 ($()E$: 4) i

$\leq[\mathbf{x},(\mathbf{y}+1)] \ \& \ \neg \mathbf{x} = (\mathbf{y}+1)$,! 6 ($\Rightarrow E$: 2,5) i

$(\leq[\mathbf{x},(\mathbf{y}+1)] \ \& \ \neg \mathbf{x} = (\mathbf{y}+1) \Rightarrow \leq[\mathbf{x},\mathbf{y}])$
,! 7 ($\forall E$: C3.53) i

$\leq[\mathbf{x},(\mathbf{y}+1)] \ \& \ \neg \mathbf{x} = (\mathbf{y}+1) \Rightarrow \leq[\mathbf{x},\mathbf{y}]$,! 8 ($()E$: 7) i

$\leq[\mathbf{x},\mathbf{y}]$,! 9 ($\Rightarrow E$: 6,8) i

$<[\mathbf{x},(\mathbf{y}+1)] \Rightarrow \leq[\mathbf{x},\mathbf{y}]$,! 10 ($\Rightarrow I$: 2,9) i

$(<[\mathbf{x},(\mathbf{y}+1)] \Rightarrow \leq[\mathbf{x},\mathbf{y}])$,! 11 ($()I$: 10) i

$\forall x \forall y (<[x,(y+1)] \Rightarrow \leq[x,y])$! 12 ($\forall I$: 1,11) i

□

! 29. i

$\vdash \forall x \forall y (\leq[x,y] \Rightarrow <[x,(y+1)])$ i

\mathbf{x}, \mathbf{y} ,! 1 (Prem) i

$\leq[x, y]$,! 2 (Prem)	i
$(\leq[x, y] \Rightarrow \omega[y])$,! 3 ($\forall E$: C3.7)	i
$\leq[x, y] \Rightarrow \omega[y]$,! 4 ($()E$: 3)	i
$\omega[y]$,! 5 ($\Rightarrow E$: 2,4)	i
$(\omega[y] \Rightarrow \langle[y, (y+1)] \rangle)$,! 6 ($\forall E$: P26)	i
$\omega[y] \Rightarrow \langle[y, (y+1)] \rangle$,! 7 ($()E$: 6)	i
$\langle[y, (y+1)] \rangle$,! 8 ($\Rightarrow E$: 5,7)	i
$\leq[x, y] \ \& \ \langle[y, (y+1)] \rangle$,! 9 ($\&I$: 2,8)	i
$\omega[y] \ \& \ \omega[1]$,! 10 ($\&I$: IV9.2)	i
$(\leq[x, y] \ \& \ \langle[y, (y+1)] \rangle \Rightarrow \langle[x, (y+1)] \rangle)$,! 11 ($\forall E$: P17; ($y+1$): C1.7,10)	i
$\leq[x, y] \ \& \ \langle[y, (y+1)] \rangle \Rightarrow \langle[x, (y+1)] \rangle$,! 12 ($()E$: 11)	i
$\langle[x, (y+1)] \rangle$,! 13 ($\Rightarrow E$: 9,12)	i
$\leq[x, y] \Rightarrow \langle[x, (y+1)] \rangle$,! 14 ($\Rightarrow I$: 2,13)	i
$(\leq[x, y] \Rightarrow \langle[x, (y+1)] \rangle)$,! 15 ($()I$: 14)	i
$\forall x \forall y (\leq[x, y] \Rightarrow \langle[x, (y+1)] \rangle)$! 16 ($\forall I$: 1,15)	i

□

! 30.

$\vdash \forall x \forall y (\langle[x, (y+1)] \rangle \Leftrightarrow \leq[x, y])$		i
x, y	,! 1 (Prem)	i
$(\langle[x, (y+1)] \rangle \Rightarrow \leq[x, y])$,! 2 ($\forall E$: P28)	i
$\langle[x, (y+1)] \rangle \Rightarrow \leq[x, y]$,! 3 ($()E$: 2)	i
$(\leq[x, y] \Rightarrow \langle[x, (y+1)] \rangle)$,! 4 ($\forall E$: P29)	i
$\leq[x, y] \Rightarrow \langle[x, (y+1)] \rangle$,! 5 ($()E$: 4)	i
$\langle[x, (y+1)] \rangle \Leftrightarrow \leq[x, y]$,! 6 ($\Leftrightarrow I$: 3,5)	i
$(\langle[x, (y+1)] \rangle \Leftrightarrow \leq[x, y])$,! 7 ($()I$: 6)	i
$\forall x \forall y (\langle[x, (y+1)] \rangle \Leftrightarrow \leq[x, y])$! 8 ($\forall I$: 1,7)	i

□

! 31.		i
$\vdash \forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \Rightarrow \langle [n,m] \rangle)$		i
n, m	, ! 1 (Prem)	i
$\omega[n] \ \& \ \sigma[n,m]$, ! 2 (Prem)	i
$(\omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1))$, ! 3 ($\forall E$: C2.48)	i
$\omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1)$, ! 4 ($(\)E$: 3)	i
$m = (n + 1)$, ! 5 ($\Rightarrow E$: 2,4)	i
$\omega[n]$, ! 6 ($\&E$: 2)	i
$(\omega[n] \Rightarrow \langle [n, (n + 1)] \rangle)$, ! 7 ($\forall E$: P26)	i
$\omega[n] \Rightarrow \langle [n, (n + 1)] \rangle$, ! 8 ($(\)E$: 7)	i
$\langle [n, (n + 1)] \rangle$, ! 9 ($\Rightarrow E$: 6,8)	i
$\langle [n,m] \rangle$, ! 10 ($=E$: 5,9)	i
$\omega[n] \ \& \ \sigma[n,m] \Rightarrow \langle [n,m] \rangle$, ! 11 ($\Rightarrow I$: 2,10)	i
$(\omega[n] \ \& \ \sigma[n,m] \Rightarrow \langle [n,m] \rangle)$, ! 12 ($(\)I$: 11)	i
$\forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \Rightarrow \langle [n,m] \rangle)$! 13 ($\forall I$: 1,12)	i
\square		

! 32.		i
$\vdash \forall x \forall n \forall m (\leq[x,n] \ \& \ \sigma[n,m] \Rightarrow \langle [x,m] \rangle)$		i
x, n, m	, ! 1 (Prem)	i
$\leq[x,n] \ \& \ \sigma[n,m]$, ! 2 (Prem)	i
$\leq[x,n]$, ! 3 ($\&E$: 2)	i
$\sigma[n,m]$, ! 4 ($\&E$: 2)	i
$(\leq[x,n] \Rightarrow \omega[n])$, ! 5 ($\forall E$: C3.7)	i
$\leq[x,n] \Rightarrow \omega[n]$, ! 6 ($(\)E$: 5)	i
$\omega[n]$, ! 7 ($\Rightarrow E$: 3,6)	i
$\omega[n] \ \& \ \sigma[n,m]$, ! 8 ($\&I$: 4,7)	i
$(\omega[n] \ \& \ \sigma[n,m] \Rightarrow \langle [n,m] \rangle)$, ! 9 ($\forall E$: P31)	i

$\omega[n] \ \& \ \sigma[n,m] \Rightarrow \langle [n,m]$,! 10 ((E: 9) i
 $\langle [n,m]$,! 11 (\Rightarrow E: 8,10) i
 $\leq[x,n] \ \& \ \langle [n,m]$,! 12 (&I: 3,11) i
 $(\leq[x,n] \ \& \ \langle [n,m] \Rightarrow \langle [x,m])$,! 13 (\forall E: P17) i
 $\leq[x,n] \ \& \ \langle [n,m] \Rightarrow \langle [x,m]$,! 14 ((E: 13) i
 $\langle [x,m]$,! 15 (\Rightarrow E: 12,14) i
 $\leq[x,n] \ \& \ \sigma[n,m] \Rightarrow \langle [x,m]$,! 16 (\Rightarrow I: 2,15) i
 $(\leq[x,n] \ \& \ \sigma[n,m] \Rightarrow \langle [x,m])$,! 17 ((I: 16) i
 $\forall x \forall n \forall m (\leq[x,n] \ \& \ \sigma[n,m] \Rightarrow \langle [x,m])$! 18 (\forall I: 1,17) i

□

! 33. i

$\vdash \forall x \forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \ \& \ \langle [x,m] \Rightarrow \leq[x,n])$ i
 x, n, m ,! 1 (Prem) i
 $\omega[n] \ \& \ \sigma[n,m] \ \& \ \langle [x,m]$,! 2 (Prem) i
 $\omega[n] \ \& \ \sigma[n,m]$,! 3 (&E: 2) i
 $\langle [x,m]$,! 4 (&E: 2) i
 $(\omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1))$,! 5 (\forall E: C2.48) i
 $\omega[n] \ \& \ \sigma[n,m] \Rightarrow m = (n + 1)$,! 6 ((E: 5) i
 $m = (n + 1)$,! 7 (\Rightarrow E: 3,5) i
 $\langle [x, (n + 1)]$,! 8 (=E: 4,7) i
 $(\langle [x, (n + 1)] \Rightarrow \leq[x,n])$,! 9 (\forall E: P28) i
 $\langle [x, (n + 1)] \Rightarrow \leq[x,n]$,! 10 ((E: 9) i
 $\leq[x,n]$,! 11 (\Rightarrow E: 8,10) i
 $\omega[n] \ \& \ \sigma[n,m] \ \& \ \langle [x,m] \Rightarrow \leq[x,n]$,! 12 (\Rightarrow I: 2,11) i
 $(\omega[n] \ \& \ \sigma[n,m] \ \& \ \langle [x,m] \Rightarrow \leq[x,n])$,! 13 ((I: 12) i
 $\forall x \forall n \forall m (\omega[n] \ \& \ \sigma[n,m] \ \& \ \langle [x,m] \Rightarrow \leq[x,n])$! 14 (\forall I: 1,13) i

□

! 34.		i
$\vdash \forall x \forall y (\leq[(x+1), y] \Rightarrow <[x, y])$		i
x, y	, ! 1 (Prem)	i
$\leq[(x+1), y]$, ! 2 (Prem)	i
$\omega[x] \ \& \ \omega[1]$, ! 3 (TE : C1.7,2)	i
$\omega[x]$, ! 4 (&E: 3)	i
$(\omega[x] \Rightarrow <[x, (x+1)])$, ! 5 (\forall E: P26)	i
$\omega[x] \Rightarrow <[x, (x+1)]$, ! 6 (()E: 5)	i
$<[x, (x+1)]$, ! 7 (\Rightarrow E: 4,6)	i
$<[x, (x+1)] \ \& \ \leq[(x+1), y]$, ! 8 (&I: 2,7)	i
$(<[x, (x+1)] \ \& \ \leq[(x+1), y] \Rightarrow <[x, y])$, ! 9 (\forall E: P16; (x+1): C1.7,3)	i
$<[x, (x+1)] \ \& \ \leq[(x+1), y] \Rightarrow <[x, y]$, ! 10 (()E: 9)	i
$<[x, y]$, ! 11 (\Rightarrow E: 8,10)	i
$\leq[(x+1), y] \Rightarrow <[x, y]$, ! 12 (\Rightarrow I: 2,11)	i
$(\leq[(x+1), y] \Rightarrow <[x, y])$, ! 13 (()I: 12)	i
$\forall x \forall y (\leq[(x+1), y] \Rightarrow <[x, y])$! 14 (\forall I: 1,13)	i

□

! 35.		i
$\vdash \forall x \forall y (\leq[(1+x), y] \Rightarrow <[x, y])$		i
x, y	, ! 1 (Prem)	i
$\leq[(1+x), y]$, ! 2 (Prem)	i
$\omega[1] \ \& \ \omega[x]$, ! 3 (TE : C1.7,2)	i
$(\omega[1] \ \& \ \omega[x] \Rightarrow (1+x) = (x+1))$, ! 4 (\forall E: C2.5)	i
$\omega[1] \ \& \ \omega[x] \Rightarrow (1+x) = (x+1)$, ! 5 (()E: 4)	i
$(1+x) = (x+1)$, ! 6 (\Rightarrow E: 3,5)	i
$\leq[(x+1), y]$, ! 7 (=E: 2,6)	i
$(\leq[(x+1), y] \Rightarrow <[x, y])$, ! 8 (\forall E: P34)	i
$\leq[(x+1), y] \Rightarrow <[x, y]$, ! 9 (()E: 8)	i

$\langle [x, y]$, ! 10 ($\Rightarrow E$: 7,9)	i
$\leq [(1+x), y] \Rightarrow \langle [x, y]$, ! 11 ($\Rightarrow I$: 2,10)	i
$(\leq [(1+x), y] \Rightarrow \langle [x, y])$, ! 12 ($() I$: 11)	i
$\forall x \forall y (\leq [(1+x), y] \Rightarrow \langle [x, y])$! 13 ($\forall I$: 1,12)	i

□

! 36.

$\vdash \forall x \forall y (\langle [x, y] \Rightarrow \leq [(x+1), y])$		i
x, y	, ! 1 (Prem)	i
$\langle [x, y]$, ! 2 (Prem)	i
$(\langle [x, y] \Rightarrow \leq [x, y] \ \& \ \neg x = y)$, ! 3 ($\forall E$: P3)	i
$\langle [x, y] \Rightarrow \leq [x, y] \ \& \ \neg x = y$, ! 4 ($() E$: 3)	i
$\leq [x, y] \ \& \ \neg x = y$, ! 5 ($\Rightarrow E$: 2,4)	i
$(\leq [x, y] \ \& \ \neg x = y \Rightarrow \leq [(x+1), y])$, ! 6 ($\forall E$: C3.52)	i
$\leq [x, y] \ \& \ \neg x = y \Rightarrow \leq [(x+1), y]$, ! 7 ($() E$: 6)	i
$\leq [(x+1), y]$, ! 8 ($\Rightarrow E$: 5,7)	i
$\langle [x, y] \Rightarrow \leq [(x+1), y]$, ! 9 ($\Rightarrow I$: 2,8)	i
$(\langle [x, y] \Rightarrow \leq [(x+1), y])$, ! 10 ($() I$: 9)	i
$\forall x \forall y (\langle [x, y] \Rightarrow \leq [(x+1), y])$! 11 ($\forall I$: 1,10)	i

□

! 37.

$\vdash \forall x \forall y (\langle [x, y] \Rightarrow \leq [(1+x), y])$		i
x, y	, ! 1 (Prem)	i
$\langle [x, y]$, ! 2 (Prem)	i
$(\langle [x, y] \Rightarrow \leq [(x+1), y])$, ! 3 ($\forall E$: P36)	i
$\langle [x, y] \Rightarrow \leq [(x+1), y]$, ! 4 ($() E$: 3)	i
$\leq [(x+1), y]$, ! 5 ($\Rightarrow E$: 2,4)	i
$\omega[x] \ \& \ \omega[1]$, ! 6 ($\mathbb{T}E$: C1.7,5)	i
$(\omega[x] \ \& \ \omega[1] \Rightarrow (x+1) = (1+x))$, ! 7 ($\forall E$: C2.5)	i

$\omega[\mathbf{x}] \ \& \ \omega[1] \Rightarrow (\mathbf{x}+1) = (1+\mathbf{x})$, ! 8 ((E: 7)	i
$(\mathbf{x}+1) = (1+\mathbf{x})$, ! 9 (\Rightarrow E: 6,8)	i
$\leq[(1+\mathbf{x}), \mathbf{y}]$, ! 10 (=E: 5,9)	i
$\langle[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[(1+\mathbf{x}), \mathbf{y}]$, ! 11 (\Rightarrow I: 2,10)	i
$(\langle[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[(1+\mathbf{x}), \mathbf{y}])$, ! 12 ((I: 11)	i
$\forall \mathbf{x} \forall \mathbf{y} (\langle[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[(1+\mathbf{x}), \mathbf{y}])$! 13 (\forall I: 1,12)	i

□

! 38.

$\vdash \forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[1, k] \Rightarrow \langle[n, (m+k)])$		i
n, m, k	, ! 1 (Prem)	i
$\leq[n, m] \ \& \ \leq[1, k]$, ! 2 (Prem)	i
$(\leq[n, m] \ \& \ \leq[1, k] \Rightarrow \leq[(n+1), (m+k)])$, ! 3 (\forall E: C3.30)	i
$\leq[n, m] \ \& \ \leq[1, k] \Rightarrow \leq[(n+1), (m+k)]$, ! 4 ((E: 3)	i
$\leq[(n+1), (m+k)]$, ! 5 (\Rightarrow E: 2,4)	i
$\omega[m] \ \& \ \omega[k]$, ! 6 (\mathbb{T} E: C1.7,5)	i
$(\leq[(n+1), (m+k)] \Rightarrow \langle[n, (m+k)])$, ! 7 (\forall E: P34; (m+k): C1.7,6)	i
$\leq[(n+1), (m+k)] \Rightarrow \langle[n, (m+k)]$, ! 8 ((E: 7)	i
$\langle[n, (m+k)]$, ! 9 (\Rightarrow E: 5,8)	i
$\leq[n, m] \ \& \ \leq[1, k] \Rightarrow \langle[n, (m+k)]$, ! 10 (\Rightarrow I: 2,9)	i
$(\leq[n, m] \ \& \ \leq[1, k] \Rightarrow \langle[n, (m+k)])$, ! 11 ((I: 10)	i
$\forall n \forall m \forall k (\leq[n, m] \ \& \ \leq[1, k] \Rightarrow \langle[n, (m+k)])$! 12 (\forall I: 1,11)	i

□

! 39.

$\vdash \forall x \forall y \forall z (\leq[(x+y), z] \ \& \ \neg y = 0 \Rightarrow \langle[x, z])$		i
x, y, z	, ! 1 (Prem)	i
$\leq[(x+y), z] \ \& \ \neg y = 0$, ! 2 (Prem)	i
$\leq[(x+y), z]$, ! 3 ($\&$ E: 2)	i
$\neg y = 0$, ! 4 ($\&$ E: 2)	i

$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 5 ($\mathbb{T}\mathbb{E}$: C1.7,3)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0$,! 6 ($\&\mathbb{I}$: 4,5)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \)$,! 7 ($\forall\mathbb{E}$: P24)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \)$,! 8 ($(\)\mathbb{E}$: 7)	i
$\langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \)$,! 9 ($\Rightarrow\mathbb{E}$: 6,8)	i
$\langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \ \& \ \leq [(\mathbf{x}+\mathbf{y}), \mathbf{z}]$,! 10 ($\&\mathbb{I}$: 3,9)	i
$(\ \langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \ \& \ \leq [(\mathbf{x}+\mathbf{y}), \mathbf{z}] \Rightarrow \langle [\mathbf{x}, \mathbf{z}] \)$,! 11 ($\forall\mathbb{E}$: P16; $(\mathbf{x}+\mathbf{y})$: C1.7,5)	i
$\langle [\mathbf{x}, (\mathbf{x}+\mathbf{y})] \ \& \ \leq [(\mathbf{x}+\mathbf{y}), \mathbf{z}] \Rightarrow \langle [\mathbf{x}, \mathbf{z}] \)$,! 12 ($(\)\mathbb{E}$: 11)	i
$\langle [\mathbf{x}, \mathbf{z}] \)$,! 13 ($\Rightarrow\mathbb{E}$: 10,12)	i
$\leq [(\mathbf{x}+\mathbf{y}), \mathbf{z}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, \mathbf{z}] \)$,! 14 ($\Rightarrow\mathbb{I}$: 2,13)	i
$(\ \leq [(\mathbf{x}+\mathbf{y}), \mathbf{z}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, \mathbf{z}] \)$,! 15 ($(\)\mathbb{I}$: 14)	i
$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} (\ \leq [(\mathbf{x}+\mathbf{y}), \mathbf{z}] \ \& \ \neg \mathbf{y} = 0 \Rightarrow \langle [\mathbf{x}, \mathbf{z}] \)$! 16 ($\forall\mathbb{I}$: 1,15)	i

□

! P40 and P41 express that addition maintains strict inequality.

i

! 40.

i

$\vdash \forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} (\ \langle [\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{z}] \Rightarrow \langle [(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z})] \)$		i
$\mathbf{x}, \mathbf{y}, \mathbf{z}$,! 1 (Prem)	i
$\langle [\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{z}]$,! 2 (Prem)	i
$\langle [\mathbf{x}, \mathbf{y}]$,! 3 ($\&\mathbb{E}$: 2)	i
$\omega[\mathbf{z}]$,! 4 ($\&\mathbb{E}$: 2)	i
$(\ \langle [\mathbf{x}, \mathbf{y}] \Rightarrow \leq [\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y} \)$,! 5 ($\forall\mathbb{E}$: P3)	i
$\langle [\mathbf{x}, \mathbf{y}] \Rightarrow \leq [\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$,! 6 ($(\)\mathbb{E}$: 5)	i
$\leq [\mathbf{x}, \mathbf{y}] \ \& \ \neg \mathbf{x} = \mathbf{y}$,! 7 ($\Rightarrow\mathbb{E}$: 3,6)	i
$\leq [\mathbf{x}, \mathbf{y}]$,! 8 ($\&\mathbb{E}$: 7)	i
$\neg \mathbf{x} = \mathbf{y}$,! 9 ($\&\mathbb{E}$: 7)	i
$\leq [\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{z}]$,! 10 ($\&\mathbb{I}$: 4,8)	i
$(\ \leq [\mathbf{x}, \mathbf{y}] \ \& \ \omega[\mathbf{z}] \Rightarrow \leq [(\mathbf{x} + \mathbf{z}), (\mathbf{y} + \mathbf{z})] \)$		

	,! 11 ($\forall E$: C3.28)	i
$\leq[x, y] \ \& \ \omega[z] \Rightarrow \leq[(x + z), (y + z)]$,! 12 ($(\)E$: 11)	i
$\leq[(x + z), (y + z)]$,! 13 ($\Rightarrow E$: 10,12)	i
$(x + z) = (y + z)$,! 14 (Prem)	i
$((x + z) = (y + z) \Rightarrow x = y)$,! 15 ($\forall E$: C2.42)	i
$(x + z) = (y + z) \Rightarrow x = y$,! 16 ($(\)E$: 15)	i
$x = y$,! 17 ($\Rightarrow E$: 14,16)	i
\mathfrak{F}	,! 18 ($\mathfrak{F}I$: 9,17)	i
$(x + z) = (y + z) \Rightarrow \mathfrak{F}$,! 19 ($\Rightarrow I$: 14,18)	i
$\neg (x + z) = (y + z)$,! 20 ($\neg I$: 19)	i
$\leq[(x + z), (y + z)] \ \& \ \neg (x + z) = (y + z)$,! 21 ($\&I$: 13,20)	i
$\omega[x] \ \& \ \omega[z]$,! 22 ($\mathbb{T}E$: C1.7,21)	i
$\omega[y] \ \& \ \omega[z]$,! 23 ($\mathbb{T}E$: C1.7,21)	i
$(\leq[(x + z), (y + z)] \ \& \ \neg (x + z) = (y + z) \Rightarrow \langle [(x + z), (y + z)] \rangle)$,! 24 ($\forall E$: P4; ($x + z$): C1.7,22; ($y + z$): C1.7,23)	i
$\leq[(x + z), (y + z)] \ \& \ \neg (x + z) = (y + z) \Rightarrow \langle [(x + z), (y + z)] \rangle$,! 25 ($(\)E$: 24)	i
$\langle [(x + z), (y + z)] \rangle$,! 26 ($\Rightarrow E$: 21,25)	i
$\langle [x, y] \ \& \ \omega[z] \Rightarrow \langle [(x + z), (y + z)] \rangle$,! 27 ($\Rightarrow I$: 2,26)	i
$(\langle [x, y] \ \& \ \omega[z] \Rightarrow \langle [(x + z), (y + z)] \rangle)$,! 28 ($(\)I$: 27)	i
$\forall x \forall y \forall z (\langle [x, y] \ \& \ \omega[z] \Rightarrow \langle [(x+z), (y+z)] \rangle)$! 29 ($\forall I$: 1,28)	i
\square		
! 41.		i
$\vdash \forall x \forall y \forall z (\langle [x, y] \ \& \ \omega[z] \Rightarrow \langle [(z+x), (z+y)] \rangle)$		i
x, y, z	,! 1 (Prem)	i
$\langle [x, y] \ \& \ \omega[z]$,! 2 (Prem)	i

$\langle \mathbf{x}, \mathbf{y} \rangle$,!	3	(&E: 2)	i
$\omega[\mathbf{z}]$,!	4	(&E: 2)	i
$(\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}] \Rightarrow \langle (\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z}) \rangle)$,!	5	(\forall E: P40)	i
$\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}] \Rightarrow \langle (\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z}) \rangle$,!	6	($()$ E: 5)	i
$\langle (\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z}) \rangle$,!	7	(\Rightarrow E: 2,6)	i
$(\langle \mathbf{x}, \mathbf{y} \rangle \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$,!	8	(\forall E: P9)	i
$\langle \mathbf{x}, \mathbf{y} \rangle \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,!	9	($()$ E: 8)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,!	10	(\Rightarrow E: 3,9)	i
$\omega[\mathbf{x}]$,!	11	(&E: 10)	i
$\omega[\mathbf{y}]$,!	12	(&E: 10)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{z}]$,!	13	(&I: 4,11)	i
$(\omega[\mathbf{x}] \ \& \ \omega[\mathbf{z}] \Rightarrow (\mathbf{x}+\mathbf{z}) = (\mathbf{z}+\mathbf{x}))$,!	14	(\forall E: C2.5)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{z}] \Rightarrow (\mathbf{x}+\mathbf{z}) = (\mathbf{z}+\mathbf{x})$,!	15	($()$ E: 14)	i
$(\mathbf{x}+\mathbf{z}) = (\mathbf{z}+\mathbf{x})$,!	16	(\Rightarrow E: 13,15)	i
$\langle (\mathbf{z}+\mathbf{x}), (\mathbf{y}+\mathbf{z}) \rangle$,!	17	(=E: 7,16)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{z}]$,!	18	(&I: 4,12)	i
$(\omega[\mathbf{y}] \ \& \ \omega[\mathbf{z}] \Rightarrow (\mathbf{y}+\mathbf{z}) = (\mathbf{z}+\mathbf{y}))$,!	19	(\forall E: C2.5)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{z}] \Rightarrow (\mathbf{y}+\mathbf{z}) = (\mathbf{z}+\mathbf{y})$,!	20	($()$ E: 19)	i
$(\mathbf{y}+\mathbf{z}) = (\mathbf{z}+\mathbf{y})$,!	21	(\Rightarrow E: 18,20)	i
$\langle (\mathbf{z}+\mathbf{x}), (\mathbf{z}+\mathbf{y}) \rangle$,!	22	(=E: 17,21)	i
$\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}] \Rightarrow \langle (\mathbf{z}+\mathbf{x}), (\mathbf{z}+\mathbf{y}) \rangle$,!	23	(\Rightarrow I: 2,22)	i
$(\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}] \Rightarrow \langle (\mathbf{z}+\mathbf{x}), (\mathbf{z}+\mathbf{y}) \rangle)$,!	24	($()$ I: 23)	i
$\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} (\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}] \Rightarrow \langle (\mathbf{z}+\mathbf{x}), (\mathbf{z}+\mathbf{y}) \rangle)$!	25	(\forall I: 1,24)	i

□

! 42. i

⊢ $\forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} \forall \mathbf{a} (\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \leq[\mathbf{z}, \mathbf{a}] \Rightarrow \langle (\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{a}) \rangle)$ i

$\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}$,! 1 (Prem) i

$\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \leq[\mathbf{z}, \mathbf{a}]$,! 2 (Prem)	i
$\langle \mathbf{x}, \mathbf{y} \rangle$,! 3 (&E: 2)	i
$\leq[\mathbf{z}, \mathbf{a}]$,! 4 (&E: 2)	i
$(\leq[\mathbf{z}, \mathbf{a}] \Rightarrow \omega[\mathbf{z}] \ \& \ \omega[\mathbf{a}])$,! 5 (\forall E: C3.5)	i
$\leq[\mathbf{z}, \mathbf{a}] \Rightarrow \omega[\mathbf{z}] \ \& \ \omega[\mathbf{a}]$,! 6 (()E: 5)	i
$\omega[\mathbf{z}] \ \& \ \omega[\mathbf{a}]$,! 7 (\Rightarrow E: 4,6)	i
$\omega[\mathbf{z}]$,! 8 (&E: 7)	i
$\omega[\mathbf{a}]$,! 9 (&E: 7)	i
$\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}]$,! 10 (&I: 3,8)	i
$(\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}] \Rightarrow \langle [(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z})]])$,! 11 (\forall E: P40)	i
$\langle \mathbf{x}, \mathbf{y} \rangle \ \& \ \omega[\mathbf{z}] \Rightarrow \langle [(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z})]]$,! 12 (()E: 11)	i
$\langle [(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z})]]$,! 13 (\Rightarrow E: 10,12)	i
$(\langle \mathbf{x}, \mathbf{y} \rangle \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}])$,! 14 (\forall E: P9)	i
$\langle \mathbf{x}, \mathbf{y} \rangle \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 15 (()E: 14)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 16 (\Rightarrow E: 3,15)	i
$\omega[\mathbf{x}]$,! 17 (&E: 16)	i
$\omega[\mathbf{y}]$,! 18 (&E: 16)	i
$\leq[\mathbf{z}, \mathbf{a}] \ \& \ \omega[\mathbf{y}]$,! 19 (&I: 4,18)	i
$(\leq[\mathbf{z}, \mathbf{a}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[(\mathbf{y}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]])$,! 20 (\forall E: C3.29)	i
$\leq[\mathbf{z}, \mathbf{a}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[(\mathbf{y}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]]$,! 21 (()E: 20)	i
$\leq[(\mathbf{y}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]]$,! 22 (\Rightarrow E: 19,21)	i
$\langle [(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z})]] \ \& \ \leq[(\mathbf{y}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]]$,! 23 (&I: 13,22)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{z}]$,! 24 (&I: 8,17)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{z}]$,! 25 (&I: 8,18)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{a}]$,! 26 (&I: 9,18)	i
$(\langle [(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z})]] \ \& \ \leq[(\mathbf{y}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]] \Rightarrow \langle [(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]])$,! 27 (\forall E: P16; ($\mathbf{x}+\mathbf{z}$): C1.7,24; ($\mathbf{y}+\mathbf{z}$): C1.7,25; ($\mathbf{y}+\mathbf{a}$): C1.7,26)	i

$$\begin{aligned} & <[(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{z})] \ \& \ \leq[(\mathbf{y}+\mathbf{z}), (\mathbf{y}+\mathbf{a})] \Rightarrow <[(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{a})] & ,! \ 28 \ (\ ()E: \ 27) & \ i \\ & <[(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{a})] & ,! \ 29 \ (\Rightarrow E: \ 23,28) & \ i \\ & <[\mathbf{x}, \mathbf{y}] \ \& \ \leq[\mathbf{z}, \mathbf{a}] \Rightarrow <[(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{a})] & ,! \ 30 \ (\Rightarrow I: \ 2,29) & \ i \\ & (<[\mathbf{x}, \mathbf{y}] \ \& \ \leq[\mathbf{z}, \mathbf{a}] \Rightarrow <[(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]) & ,! \ 31 \ (\ ()I: \ 30) & \ i \\ \forall \mathbf{x} \forall \mathbf{y} \forall \mathbf{z} \forall \mathbf{a} \ (<[\mathbf{x}, \mathbf{y}] \ \& \ \leq[\mathbf{z}, \mathbf{a}] \Rightarrow <[(\mathbf{x}+\mathbf{z}), (\mathbf{y}+\mathbf{a})]) & & ! \ 32 \ (\forall I: \ 1,32) & \ i \end{aligned}$$

□

! 43. i

$$\begin{aligned} \vdash \forall \mathbf{x} \forall \mathbf{y} \ (\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}]) & \ i \\ \mathbf{x}, \mathbf{y} & ,! \ 1 \ (\text{Prem}) & \ i \\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] & ,! \ 2 \ (\text{Prem}) & \ i \\ (\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ \leq[(\mathbf{y}+1), \mathbf{x}]) & ,! \ 3 \ (\forall E: \ C3.65) & \ i \\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ \leq[(\mathbf{y}+1), \mathbf{x}] & ,! \ 4 \ (\ ()E: \ 3) & \ i \\ \leq[\mathbf{x}, \mathbf{y}] \ \vee \ \leq[(\mathbf{y}+1), \mathbf{x}] & ,! \ 5 \ (\Rightarrow E: \ 2,4) & \ i \\ \leq[\mathbf{x}, \mathbf{y}] & ,! \ 6 \ (\text{Prem}) & \ i \\ \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}] & ,! \ 7 \ (\forall I: \ 6) & \ i \\ \leq[\mathbf{x}, \mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}] & ,! \ 8 \ (\Rightarrow I: \ 6,7) & \ i \\ \leq[(\mathbf{y}+1), \mathbf{x}] & ,! \ 9 \ (\text{Prem}) & \ i \\ (\leq[(\mathbf{y}+1), \mathbf{x}] \Rightarrow <[\mathbf{y}, \mathbf{x}]) & ,! \ 10 \ (\forall E: \ P34) & \ i \\ \leq[(\mathbf{y}+1), \mathbf{x}] \Rightarrow <[\mathbf{y}, \mathbf{x}] & ,! \ 11 \ (\ ()E: \ 10) & \ i \\ <[\mathbf{y}, \mathbf{x}] & ,! \ 12 \ (\Rightarrow E: \ 9,11) & \ i \\ \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}] & ,! \ 13 \ (\forall I: \ 12) & \ i \\ \leq[(\mathbf{y}+1), \mathbf{x}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}] & ,! \ 14 \ (\Rightarrow I: \ 9,13) & \ i \\ \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}] & ,! \ 15 \ (\forall E: \ 5,8,14) & \ i \\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}] & ,! \ 16 \ (\Rightarrow I: \ 2,15) & \ i \\ (\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}]) & ,! \ 17 \ (\ ()I: \ 16) & \ i \\ \forall \mathbf{x} \forall \mathbf{y} \ (\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \Rightarrow \leq[\mathbf{x}, \mathbf{y}] \ \vee \ <[\mathbf{y}, \mathbf{x}]) & ! \ 18 \ (\forall I: \ 1,17) & \ i \end{aligned}$$

□

! 44. Trichotomy. i

$\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \Rightarrow <[x,y] \vee x = y \vee <[y,x])$		i
x, y	,! 1 (Prem)	i
$\omega[x] \ \& \ \omega[y]$,! 2 (Prem)	i
$(\omega[x] \ \& \ \omega[y] \Rightarrow \leq[x,y] \vee <[y,x])$,! 3 ($\forall E$: P43)	i
$\omega[x] \ \& \ \omega[y] \Rightarrow \leq[x,y] \vee <[y,x]$,! 4 ($(\)E$: 3)	i
$\leq[x,y] \vee <[y,x]$,! 5 ($\Rightarrow E$: 2,4)	i
$\leq[x,y]$,! 6 (Prem)	i
$(\leq[x,y] \Rightarrow <[x,y] \vee x = y)$,! 7 ($\forall E$: P8)	i
$\leq[x,y] \Rightarrow <[x,y] \vee x = y$,! 8 ($(\)E$: 7)	i
$<[x,y] \vee x = y$,! 9 ($\Rightarrow E$: 6,8)	i
$<[x,y] \vee x = y \vee <[y,x]$,! 10 ($\vee I$: 9)	i
$\leq[x,y] \Rightarrow <[x,y] \vee x = y \vee <[y,x]$,! 11 ($\Rightarrow I$: 6,10)	i
$<[y,x]$,! 12 (Prem)	i
$<[x,y] \vee x = y \vee <[y,x]$,! 13 ($\vee I$: 12)	i
$<[y,x] \Rightarrow <[x,y] \vee x = y \vee <[y,x]$,! 14 ($\Rightarrow I$: 12,13)	i
$<[x,y] \vee x = y \vee <[y,x]$,! 15 ($\vee E$: 5,11,14)	i
$\omega[x] \ \& \ \omega[y] \Rightarrow <[x,y] \vee x = y \vee <[y,x]$,! 16 ($\Rightarrow I$: 2,15)	i
$(\omega[x] \ \& \ \omega[y] \Rightarrow <[x,y] \vee x = y \vee <[y,x])$,! 17 ($(\)I$: 16)	i
$\forall x \forall y (\omega[x] \ \& \ \omega[y] \Rightarrow <[x,y] \vee x = y \vee <[y,x])$! 18 ($\forall I$: 1,17)	i

□

! P45 through P48 are lemmas to P43 or P44. i

! 45. i

$\vdash \forall x \forall y (\omega[x] \ \& \ \omega[y] \ \& \ \neg <[x,y] \Rightarrow \leq[y,x])$		i
x, y	,! 1 (Prem)	i
$\omega[x] \ \& \ \omega[y] \ \& \ \neg <[x,y]$,! 2 (Prem)	i

$\omega[\mathbf{x}]$,! 3 (&E: 2)	i
$\omega[\mathbf{y}]$,! 4 (&E: 2)	i
$\neg <[\mathbf{x}, \mathbf{y}]$,! 5 (&E: 2)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}]$,! 6 (&I: 3,4)	i
$(\ \omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}] \ \vee \ <[\mathbf{x}, \mathbf{y}] \)$,! 7 (\forall E: P43)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}] \ \vee \ <[\mathbf{x}, \mathbf{y}]$,! 8 (()E: 7)	i
$\leq[\mathbf{y}, \mathbf{x}] \ \vee \ <[\mathbf{x}, \mathbf{y}]$,! 9 (\Rightarrow E: 6,8)	i
$\leq[\mathbf{y}, \mathbf{x}]$,! 10 (Prem)	i
$\leq[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}]$,! 11 (\Rightarrow I: 10,10)	i
$<[\mathbf{x}, \mathbf{y}]$,! 12 (Prem)	i
$\neg \leq[\mathbf{y}, \mathbf{x}]$,! 13 (Prem)	i
\mathfrak{F}	,! 14 (\mathfrak{F} I: 5,12)	i
$\neg \leq[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \mathfrak{F}$,! 15 (\Rightarrow I: 13,14)	i
$\neg \neg \leq[\mathbf{y}, \mathbf{x}]$,! 16 (\neg I: 15)	i
$\leq[\mathbf{y}, \mathbf{x}]$,! 17 (\neg E: 16)	i
$<[\mathbf{x}, \mathbf{y}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}]$,! 18 (\Rightarrow I: 12,17)	i
$\leq[\mathbf{y}, \mathbf{x}]$,! 19 (\vee E: 9,11,18)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg <[\mathbf{x}, \mathbf{y}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}]$,! 20 (\Rightarrow I: 2,19)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg <[\mathbf{x}, \mathbf{y}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}] \)$,! 21 (()I: 20)	i
$\forall \mathbf{x} \forall \mathbf{y} (\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg <[\mathbf{x}, \mathbf{y}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}] \)$! 22 (\forall I: 1,21)	i
\square		

! 46.

$\vdash \forall \mathbf{x} \forall \mathbf{y} (\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \leq[\mathbf{x}, \mathbf{y}] \ \Rightarrow \ <[\mathbf{y}, \mathbf{x}] \)$		i
\mathbf{x}, \mathbf{y}	,! 1 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \leq[\mathbf{x}, \mathbf{y}]$,! 2 (Prem)	i
$\omega[\mathbf{x}]$,! 3 (&E: 2)	i
$\omega[\mathbf{y}]$,! 4 (&E: 2)	i
$\neg \leq[\mathbf{x}, \mathbf{y}]$,! 5 (&E: 2)	i
$\neg <[\mathbf{y}, \mathbf{x}]$,! 6 (Prem)	i

$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}]$,! 7 (&I: 3,4)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \& \ \neg \ <[\mathbf{y},\mathbf{x}]$,! 8 (&I: 6,7)	i
$(\ \omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \& \ \neg \ <[\mathbf{y},\mathbf{x}] \ \Rightarrow \ \leq[\mathbf{x},\mathbf{y}] \)$,! 9 (\forall E: P45)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \& \ \neg \ <[\mathbf{y},\mathbf{x}] \ \Rightarrow \ \leq[\mathbf{x},\mathbf{y}]$,! 10 (()E: 9)	i
$\leq[\mathbf{x},\mathbf{y}]$,! 11 (\Rightarrow E: 8,10)	i
\mathfrak{F}	,! 12 (\mathfrak{F} I: 5,11)	i
$\neg \ <[\mathbf{y},\mathbf{x}] \ \Rightarrow \ \mathfrak{F}$,! 13 (\Rightarrow I: 6,12)	i
$\neg \neg \ <[\mathbf{y},\mathbf{x}]$,! 14 (\neg I: 13)	i
$<[\mathbf{y},\mathbf{x}]$,! 15 (\neg E: 14)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ \leq[\mathbf{x},\mathbf{y}] \ \Rightarrow \ <[\mathbf{y},\mathbf{x}]$,! 16 (\Rightarrow I: 2,15)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ \leq[\mathbf{x},\mathbf{y}] \ \Rightarrow \ <[\mathbf{y},\mathbf{x}] \)$,! 17 (()I: 16)	i
$\forall \mathbf{x} \forall \mathbf{y} \ (\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ \leq[\mathbf{x},\mathbf{y}] \ \Rightarrow \ <[\mathbf{y},\mathbf{x}] \)$! 18 (\forall I: 1,17)	i
\square		
! 47.		i
$\vdash \ \forall \mathbf{x} \forall \mathbf{y} \ (\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ \mathbf{x} = \mathbf{y} \ \Rightarrow \ <[\mathbf{x},\mathbf{y}] \ \vee \ <[\mathbf{y},\mathbf{x}] \)$		i
\mathbf{x}, \mathbf{y}	,! 1 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ \mathbf{x} = \mathbf{y}$,! 2 (Prem)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}]$,! 3 (&E: 2)	i
$\neg \ \mathbf{x} = \mathbf{y}$,! 4 (&E: 2)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \Rightarrow \ <[\mathbf{x},\mathbf{y}] \ \vee \ \mathbf{x} = \mathbf{y} \ \vee \ <[\mathbf{y},\mathbf{x}] \)$,! 5 (\forall E: P44)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \Rightarrow \ <[\mathbf{x},\mathbf{y}] \ \vee \ \mathbf{x} = \mathbf{y} \ \vee \ <[\mathbf{y},\mathbf{x}]$,! 6 (()E: 5)	i
$<[\mathbf{x},\mathbf{y}] \ \vee \ \mathbf{x} = \mathbf{y} \ \vee \ <[\mathbf{y},\mathbf{x}]$,! 7 (\Rightarrow E: 4,6)	i
$<[\mathbf{x},\mathbf{y}]$,! 8 (Prem)	i
$<[\mathbf{x},\mathbf{y}] \ \vee \ <[\mathbf{y},\mathbf{x}]$,! 9 (\vee I: 8)	i
$<[\mathbf{x},\mathbf{y}] \ \Rightarrow \ <[\mathbf{x},\mathbf{y}] \ \vee \ <[\mathbf{y},\mathbf{x}]$,! 10 (\Rightarrow I: 8,9)	i
$\mathbf{x} = \mathbf{y} \ \vee \ <[\mathbf{y},\mathbf{x}]$,! 11 (Prem)	i

$x = y$,! 12 (Prem)	i
$\neg <[y, x]$,! 13 (Prem)	i
\mathfrak{F}	,! 14 (\mathfrak{F} I: 4,12)	i
$\neg <[y, x] \Rightarrow \mathfrak{F}$,! 15 (\Rightarrow I: 13,14)	i
$\neg\neg <[y, x]$,! 16 (\neg I: 15)	i
$<[y, x]$,! 17 (\neg E: 16)	i
$<[x, y] \vee <[y, x]$,! 18 (\vee I: 17)	i
$x = y \Rightarrow <[x, y] \vee <[y, x]$,! 19 (\Rightarrow I: 12,18)	i
$<[y, x]$,! 20 (Prem)	i
$<[x, y] \vee <[y, x]$,! 21 (\vee I: 20)	i
$<[y, x] \Rightarrow <[x, y] \vee <[y, x]$,! 22 (\Rightarrow I: 20,21)	i
$<[x, y] \vee <[y, x]$,! 23 (\vee E: 11,19,22)	i
$x = y \vee <[y, x] \Rightarrow <[x, y] \vee <[y, x]$,! 24 (\Rightarrow I: 11,23)	i
$<[x, y] \vee <[y, x]$,! 25 (\vee E: 7,10,24)	i
$\omega[x] \ \& \ \omega[y] \ \& \ \neg x = y \Rightarrow <[x, y] \vee <[y, x]$,! 26 (\Rightarrow I: 2,25)	i
$(\ \omega[x] \ \& \ \omega[y] \ \& \ \neg x = y \Rightarrow <[x, y] \vee <[y, x] \)$,! 27 ($(\)$ I; 26)	i
$\forall x \forall y (\ \omega[x] \ \& \ \omega[y] \ \& \ \neg x = y \Rightarrow <[x, y] \vee <[y, x] \)$! 28 (\forall I: 1,27)	i

□

! 48. i

⊢ $\forall x \forall y (\ \omega[x] \ \& \ \omega[y] \ \& \ \neg <[x, y] \ \& \ \neg <[y, x] \Rightarrow x = y \)$ i

x, y ,! 1 (Prem) i

$\omega[x] \ \& \ \omega[y] \ \& \ \neg <[x, y] \ \& \ \neg <[y, x]$,! 2 (Prem) i

$\omega[x]$,! 3 ($\&$ E: 2) i

$\omega[y]$,! 4 ($\&$ E: 2) i

$\neg <[x, y]$,! 5 ($\&$ E: 2) i

$\neg <[y, x]$,! 6 ($\&$ E: 2) i

$\omega[x] \ \& \ \omega[y]$,! 7 ($\&$ I: 3,4) i

$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ <[\mathbf{x}, \mathbf{y}]$, ! 8 (&I: 5,7)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ <[\mathbf{x}, \mathbf{y}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}] \)$, ! 9 (\forall E: P45)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ <[\mathbf{x}, \mathbf{y}] \ \Rightarrow \ \leq[\mathbf{y}, \mathbf{x}]$, ! 10 ((E: 9)	i
$\leq[\mathbf{y}, \mathbf{x}]$, ! 11 (\Rightarrow E: 8,10)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}]$, ! 12 (&I: 3,4)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \& \ \neg \ <[\mathbf{y}, \mathbf{x}]$, ! 13 (&I: 6,12)	i
$(\ \omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \& \ \neg \ <[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \leq[\mathbf{x}, \mathbf{y}] \)$, ! 14 (\forall E: P45)	i
$\omega[\mathbf{y}] \ \& \ \omega[\mathbf{x}] \ \& \ \neg \ <[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \leq[\mathbf{x}, \mathbf{y}]$, ! 15 ((E: 14)	i
$\leq[\mathbf{x}, \mathbf{y}]$, ! 16 (\Rightarrow E: 13,15)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ \leq[\mathbf{y}, \mathbf{x}]$, ! 17 (&I: 11,16)	i
$(\ \leq[\mathbf{x}, \mathbf{y}] \ \& \ \leq[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \mathbf{x} = \mathbf{y} \)$, ! 18 (\forall E: C3.22)	i
$\leq[\mathbf{x}, \mathbf{y}] \ \& \ \leq[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \mathbf{x} = \mathbf{y}$, ! 19 ((E: 18)	i
$\mathbf{x} = \mathbf{y}$, ! 20 (\Rightarrow E: 17,19)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ <[\mathbf{x}, \mathbf{y}] \ \& \ \neg \ <[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \mathbf{x} = \mathbf{y}$, ! 21 (\Rightarrow I: 2,20)	i
$(\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ <[\mathbf{x}, \mathbf{y}] \ \& \ \neg \ <[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \mathbf{x} = \mathbf{y} \)$, ! 22 ((I: 21)	i
$\forall \mathbf{x} \forall \mathbf{y} (\ \omega[\mathbf{x}] \ \& \ \omega[\mathbf{y}] \ \& \ \neg \ <[\mathbf{x}, \mathbf{y}] \ \& \ \neg \ <[\mathbf{y}, \mathbf{x}] \ \Rightarrow \ \mathbf{x} = \mathbf{y} \)$! 23 (\forall I: 1,22)	i

□

! 49. Well-Ordering Principle, Strict Inequality Version.

$\vdash \forall P (\ \exists x (\ \omega[x] \ \& \ P[x])$		
$\Rightarrow \exists x (\ \omega[x] \ \& \ P[x] \ \& \ \forall y (\ <[y, x] \ \Rightarrow \ \neg \ P[y]))$		i
P	, ! 1 (Prem)	i
$\exists x (\ \omega[x] \ \& \ P[x])$, ! 2 (Prem)	i
$(\ \exists x (\ \omega[x] \ \& \ P[x])$		
$\Rightarrow \exists x (\ \omega[x] \ \& \ P[x] \ \& \ \forall y (\ \omega[y] \ \& \ P[y] \ \Rightarrow \ \leq[x, y]))$, ! 3 (\forall E: C3.73)	i
$\exists x (\ \omega[x] \ \& \ P[x])$		
$\Rightarrow \exists x (\ \omega[x] \ \& \ P[x] \ \& \ \forall y (\ \omega[y] \ \& \ P[y] \ \Rightarrow \ \leq[x, y]))$		

	,! 4 ((E: 3)	i
$\exists x(\omega[x] \ \& \ P[x] \ \& \ \forall y(\omega[y] \ \& \ P[y] \Rightarrow \leq[x,y]))$,! 5 (\Rightarrow E: 2,4)	i
$(\omega[\mathbf{x}] \ \& \ P[\mathbf{x}] \ \& \ \forall y(\omega[y] \ \& \ P[y] \Rightarrow \leq[\mathbf{x},y]))$,! 6 (\exists E: 5)	i
$\omega[\mathbf{x}] \ \& \ P[\mathbf{x}] \ \& \ \forall y(\omega[y] \ \& \ P[y] \Rightarrow \leq[\mathbf{x},y])$,! 7 ((E: 6)	i
$\omega[\mathbf{x}] \ \& \ P[\mathbf{x}]$,! 8 ($\&$ E: 7)	i
$\forall y(\omega[y] \ \& \ P[y] \Rightarrow \leq[\mathbf{x},y])$,! 9 ($\&$ E: 7)	i
\mathbf{y}	,! 10 (Prem)	i
$\langle[\mathbf{y},\mathbf{x}]$,! 11 (Prem)	i
$(\langle[\mathbf{y},\mathbf{x}] \Rightarrow \omega[\mathbf{y}])$,! 12 (\forall E: P10)	i
$\langle[\mathbf{y},\mathbf{x}] \Rightarrow \omega[\mathbf{y}]$,! 13 ((E: 12)	i
$\omega[\mathbf{y}]$,! 14 (\Rightarrow E: 11,13)	i
$P[\mathbf{y}]$,! 15 (Prem)	i
$\omega[\mathbf{y}] \ \& \ P[\mathbf{y}]$,! 16 ($\&$ I: 14,15)	i
$(\omega[\mathbf{y}] \ \& \ P[\mathbf{y}] \Rightarrow \leq[\mathbf{x},\mathbf{y}])$,! 17 (\forall E: 9)	i
$\omega[\mathbf{y}] \ \& \ P[\mathbf{y}] \Rightarrow \leq[\mathbf{x},\mathbf{y}]$,! 18 ((E: 17)	i
$\leq[\mathbf{x},\mathbf{y}]$,! 19 (\Rightarrow E: 16,18)	i
$\langle[\mathbf{y},\mathbf{x}] \ \& \ \leq[\mathbf{x},\mathbf{y}]$,! 20 ($\&$ I: 11,19)	i
$(\langle[\mathbf{y},\mathbf{x}] \ \& \ \leq[\mathbf{x},\mathbf{y}] \Rightarrow \mathfrak{F})$,! 21 (\forall E: P19)	i
$\langle[\mathbf{y},\mathbf{x}] \ \& \ \leq[\mathbf{x},\mathbf{y}] \Rightarrow \mathfrak{F}$,! 22 ((E: 21)	i
\mathfrak{F}	,! 23 (\Rightarrow E: 20,22)	i
$P[\mathbf{y}] \Rightarrow \mathfrak{F}$,! 24 (\mathfrak{F} I: 15,23)	i
$\neg P[\mathbf{y}]$,! 25 (\neg I: 24)	i
$\langle[\mathbf{y},\mathbf{x}] \Rightarrow \neg P[\mathbf{y}]$,! 26 (\Rightarrow I: 11,25)	i
$(\langle[\mathbf{y},\mathbf{x}] \Rightarrow \neg P[\mathbf{y}])$,! 27 ((I: 26)	i
$\forall y (\langle[\mathbf{y},\mathbf{x}] \Rightarrow \neg P[\mathbf{y}])$,! 28 (\forall I: 10,27)	i
$\omega[\mathbf{x}] \ \& \ P[\mathbf{x}] \ \& \ \forall y (\langle[\mathbf{y},\mathbf{x}] \Rightarrow \neg P[\mathbf{y}])$,! 29 ($\&$ I: 8,28)	i

$(\omega[\mathbf{x}] \ \& \ \mathbf{P}[\mathbf{x}] \ \& \ \forall y \ (\langle [y, \mathbf{x}] \Rightarrow \neg \mathbf{P}[y]))$, ! 30 (() I : 29)	i
$\exists x \ (\omega[x] \ \& \ \mathbf{P}[x] \ \& \ \forall y \ (\langle [y, x] \Rightarrow \neg \mathbf{P}[y]))$, ! 31 (\exists I : 30)	i
$\exists x \ (\omega[x] \ \& \ \mathbf{P}[x]) \Rightarrow \exists x \ (\omega[x] \ \& \ \mathbf{P}[x] \ \& \ \forall y \ (\langle [y, x] \Rightarrow \neg \mathbf{P}[y]))$, ! 32 (\Rightarrow I : 2, 31)	i
$(\exists x \ (\omega[x] \ \& \ \mathbf{P}[x]) \Rightarrow \exists x \ (\omega[x] \ \& \ \mathbf{P}[x] \ \& \ \forall y \ (\langle [y, x] \Rightarrow \neg \mathbf{P}[y])))$, ! 33 (() I : 32)	i
$\forall P \ (\exists x \ (\omega[x] \ \& \ P[x])$		
$\Rightarrow \exists x \ (\omega[x] \ \& \ P[x] \ \& \ \forall y \ (\langle [y, x] \Rightarrow \neg P[y]))$! 34 (\forall I : 1, 33)	i
\square		
! 50.		i
$\vdash \forall P \ (\forall y \ (\omega[y] \Rightarrow \exists x \ (P[x] \ \& \ \langle [y, x])) \Rightarrow \neg P)$		i
\mathbf{P}	, ! 1 (Prem)	i
$\forall y \ (\omega[y] \Rightarrow \exists x \ (\mathbf{P}[x] \ \& \ \langle [y, x]))$, ! 2 (Prem)	i
$f \ \mathbf{P}$, ! 3 (Prem)	i
$(\omega[0] \Rightarrow \exists x \ (\mathbf{P}[x] \ \& \ \langle [0, x]))$, ! 4 (\forall E : 2)	i
$\omega[0] \Rightarrow \exists x \ (\mathbf{P}[x] \ \& \ \langle [0, x])$, ! 5 (() E : 4)	i
$\exists x \ (\mathbf{P}[x] \ \& \ \langle [0, x])$, ! 6 (\Rightarrow E : $\omega 0, 5$)	i
$(\mathbf{P}[\mathbf{x}] \ \& \ \langle [0, \mathbf{x}])$, ! 7 (\exists E : 6)	i
$\mathbf{P}[\mathbf{x}] \ \& \ \langle [0, \mathbf{x}]$, ! 8 (() E : 7)	i
$\mathbf{P}[\mathbf{x}]$, ! 9 ($\&$ E : 8)	i
$\langle [0, \mathbf{x}]$, ! 10 ($\&$ E : 8)	i
$(\langle [0, \mathbf{x}] \Rightarrow \omega[\mathbf{x}])$, ! 11 (\forall E : P11)	i
$\langle [0, \mathbf{x}] \Rightarrow \omega[\mathbf{x}]$, ! 12 (() E : 11)	i
$\omega[\mathbf{x}]$, ! 13 (\Rightarrow E : 10, 12)	i
$\omega[\mathbf{x}] \ \& \ \mathbf{P}[\mathbf{x}]$, ! 14 ($\&$ I : 9, 13)	i
$(\omega[\mathbf{x}] \ \& \ \mathbf{P}[\mathbf{x}])$, ! 15 (() I : 14)	i
$\exists x \ (\omega[x] \ \& \ \mathbf{P}[x])$, ! 16 (\exists I : 15)	i
$(\exists x \ (\omega[x] \ \& \ \mathbf{P}[x]) \Rightarrow \neg (\omega \cap \mathbf{P}) \equiv \phi)$, ! 17 (\forall E : II5.26)	i

$\exists x (\omega[x] \ \& \ P[x]) \Rightarrow \neg (\omega \cap P) \equiv \phi$,! 18 ((E: 17) i
 $\neg (\omega \cap P) \equiv \phi$,! 19 (\Rightarrow E: 16,18) i
 $(\omega \cap P) \subseteq P$,! 20 (\forall E: II3.11) i
 $f P \ \& \ (\omega \cap P) \subseteq P$,! 21 (&I: 3,20) i
 $(f P \ \& \ (\omega \cap P) \subseteq P \Rightarrow f (\omega \cap P))$,! 22 (\forall E: IV5.12) i
 $f P \ \& \ (\omega \cap P) \subseteq P \Rightarrow f (\omega \cap P)$,! 23 ((E: 22) i
 $f (\omega \cap P)$,! 24 (\Rightarrow E: 21,23) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$,! 25 (&I: 19,24) i
 $(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x])))$,! 26 (\forall E: C3.75) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x]))$,! 27 ((E: 26) i
 $\exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x]))$,! 28 (\Rightarrow E: 25,27) i
 $(\omega[z] \ \& \ P[z] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,z]))$,! 29 (\exists E: 28) i
 $\omega[z] \ \& \ P[z] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,z])$,! 30 ((E: 29) i
 $\omega[z]$,! 31 (&E: 30) i
 $\forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,z])$,! 32 (&E: 30) i
 $(\omega[z] \Rightarrow \exists x (P[x] \ \& \ <[z,x]))$,! 33 (\forall E: 2) i
 $\omega[z] \Rightarrow \exists x (P[x] \ \& \ <[z,x])$,! 34 ((E: 33) i
 $\exists x (P[x] \ \& \ <[z,x])$,! 35 (\Rightarrow E: 31,34) i
 $(P[a] \ \& \ <[z,a])$,! 36 (\exists E: 35) i
 $P[a] \ \& \ <[z,a]$,! 37 ((E: 36) i
 $P[a]$,! 38 (&E: 37) i
 $<[z,a]$,! 39 (&E: 37) i
 $(<[z,a] \Rightarrow \neg \leq[a,z])$,! 40 (\forall E: P20) i

$<[z, a] \Rightarrow \neg \leq[a, z]$,!	41	(()E: 40)	i
$\neg \leq[a, z]$,!	42	(\Rightarrow E: 39,41)	i
$(<[z, a] \Rightarrow \omega[a])$,!	43	(\forall E: P11)	i
$<[z, a] \Rightarrow \omega[a]$,!	44	(()E: 43)	i
$\omega[a]$,!	45	(\Rightarrow E: 39,44)	i
$\omega[a] \ \& \ P[a]$,!	46	(&I: 38,45)	i
$(\omega[a] \ \& \ P[a] \Rightarrow \leq[a, z])$,!	47	(\forall E: 32)	i
$\omega[a] \ \& \ P[a] \Rightarrow \leq[a, z]$,!	48	(()E: 47)	i
$\leq[a, z]$,!	49	(\Rightarrow E: 46,48)	i
\mathfrak{F}	,!	50	(\mathfrak{F} I: 42,49)	i
$f \ P \Rightarrow \mathfrak{F}$,!	51	(\Rightarrow I: 3,50)	i
$\neg f \ P$,!	52	(\neg I: 51)	i
$\iota \ P$,!	53	(\mathfrak{S} I: IV6.1,52)	i
$\forall y (\omega[y] \Rightarrow \exists x (P[x] \ \& \ <[y,x])) \Rightarrow \iota \ P$,!	54	(\Rightarrow I: 2,53)	i
$(\forall y (\omega[y] \Rightarrow \exists x (P[x] \ \& \ <[y,x])) \Rightarrow \iota \ P)$,!	55	(()I: 54)	i
$\forall P (\forall y (\omega[y] \Rightarrow \exists x (P[x] \ \& \ <[y,x])) \Rightarrow \iota \ P)$!	56	(\forall I: 1,55)	i
\square				
! 51. The finite numbers are infinite.				i
$\vdash \iota \ \omega$				i
$(\forall y (\omega[y] \Rightarrow \exists x (\omega[x] \ \& \ <[y,x])) \Rightarrow \iota \ \omega)$,!	1	(\forall E Prop)	i
$\forall y (\omega[y] \Rightarrow \exists x (\omega[x] \ \& \ <[y,x])) \Rightarrow \iota \ \omega$,!	2	(()E: 1)	i
y	,!	3	(Prem)	i
$\omega[y]$,!	4	(Prem)	i
$(\omega[y] \Rightarrow <[y, (y+1)])$,!	5	(\forall E: P26)	i
$\omega[y] \Rightarrow <[y, (y+1)]$,!	6	(()E: 5)	i

$\langle [y, (y+1)] \rangle$, ! 7 (\Rightarrow E: 4,6)	i
$\omega[y] \ \& \ \omega[1]$, ! 8 ($\&$ I: IV9.2,4)	i
$(\ \omega[y] \ \& \ \omega[1] \Rightarrow \omega[(y+1)] \)$, ! 9 (\forall E: C1.8)	i
$\omega[y] \ \& \ \omega[1] \Rightarrow \omega[(y+1)]$, ! 10 ($(\)$ E: 9)	i
$\omega[(y+1)]$, ! 11 (\Rightarrow E: 8,10)	i
$\omega[(y+1)] \ \& \ \langle [y, (y+1)] \rangle$, ! 12 ($\&$ I: 7,11)	i
$(\ \omega[(y+1)] \ \& \ \langle [y, (y+1)] \rangle \)$, ! 13 ($(\)$ I: 12)	i
$\exists x (\ \omega[x] \ \& \ \langle [y, x] \rangle \)$, ! 14 (\exists I: 13; (y+1): C1.7,8)	i
$\omega[y] \Rightarrow \exists x (\ \omega[x] \ \& \ \langle [y, x] \rangle \)$, ! 15 (\Rightarrow I: 4,14)	i
$(\ \omega[y] \Rightarrow \exists x (\ \omega[x] \ \& \ \langle [y, x] \rangle \) \)$, ! 16 ($(\)$ I: 15)	i
$\forall y (\ \omega[y] \Rightarrow \exists x (\ \omega[x] \ \& \ \langle [y, x] \rangle \) \)$, ! 17 (\forall I: 1,16)	i
$\iota \ \omega$! 18 (\Rightarrow I: 2,17)	i

□