

! CHAPTER 5

SUBTRACTION;

! This chapter introduces and develops subtraction. P1 reformulates the Subtr Axiom, and is then followed by some corollaries (P2 through P4). P5 and P6 prepare the introduction of the subtraction term in P7. Simple consequences fill out the rest of the chapter. i

! 1. i

$\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\Rightarrow (\Theta[m,n,k] \Leftrightarrow \mathfrak{N}[k, (B \setminus A)]))$ i

n, m, k, A, B ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 ,! 2 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B]$,! 3 (&E: 2) i

$\omega[k]$,! 4 (&E: 2) i

$A \subseteq B$,! 5 (&E: 2) i

$\forall x (A[x] \Rightarrow B[x])$,! 6 (§E: III1.1,5) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ \forall x (A[x] \Rightarrow B[x])$
 ,! 7 (&I: 3,6) i

$(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ \forall x (A[x] \Rightarrow B[x])$
 $\Rightarrow (\Theta[m,n,k] \Leftrightarrow \mathfrak{N}[k, \{x : B[x] \ \& \ \neg A[x]\}]))$
 ,! 8 (\forall E: Subtr) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ \forall x (A[x] \Rightarrow B[x])$
 $\Rightarrow (\Theta[m,n,k] \Leftrightarrow \mathfrak{N}[k, \{x : B[x] \ \& \ \neg A[x]\}])$
 ,! 9 ((E): 8) i

$(\Theta[m,n,k] \Leftrightarrow \mathfrak{N}[k, \{x : B[x] \ \& \ \neg A[x]\}])$
 ,! 10 (\Rightarrow E: 7,9) i

$\Theta[m,n,k] \Leftrightarrow \mathfrak{N}[k, \{x : B[x] \ \& \ \neg A[x]\}]$,! 11 ((E): 10) i

! To show: $(B \setminus A) \equiv \{x : B[x] \ \& \ \neg A[x]\}$ i

x ,! 12 (Prem) i

$((B \setminus A)[x] \Leftrightarrow B[x] \ \& \ \neg A[x])$,! 13 (\forall E: II7.2) i

$(B \setminus A)[x] \Leftrightarrow B[x] \ \& \ \neg A[x]$,! 14 ((E): 13) i

$\forall x (\{a : B[a] \ \& \ \neg A[a]\}[x] \Leftrightarrow B[x] \ \& \ \neg A[x])$
 ,! 15 (Pred) i

$(\{a : B[a] \ \& \ \neg A[a]\}[x] \Leftrightarrow B[x] \ \& \ \neg A[x])$

	,! 16 (\forall E: 15)	i
$\{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}] \Leftrightarrow \mathbf{B}[\mathbf{x}] \ \& \ \neg \ \mathbf{A}[\mathbf{x}]$,! 17 ((E): 16)	i
$(\mathbf{B} \setminus \mathbf{A})[\mathbf{x}]$,! 18 (Prem)	i
$(\mathbf{B} \setminus \mathbf{A})[\mathbf{x}] \Rightarrow \mathbf{B}[\mathbf{x}] \ \& \ \neg \ \mathbf{A}[\mathbf{x}]$,! 19 (\Leftrightarrow E: 14)	i
$\mathbf{B}[\mathbf{x}] \ \& \ \neg \ \mathbf{A}[\mathbf{x}]$,! 20 (\Rightarrow E: 18,19)	i
$\mathbf{B}[\mathbf{x}] \ \& \ \neg \ \mathbf{A}[\mathbf{x}] \Rightarrow \{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}]$,! 21 (\Leftrightarrow E: 17)	i
$\{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}]$,! 22 (\Rightarrow E: 20,21)	i
$(\mathbf{B} \setminus \mathbf{A})[\mathbf{x}] \Rightarrow \{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}]$,! 23 (\Rightarrow I: 18,22)	i
$\{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}]$,! 24 (Prem)	i
$\{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}] \Rightarrow \mathbf{B}[\mathbf{x}] \ \& \ \neg \ \mathbf{A}[\mathbf{x}]$,! 25 (\Leftrightarrow E: 17)	i
$\mathbf{B}[\mathbf{x}] \ \& \ \neg \ \mathbf{A}[\mathbf{x}]$,! 26 (\Rightarrow E: 24,25)	i
$\mathbf{B}[\mathbf{x}] \ \& \ \neg \ \mathbf{A}[\mathbf{x}] \Rightarrow (\mathbf{B} \setminus \mathbf{A})[\mathbf{x}]$,! 27 (\Leftrightarrow E: 14)	i
$(\mathbf{B} \setminus \mathbf{A})[\mathbf{x}]$,! 28 (\Rightarrow E: 26,27)	i
$\{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}] \Rightarrow (\mathbf{B} \setminus \mathbf{A})[\mathbf{x}]$,! 29 (\Rightarrow I: 24,28)	i
$(\mathbf{B} \setminus \mathbf{A})[\mathbf{x}] \Leftrightarrow \{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}]$,! 30 (\Leftrightarrow I: 23,29)	i
$((\mathbf{B} \setminus \mathbf{A})[\mathbf{x}] \Leftrightarrow \{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}])$,! 31 ((I): 30)	i
$\forall \mathbf{x} ((\mathbf{B} \setminus \mathbf{A})[\mathbf{x}] \Leftrightarrow \{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}[\mathbf{x}])$,! 32 (\forall I: 12,31)	i
$(\mathbf{B} \setminus \mathbf{A}) \equiv \{a : \mathbf{B}[a] \ \& \ \neg \ \mathbf{A}[a]\}$,! 33 (\mathbb{S} E: III1.7,32)	i
$(\mathbf{B} \setminus \mathbf{A}) \equiv \{x : \mathbf{B}[x] \ \& \ \neg \ \mathbf{A}[x]\}$,! 34 (Exch: 33)	i
$\omega[\mathbf{k}] \ \& \ (\mathbf{B} \setminus \mathbf{A}) \equiv \{x : \mathbf{B}[x] \ \& \ \neg \ \mathbf{A}[x]\}$,! 35 ($\&$ I: 4,34)	i
$\Theta[\mathbf{m},\mathbf{n},\mathbf{k}]$,! 36 (Prem)	i
$\Theta[\mathbf{m},\mathbf{n},\mathbf{k}] \Rightarrow \mathfrak{N}[\mathbf{k},\{x : \mathbf{B}[x] \ \& \ \neg \ \mathbf{A}[x]\}]$,! 37 (\Leftrightarrow E: 11)	i
$\mathfrak{N}[\mathbf{k},\{x : \mathbf{B}[x] \ \& \ \neg \ \mathbf{A}[x]\}]$,! 38 (\Rightarrow E: 36,37)	i
$\omega[\mathbf{k}] \ \& \ \mathfrak{N}[\mathbf{k},\{x : \mathbf{B}[x] \ \& \ \neg \ \mathbf{A}[x]\}]$		

$\& (B \setminus A) \equiv \{x : B[x] \& \neg A[x]\}$,! 39 (&I: 35,38) ;

($\omega[k] \& \mathfrak{N}[k, \{x : B[x] \& \neg A[x]\}]$
 $\& (B \setminus A) \equiv \{x : B[x] \& \neg A[x]\}$
 $\Rightarrow \mathfrak{N}[k, (B \setminus A)]$) ,! 40 ($\forall E$: IV4.6) ;

$\omega[k] \& \mathfrak{N}[k, \{x : B[x] \& \neg A[x]\}]$
 $\& (B \setminus A) \equiv \{x : B[x] \& \neg A[x]\}$
 $\Rightarrow \mathfrak{N}[k, (B \setminus A)]$,! 41 (($)E$: 40) ;

$\mathfrak{N}[k, (B \setminus A)]$,! 42 ($\Rightarrow E$: 39,41) ;

$\Theta[m, n, k] \Rightarrow \mathfrak{N}[k, (B \setminus A)]$,! 43 ($\Rightarrow I$: 36,42) ;

$\mathfrak{N}[k, (B \setminus A)]$,! 44 (Prem) ;

$\omega[k] \& \mathfrak{N}[k, (B \setminus A)] \& (B \setminus A) \equiv \{x : B[x] \& \neg A[x]\}$
 ,! 45 (&I: 4,44) ;

($\omega[k] \& \mathfrak{N}[k, (B \setminus A)] \& (B \setminus A) \equiv \{x : B[x] \& \neg A[x]\}$
 $\Rightarrow \mathfrak{N}[k, \{x : B[x] \& \neg A[x]\}]$) ,! 46 ($\forall E$: IV4.5) ;

$\omega[k] \& \mathfrak{N}[k, (B \setminus A)] \& (B \setminus A) \equiv \{x : B[x] \& \neg A[x]\}$
 $\Rightarrow \mathfrak{N}[k, \{x : B[x] \& \neg A[x]\}]$,! 47 (($)E$: 46) ;

$\mathfrak{N}[k, \{x : B[x] \& \neg A[x]\}]$,! 48 ($\Rightarrow E$: 45,47) ;

$\mathfrak{N}[k, \{x : B[x] \& \neg A[x]\}] \Rightarrow \Theta[m, n, k]$
 ,! 49 ($\Leftrightarrow E$: 11) ;

$\Theta[m, n, k]$,! 50 ($\Rightarrow E$: 48,49) ;

$\mathfrak{N}[k, (B \setminus A)] \Rightarrow \Theta[m, n, k]$,! 51 ($\Rightarrow I$: 44,50) ;

$\Theta[m, n, k] \Leftrightarrow \mathfrak{N}[k, (B \setminus A)]$,! 52 ($\Leftrightarrow I$: 43,51) ;

($\Theta[m, n, k] \Leftrightarrow \mathfrak{N}[k, (B \setminus A)]$) ,! 53 (($)I$: 52) ;

$\omega[n] \& \omega[m] \& \omega[k] \& \mathfrak{N}[n, A] \& \mathfrak{N}[m, B] \& A \subseteq B$
 $\Rightarrow (\Theta[m, n, k] \Leftrightarrow \mathfrak{N}[k, (B \setminus A)])$,! 54 ($\Rightarrow I$: 2,53) ;

($\omega[n] \& \omega[m] \& \omega[k] \& \mathfrak{N}[n, A] \& \mathfrak{N}[m, B] \& A \subseteq B$
 $\Rightarrow (\Theta[m, n, k] \Leftrightarrow \mathfrak{N}[k, (B \setminus A)])$) ,! 55 (($)I$: 54) ;

$\forall n \forall m \forall k \forall A \forall B (\omega[n] \& \omega[m] \& \omega[k] \& \mathfrak{N}[n, A] \& \mathfrak{N}[m, B] \& A \subseteq B$

$\Rightarrow (\Theta_{[m,n,k]} \Leftrightarrow \mathcal{N}_{[k,(B \setminus A)]}))$
 ! 56 ($\forall I$: 1,55) i

□

! 2. i

$\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B$
 $\ \& \ \Theta_{[m,n,k]}$
 $\ \Rightarrow \ \mathcal{N}_{[k,(B \setminus A)]})$ i

n, m, k, A, B ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B \ \& \ \Theta_{[m,n,k]}$
 ,! 2 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B$
 ,! 3 ($\&E$: 2) i

$\Theta_{[m,n,k]}$,! 4 ($\&E$: 2) i

$(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B$
 $\ \Rightarrow \ (\Theta_{[m,n,k]} \Leftrightarrow \mathcal{N}_{[k,(B \setminus A)]}))$
 ,! 5 ($\forall E$: P1) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B$
 $\Rightarrow (\Theta_{[m,n,k]} \Leftrightarrow \mathcal{N}_{[k,(B \setminus A)]})$
 ,! 6 ($(\)E$: 5) i

$(\Theta_{[m,n,k]} \Leftrightarrow \mathcal{N}_{[k,(B \setminus A)]})$,! 7 ($\Rightarrow E$: 3,6) i

$\Theta_{[m,n,k]} \Leftrightarrow \mathcal{N}_{[k,(B \setminus A)]}$,! 8 ($(\)E$: 7) i

$\Theta_{[m,n,k]} \Rightarrow \mathcal{N}_{[k,(B \setminus A)]}$,! 9 ($\Leftrightarrow E$: 8) i

$\mathcal{N}_{[k,(B \setminus A)]}$,! 10 ($\Rightarrow E$: 4,9) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B \ \& \ \Theta_{[m,n,k]}$
 $\Rightarrow \mathcal{N}_{[k,(B \setminus A)]}$
 ,! 11 ($\Rightarrow I$: 2,10) i

$(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B \ \& \ \Theta_{[m,n,k]}$
 $\Rightarrow \mathcal{N}_{[k,(B \setminus A)]})$
 ,! 12 ($(\)I$: 11) i

$\forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B$
 $\ \& \ \Theta_{[m,n,k]}$
 $\ \Rightarrow \ \mathcal{N}_{[k,(B \setminus A)]})$
 ! 13 ($\forall I$: 1,12) i

□

! 3. i

$\vdash \forall n \forall m \forall k \forall A \forall B (\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ A \subseteq B$

$\& \mathcal{N}[k, (B \setminus A)]$		
$\Rightarrow \Theta[m, n, k]$)	i
A, B, n, m, k	, ! 1 (Prem)	i
$\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B$		
$\& \mathcal{N}[k, (B \setminus A)]$, ! 2 (Prem)	i
$\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B$, ! 3 (&E: 2)	i
$\mathcal{N}[k, (B \setminus A)]$, ! 4 (&E: 2)	i
$(\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B$		
$\Rightarrow (\Theta[m, n, k] \Leftrightarrow \mathcal{N}[k, (B \setminus A)])$, ! 5 (\forall E: P1)	i
$\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B$		
$\Rightarrow (\Theta[m, n, k] \Leftrightarrow \mathcal{N}[k, (B \setminus A)])$, ! 6 ($(())$ E: 5)	i
$(\Theta[m, n, k] \Leftrightarrow \mathcal{N}[k, (B \setminus A)])$, ! 7 (\Rightarrow E: 3, 6)	i
$\Theta[m, n, k] \Leftrightarrow \mathcal{N}[k, (B \setminus A)]$, ! 8 ($(())$ E: 7)	i
$\mathcal{N}[k, (B \setminus A)] \Rightarrow \Theta[m, n, k]$, ! 9 (\Leftrightarrow E: 8)	i
$\Theta[m, n, k]$, ! 10 (\Rightarrow E: 4, 9)	i
$\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[k, (B \setminus A)]$		
$\Rightarrow \Theta[m, n, k]$, ! 11 (\Rightarrow I: 2, 10)	i
$(\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B$		
$\& \mathcal{N}[k, (B \setminus A)]$		
$\Rightarrow \Theta[m, n, k]$, ! 12 ($(())$ I: 11)	i
$\forall n \forall m \forall k \forall A \forall B (\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B$		
$\& \mathcal{N}[k, (B \setminus A)]$		
$\Rightarrow \Theta[m, n, k]$! 13 (\forall I: 1, 12)	i
\square		
! 4.		i
$\vdash \forall n \forall m \forall k (\omega[k] \& \leq[n, m] \& \Theta[m, n, k]$		
$\Rightarrow \exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[k, (B \setminus A)])$		i
n, m, k	, ! 1 (Prem)	i

$\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k]$,! 2 (Prem)	i
$\omega[k]$,! 3 (&E: 2)	i
$\leq[n,m]$,! 4 (&E: 2)	i
$\Theta[m,n,k]$,! 5 (&E: 2)	i
$(\leq[n,m] \Rightarrow \omega[n] \ \& \ \omega[m])$,! 6 (\forall E: C3.5)	i
$\leq[n,m] \Rightarrow \omega[n] \ \& \ \omega[m]$,! 7 (()E: 6)	i
$\omega[n] \ \& \ \omega[m]$,! 8 (\Rightarrow E: 4,7)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k]$,! 9 (&I: 3,8)	i
$(\leq[n,m] \Rightarrow \exists A \exists B (\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B))$,! 10 (\forall E: C3.13)	i
$\leq[n,m] \Rightarrow \exists A \exists B (\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B)$,! 11 (()E: 10)	i
$\exists A \exists B (\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B)$,! 12 (\Rightarrow E: 4,11)	i
$\exists B (\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B)$,! 13 (\exists E: 12)	i
$(\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B)$,! 14 (\exists E: 13)	i
$\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B$,! 15 (()E: 14)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B$,! 16 (&I: 9,15)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B \ \& \ \Theta[m,n,k]$,! 17 (&I: 5,16)	i
$(\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B \ \& \ \Theta[m,n,k] \Rightarrow \mathcal{N}[k, (B \setminus A)])$,! 18 (\forall E: P2)	i
$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B \ \& \ \Theta[m,n,k] \Rightarrow \mathcal{N}[k, (B \setminus A)]$,! 19 (()E: 18)	i
$\mathcal{N}[k, (B \setminus A)]$,! 20 (\Rightarrow E: 17,19)	i
$\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[k, (B \setminus A)]$,! 21 (&I: 15,20)	i
$(\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[k, (B \setminus A)])$,! 22 (()I: 21)	i
$\exists B (\mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[k, (B \setminus A)])$,! 23 (\exists I: 22)	i

$\exists A \exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B \ \& \ \mathcal{P}[k, (B \setminus A)])$
, ! 24 ($\exists I$: 23) i

$\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k]$
 $\Rightarrow \exists A \exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B \ \& \ \mathcal{P}[k, (B \setminus A)])$
, ! 25 ($\Rightarrow I$: 2,24) i

$(\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k]$
 $\Rightarrow \exists A \exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B \ \& \ \mathcal{P}[k, (B \setminus A)]))$
, ! 26 ($() I$: 25) i

$\forall n \forall m \forall k (\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k]$
 $\Rightarrow \exists A \exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B \ \& \ \mathcal{P}[k, (B \setminus A)]))$
! 27 ($\forall I$: 1,26) i

□

! 5. i

$\vdash \forall n \forall m (\leq[n,m] \Rightarrow \exists a (\omega[a] \ \& \ \Theta[m,n,a]))$ i

n, m , ! 1 (Prem) i

$\leq[n,m]$, ! 2 (Prem) i

$(\leq[n,m] \Rightarrow \omega[n] \ \& \ \omega[m])$, ! 3 ($\forall E$: C3.5) i

$\leq[n,m] \Rightarrow \omega[n] \ \& \ \omega[m]$, ! 4 ($() E$: 3) i

$\omega[n] \ \& \ \omega[m]$, ! 5 ($\Rightarrow E$: 2,4) i

$\omega[m]$, ! 6 ($\& E$: 5) i

$(\leq[n,m] \Rightarrow \exists A \exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B))$
, ! 7 ($\forall E$: C3.13) i

$\leq[n,m] \Rightarrow \exists A \exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B)$
, ! 8 ($() E$: 7) i

$\exists A \exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B)$, ! 9 ($\Rightarrow E$: 2,8) i

$\exists B (\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B)$, ! 10 ($\exists E$: 9) i

$(\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B)$, ! 11 ($\exists E$: 10) i

$\mathcal{P}[n,A] \ \& \ \mathcal{P}[m,B] \ \& \ A \subseteq B$, ! 12 ($() E$: 11) i

$\mathcal{P}[m,B]$, ! 13 ($\& E$: 12) i

$\omega[m] \ \& \ \mathcal{P}[m,B]$, ! 14 ($\& I$: 6,13) i

$(B \setminus A) \subseteq B$, ! 15 ($\forall E$: II7.13) i

$\omega[m] \ \& \ \mathfrak{N}[m,B] \ \& \ (B \setminus A) \subseteq B$,! 16 (&I: 14,15) ;
 $(\omega[m] \ \& \ \mathfrak{N}[m,B] \ \& \ (B \setminus A) \subseteq B \Rightarrow f (B \setminus A))$
, ! 17 ($\forall E$: IV5.11) ;
 $\omega[m] \ \& \ \mathfrak{N}[m,B] \ \& \ (B \setminus A) \subseteq B \Rightarrow f (B \setminus A)$
, ! 18 ((E): 17) ;
 $f (B \setminus A)$, ! 19 ($\Rightarrow E$: 16,18) ;
 $\exists n (\omega[n] \ \& \ \mathfrak{N}[n,(B \setminus A)])$, ! 20 ($\exists E$: IV5.1,19) ;
 $(\omega[a] \ \& \ \mathfrak{N}[a,(B \setminus A)])$, ! 21 ($\exists E$: 20) ;
 $\omega[a] \ \& \ \mathfrak{N}[a,(B \setminus A)]$, ! 22 ((E): 21) ;
 $\omega[n] \ \& \ \omega[m] \ \& \ \omega[a] \ \& \ \mathfrak{N}[a,(B \setminus A)]$, ! 23 (&I: 5,22) ;
 $\omega[n] \ \& \ \omega[m] \ \& \ \omega[a] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \mathfrak{N}[a,(B \setminus A)]$, ! 24 (&I: 12,23) ;
 $(\omega[n] \ \& \ \omega[m] \ \& \ \omega[a] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \mathfrak{N}[a,(B \setminus A)]$
 $\Rightarrow \Theta[m,n,a])$
, ! 25 ($\forall E$: P3) ;
 $\omega[n] \ \& \ \omega[m] \ \& \ \omega[a] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \mathfrak{N}[a,(B \setminus A)]$
 $\Rightarrow \Theta[m,n,a]$
, ! 26 ((E): 25) ;
 $\Theta[m,n,a]$, ! 27 ($\Rightarrow E$: 24,26) ;
 $\omega[a]$, ! 28 (&E: 22) ;
 $\omega[a] \ \& \ \Theta[m,n,a]$, ! 29 (&I: 27,28) ;
 $(\omega[a] \ \& \ \Theta[m,n,a])$, ! 30 ((I): 29) ;
 $\exists a (\omega[a] \ \& \ \Theta[m,n,a])$, ! 31 ($\exists I$: 30) ;
 $\leq[n,m] \Rightarrow \exists a (\omega[a] \ \& \ \Theta[m,n,a])$, ! 32 ($\Rightarrow I$: 2,31) ;
 $(\leq[n,m] \Rightarrow \exists a (\omega[a] \ \& \ \Theta[m,n,a]))$, ! 33 ((I): 32) ;
 $\forall n \forall m (\leq[n,m] \Rightarrow \exists a (\omega[a] \ \& \ \Theta[m,n,a]))$! 34 ($\forall I$: 1,33) ;
 \square
! 6. ;
 $\vdash \forall n \forall m (\leq[n,m] \Rightarrow \forall a \forall b ((\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b])$
 $\Rightarrow a = b))$;

n, m	, ! 1 (Prem)	i
$\leq[n, m]$, ! 2 (Prem)	i
$(\leq[n, m] \Rightarrow \omega[n] \ \& \ \omega[m])$, ! 3 ($\forall E$: C3.5)	i
$\leq[n, m] \Rightarrow \omega[n] \ \& \ \omega[m]$, ! 4 ($(\)E$: 3)	i
$\omega[n] \ \& \ \omega[m]$, ! 5 ($\Rightarrow E$: 2, 4)	i
a, b	, ! 6 (Prem)	i
$(\omega[a] \ \& \ \Theta[m, n, a]) \ \& \ (\omega[b] \ \& \ \Theta[m, n, b])$, ! 7 (Prem)	i
$(\omega[a] \ \& \ \Theta[m, n, a])$, ! 8 ($\&E$: 7)	i
$(\omega[b] \ \& \ \Theta[m, n, b])$, ! 9 ($\&E$: 7)	i
$\omega[a] \ \& \ \Theta[m, n, a]$, ! 10 ($(\)E$: 8)	i
$\omega[a]$, ! 11 ($\&E$: 10)	i
$\omega[b] \ \& \ \Theta[m, n, b]$, ! 12 ($(\)E$: 9)	i
$\omega[b]$, ! 13 ($\&E$: 12)	i
$\omega[a] \ \& \ \omega[b]$, ! 14 ($\&I$: 11, 13)	i
$\omega[a] \ \& \ \leq[n, m] \ \& \ \Theta[m, n, a]$, ! 15 ($(\)E$: 2, 10)	i
$(\omega[a] \ \& \ \leq[n, m] \ \& \ \Theta[m, n, a] \Rightarrow \exists A \exists B (\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[a, (B \setminus A)]))$, ! 16 ($\forall E$: P4)	i
$\omega[a] \ \& \ \leq[n, m] \ \& \ \Theta[m, n, a] \Rightarrow \exists A \exists B (\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[a, (B \setminus A)])$, ! 17 ($(\)E$: 16)	i
$\exists A \exists B (\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[a, (B \setminus A)])$, ! 18 ($\Rightarrow E$: 15, 17)	i
$\exists B (\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[a, (B \setminus A)])$, ! 19 ($\exists E$: 18)	i
$(\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[a, (B \setminus A)])$, ! 20 ($\exists E$: 19)	i
$\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[a, (B \setminus A)]$, ! 21 ($(\)E$: 20)	i
$\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B$, ! 22 ($\&E$: 21)	i

$\mathcal{N}[a, (B \setminus A)]$,! 23 (&E: 21) ;
 $\omega[a] \ \& \ \omega[b] \ \& \ \mathcal{N}[a, (B \setminus A)]$,! 24 (&I: 14,23) ;
 $\omega[n] \ \& \ \omega[m] \ \& \ \omega[b] \ \& \ \Theta[m,n,b]$,! 25 (&I: 5,12) ;
 $\omega[n] \ \& \ \omega[m] \ \& \ \omega[b] \ \& \ \mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \Theta[m,n,b]$,! 26 (&I: 22,25) ;
 $(\ \omega[n] \ \& \ \omega[m] \ \& \ \omega[b] \ \& \ \mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \Theta[m,n,b]$
 $\Rightarrow \mathcal{N}[b, (B \setminus A)] \)$,! 27 (\forall E: P2) ;
 $\omega[n] \ \& \ \omega[m] \ \& \ \omega[b] \ \& \ \mathcal{N}[n,A] \ \& \ \mathcal{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \Theta[m,n,b]$
 $\Rightarrow \mathcal{N}[b, (B \setminus A)]$,! 28 ((E: 27) ;
 $\mathcal{N}[b, (B \setminus A)]$,! 29 (\Rightarrow E: 26,28) ;
 $\omega[a] \ \& \ \omega[b] \ \& \ \mathcal{N}[a, (B \setminus A)] \ \& \ \mathcal{N}[b, (B \setminus A)]$,! 30 (&I: 24,29) ;
 $(\ \omega[a] \ \& \ \omega[b] \ \& \ \mathcal{N}[a, (B \setminus A)] \ \& \ \mathcal{N}[b, (B \setminus A)] \Rightarrow a = b \)$,! 31 (\forall E: IV2.10) ;
 $\omega[a] \ \& \ \omega[b] \ \& \ \mathcal{N}[a, (B \setminus A)] \ \& \ \mathcal{N}[b, (B \setminus A)] \Rightarrow a = b$,! 32 ((E: 31) ;
 $a = b$,! 33 (\Rightarrow E: 30,32) ;
 $(\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b]) \Rightarrow a = b$,! 34 (\Rightarrow I: 7,33) ;
 $((\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b])) \Rightarrow a = b$,! 35 ((I: 34) ;
 $\forall a \forall b ((\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b])) \Rightarrow a = b$,! 36 (\forall I: 6,35) ;
 $\leq[n,m]$
 $\Rightarrow \forall a \forall b ((\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b])) \Rightarrow a = b$,! 37 (\Rightarrow I: 2,36) ;
 $(\leq[n,m]$
 $\Rightarrow \forall a \forall b ((\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b])) \Rightarrow a = b \)$,! 38 ((I: 37) ;
 $\forall n \forall m (\leq[n,m] \Rightarrow \forall a \forall b ((\omega[a] \ \& \ \Theta[m,n,a]) \ \& \ (\omega[b] \ \& \ \Theta[m,n,b])) \Rightarrow a = b \)$! 39 (\forall I: 1,38) ;

□

! 7. i

$\mathbb{T} - ; (m - n) ; \leq[n,m] ; (\omega[a] \ \& \ \Theta[m,n,a])$ i!
($\mathbb{T}D$: P6,P7) i

! 8. i

$\vdash \forall n \forall m (\leq[n,m] \Rightarrow \omega[(m-n)])$ i

n, m ,! 1 (Prem) i

$\leq[n, m]$,! 2 (Prem) i

$(\omega[(m-n)] \ \& \ \Theta[m, n, (m-n)])$,! 3 ($\mathbb{T}I$: P7,2) i

$\omega[(m-n)] \ \& \ \Theta[m, n, (m-n)]$,! 4 ((E): 3) i

$\omega[(m-n)]$,! 5 (&E: 4) i

$\leq[n, m] \Rightarrow \omega[(m-n)]$,! 6 ($\Rightarrow I$: 2,5) i

$(\leq[n, m] \Rightarrow \omega[(m-n)])$,! 7 ((I): 6) i

$\forall n \forall m (\leq[n, m] \Rightarrow \omega[(m-n)])$! 8 ($\forall I$: 1,7) i

□

! 9. i

$\vdash \forall n \forall m \forall k ((m-n) = k \Rightarrow \omega[k])$ i

n, m, k ,! 1 (Prem) i

$(m-n) = k$,! 2 (Prem) i

$\leq[n, m]$,! 3 ($\mathbb{T}E$: P7,2) i

$(\leq[n, m] \Rightarrow \omega[(m-n)])$,! 4 ($\forall E$: P8) i

$\leq[n, m] \Rightarrow \omega[(m-n)]$,! 5 ((E): 4) i

$\omega[(m-n)]$,! 6 ($\Rightarrow E$: 3,5) i

$\omega[k]$,! 7 (=E: 2,6) i

$(m-n) = k \Rightarrow \omega[k]$,! 8 ($\Rightarrow I$: 2,7) i

$((m-n) = k \Rightarrow \omega[k])$,! 9 ((I): 8) i

$\forall n \forall m \forall k ((m-n) = k \Rightarrow \omega[k])$! 10 ($\forall I$: 1,9) i

□

! 10. i

$\vdash \forall n \forall m (\leq[n,m] \Rightarrow \Theta[m,n,(m-n)])$		i
n, m	, ! 1 (Prem)	i
$\leq[n,m]$, ! 2 (Prem)	i
$(\omega[(m-n)] \& \Theta[m,n,(m-n)])$, ! 3 ($\mathbb{T}I$: P7,2)	i
$\omega[(m-n)] \& \Theta[m,n,(m-n)]$, ! 4 ($(\)E$: 3)	i
$\Theta[m,n,(m-n)]$, ! 5 ($\&E$: 4)	i
$\leq[n,m] \Rightarrow \Theta[m,n,(m-n)]$, ! 6 ($\Rightarrow I$: 2,5)	i
$(\leq[n,m] \Rightarrow \Theta[m,n,(m-n)])$, ! 7 ($(\)I$: 6)	i
$\forall n \forall m (\leq[n,m] \Rightarrow \Theta[m,n,(m-n)])$! 8 ($\forall I$: 1,7)	i

□

! 11.

$\vdash \forall n \forall m \forall k (\omega[k] \& \leq[n,m] \& \Theta[m,n,k] \Rightarrow k = (m-n))$		i
n, m, k	, ! 1 (Prem)	i
$\omega[k] \& \leq[n,m] \& \Theta[m,n,k]$, ! 2 (Prem)	i
$\omega[k]$, ! 3 ($\&E$: 2)	i
$\leq[n,m]$, ! 4 ($\&E$: 2)	i
$\Theta[m,n,k]$, ! 5 ($\&E$: 2)	i
$\omega[k] \& \Theta[m,n,k]$, ! 6 ($\&I$: 3,5)	i
$(\omega[k] \& \Theta[m,n,k])$, ! 7 ($(\)I$: 6)	i
$(\omega[(m-n)] \& \Theta[m,n,(m-n)])$, ! 8 ($\mathbb{T}I$: P7,4)	i
$(\omega[k] \& \Theta[m,n,k]) \& (\omega[(m-n)] \& \Theta[m,n,(m-n)])$, ! 9 ($\&I$: 7,8)	i
$(\leq[n,m] \Rightarrow \forall a \forall b ((\omega[a] \& \Theta[m,n,a]) \& (\omega[b] \& \Theta[m,n,b]) \Rightarrow a = b))$, ! 10 ($\forall E$: P6)	i
$\leq[n,m] \Rightarrow \forall a \forall b ((\omega[a] \& \Theta[m,n,a]) \& (\omega[b] \& \Theta[m,n,b]) \Rightarrow a = b)$, ! 11 ($(\)E$: 10)	i
$\forall a \forall b ((\omega[a] \& \Theta[m,n,a]) \& (\omega[b] \& \Theta[m,n,b]) \Rightarrow a = b)$, ! 12 ($\Rightarrow E$: 4,11)	i
$((\omega[k] \& \Theta[m,n,k]) \& (\omega[(m-n)] \& \Theta[m,n,(m-n)]))$		
$\Rightarrow k = (m-n)$		

,! 13 ($\forall E$: 12;
($m-n$): P7,4) i

($\omega[k] \ \& \ \Theta[m,n,k]$) $\&$ ($\omega[(m-n)] \ \& \ \Theta[m,n,(m-n)]$) $\Rightarrow k = (m-n)$
,! 14 ($(\)E$: 13) i

$k = (m-n)$,! 15 ($\Rightarrow E$: 9,14) i

$\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow k = (m-n)$,! 16 ($\Rightarrow I$: 2,15) i

($\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow k = (m-n)$)
,! 17 ($(\)I$: 16) i

$\forall n \forall m \forall k$ ($\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow k = (m-n)$)
! 18 ($\forall I$: 1,17) i

□

! 12. i

$\vdash \forall n \forall m \forall k$ ($\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow (m-n) = k$) i

n, m, k ,! 1 (Prem) i

$\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k]$,! 2 (Prem) i

($\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow k = (m-n)$)
,! 3 ($\forall E$: P11) i

$\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow k = (m-n)$,! 4 ($(\)E$: 3) i

$k = (m-n)$,! 5 ($\Rightarrow E$: 2,4) i

$k = k$,! 6 ($=I$) i

$(m-n) = k$,! 7 ($=E$: 5,6) i

$\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow (m-n) = k$,! 8 ($\Rightarrow I$: 2,7) i

($\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow (m-n) = k$)
,! 9 ($(\)I$: 8) i

$\forall n \forall m \forall k$ ($\omega[k] \ \& \ \leq[n,m] \ \& \ \Theta[m,n,k] \Rightarrow (m-n) = k$)
! 10 ($\forall I$: 1,9) i

□

! 13. i

$\vdash \forall n \forall m \forall A \forall B$ ($\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}_\omega[n,A] \ \& \ \mathfrak{N}_\omega[m,B] \ \& \ A \subseteq B$
 $\Rightarrow \mathfrak{N}_\omega[(m-n), (B \setminus A)]$) i

n, m, A, B ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}_\omega[n,A] \ \& \ \mathfrak{N}_\omega[m,B] \ \& \ A \subseteq B$
,! 2 (Prem) i

($\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B \Rightarrow \leq[n,m]$)
, ! 3 ($\forall E$: C3.9) i

$\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B \Rightarrow \leq[n,m]$
, ! 4 (() E : 3) i

$\leq[n,m]$
, ! 5 ($\Rightarrow E$: 2,4) i

($\omega[(m-n)] \ \& \ \Theta[m,n,(m-n)]$)
, ! 6 ($\mathbb{T}I$: P7,5) i

$\omega[(m-n)] \ \& \ \Theta[m,n,(m-n)]$
, ! 7 (() E : 6) i

$\omega[(m-n)]$
, ! 8 ($\&E$: 7) i

$\Theta[m,n,(m-n)]$
, ! 9 ($\&E$: 7) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[(m-n)] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
, ! 10 ($\&I$: 2,8) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[(m-n)] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \Theta[m,n,(m-n)]$
, ! 11 ($\&I$: 9,10) i

($\omega[n] \ \& \ \omega[m] \ \& \ \omega[(m-n)] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \Theta[m,n,(m-n)]$
 $\Rightarrow \mathfrak{N}[(m-n),(B \setminus A)]$)
, ! 12 ($\forall E$: P2;
 $(m-n)$: P7,5) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[(m-n)] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \Theta[m,n,(m-n)]$
 $\Rightarrow \mathfrak{N}[(m-n),(B \setminus A)]$
, ! 13 (() E : 12) i

$\mathfrak{N}[(m-n),(B \setminus A)]$
, ! 14 ($\Rightarrow E$: 11,13) i

$\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B \Rightarrow \mathfrak{N}[(m-n),(B \setminus A)]$
, ! 15 ($\Rightarrow I$: 2,14) i

($\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\Rightarrow \mathfrak{N}[(m-n),(B \setminus A)]$)
, ! 16 (() I : 15) i

$\forall n \forall m \forall A \forall B$ ($\omega[n] \ \& \ \omega[m] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\Rightarrow \mathfrak{N}[(m-n),(B \setminus A)]$)
! 17 ($\forall I$: 1,16) i

□

! 14. i

$\vdash \forall n \forall m \forall k \forall A \forall B$ ($\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathfrak{N}[n,A] \ \& \ \mathfrak{N}[m,B] \ \& \ A \subseteq B$
 $\& \ \mathfrak{N}[k,(B \setminus A)]$)

$$\Rightarrow k = (m-n) \quad i$$

n, m, k, A, B ,! 1 (Prem) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B$
 $\ \& \ \mathcal{N}[k, (B \setminus A)]$,! 2 (Prem) i

$\omega[n] \ \& \ \omega[m]$,! 3 (&E: 2) i

$\omega[k]$,! 4 (&E: 2) i

$\mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B$,! 5 (&E: 2) i

$\omega[n] \ \& \ \omega[m] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B$
,! 6 (&I: 3,5) i

$(\ \omega[n] \ \& \ \omega[m] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \Rightarrow \leq[n, m] \)$
,! 7 (\forall E: C3.9) i

$\omega[n] \ \& \ \omega[m] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \Rightarrow \leq[n, m]$
,! 8 ((E: 7) i

$\leq[n, m]$,! 9 (\Rightarrow E: 6,8) i

$\omega[k] \ \& \ \leq[n, m]$,! 10 (&I: 4,9) i

$(\ \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B$
 $\ \& \ \mathcal{N}[k, (B \setminus A)]$
 $\Rightarrow \Theta[m, n, k] \)$
,! 11 (\forall E: P3) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B$
 $\ \& \ \mathcal{N}[k, (B \setminus A)]$
 $\Rightarrow \Theta[m, n, k]$
,! 12 ((E: 11) i

$\Theta[m, n, k]$,! 13 (\Rightarrow E: 2,12) i

$\omega[k] \ \& \ \leq[n, m] \ \& \ \Theta[m, n, k]$,! 14 (&I: 10,13) i

$(\ \omega[k] \ \& \ \leq[n, m] \ \& \ \Theta[m, n, k] \Rightarrow k = (m-n) \)$
,! 15 (\forall E: P11) i

$\omega[k] \ \& \ \leq[n, m] \ \& \ \Theta[m, n, k] \Rightarrow k = (m-n)$,! 16 ((E: 15) i

$k = (m-n)$,! 17 (\Rightarrow E: 14,16) i

$\omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B \ \& \ \mathcal{N}[k, (B \setminus A)]$
 $\Rightarrow k = (m-n)$
,! 18 (\Rightarrow I: 2,17) i

$(\ \omega[n] \ \& \ \omega[m] \ \& \ \omega[k] \ \& \ \mathcal{N}[n, A] \ \& \ \mathcal{N}[m, B] \ \& \ A \subseteq B$

$\& \mathcal{N}[k, (B \setminus A)]$

$\Rightarrow k = (m-n)$

, ! 19 ((I: 18) i

$\forall n \forall m \forall k \forall A \forall B (\omega[n] \& \omega[m] \& \omega[k] \& \mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B$
 $\& \mathcal{N}[k, (B \setminus A)]$

$\Rightarrow k = (m-n)$

! 20 (\forall I: 1,19) i

□

! 15. i

$\vdash \forall n \forall m (\leq[n, m]$

$\Rightarrow \exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[(m-n), (B \setminus A)])$

i

n, m

, ! 1 (Prem) i

$\leq[n, m]$

, ! 2 (Prem) i

$(\omega[(m-n)] \& \Theta[m, n, (m-n)])$

, ! 3 (\mathbb{T} I: P7,2) i

$\omega[(m-n)] \& \Theta[m, n, (m-n)]$

, ! 4 ((I: 3) i

$\omega[(m-n)] \& \leq[n, m] \& \Theta[m, n, (m-n)]$

, ! 5 (&I: 2,4) i

$(\omega[(m-n)] \& \leq[n, m] \& \Theta[m, n, (m-n)]$

$\Rightarrow \exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[(m-n), (B \setminus A)])$

, ! 6 (\forall E: P4;
 $(m-n)$: P7,2) i

$\omega[(m-n)] \& \leq[n, m] \& \Theta[m, n, (m-n)]$

$\Rightarrow \exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[(m-n), (B \setminus A)])$

, ! 7 ((E: 6) i

$\exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[(m-n), (B \setminus A)])$

, ! 8 (\Rightarrow E: 5,7) i

$\leq[n, m]$

$\Rightarrow \exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[(m-n), (B \setminus A)])$

, ! 9 (\Rightarrow I: 2,8) i

$(\leq[n, m]$

$\Rightarrow \exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[(m-n), (B \setminus A)])$

, ! 10 ((I: 9) i

$\forall n \forall m (\leq[n, m]$

$\Rightarrow \exists A \exists B (\mathcal{N}[n, A] \& \mathcal{N}[m, B] \& A \subseteq B \& \mathcal{N}[(m-n), (B \setminus A)])$

! 11 (\forall I: 1,10) i

□

! 16. i

$\exists B (\mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ \mathbf{A} \subseteq B \ \& \ \mathcal{N}_{[(m-n), (B \setminus A)]})$
, ! 6 ($\exists E$: 5) i

$(\mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ \mathbf{A} \subseteq B \ \& \ \mathcal{N}_{[(m-n), (B \setminus A)]})$
, ! 7 ($\exists E$: 6) i

$\mathcal{N}_{[n,A]} \ \& \ \mathcal{N}_{[m,B]} \ \& \ \mathbf{A} \subseteq B \ \& \ \mathcal{N}_{[(m-n), (B \setminus A)]}$
, ! 8 ($()E$: 7) i

$\mathcal{N}_{[m,B]}$
, ! 9 ($\&E$: 8) i

$\mathcal{N}_{[(m-n), (B \setminus A)]}$
, ! 10 ($\&E$: 8) i

$\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), (B \setminus A)]}$
, ! 11 ($\&I$: 9,10) i

$(B \setminus A) \subseteq B$
, ! 12 ($\forall E$: II7.13) i

$\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), (B \setminus A)]} \ \& \ (B \setminus A) \subseteq B$
, ! 13 ($\&I$: 11,12) i

$(\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), (B \setminus A)]} \ \& \ (B \setminus A) \subseteq B)$
, ! 14 ($()I$: 13) i

$\exists B (\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), (B \setminus A)]} \ \& \ (B \setminus A) \subseteq B)$
, ! 15 ($\exists I$: 14) i

$\exists A \exists B (\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), A]} \ \& \ A \subseteq B)$, ! 16 ($\exists I$: 15) i

$\leq_{[n,m]} \Rightarrow \exists A \exists B (\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), A]} \ \& \ A \subseteq B)$
, ! 17 ($\Rightarrow I$: 2,16) i

$(\leq_{[n,m]} \Rightarrow \exists A \exists B (\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), A]} \ \& \ A \subseteq B))$
, ! 18 ($()E$: 17) i

$\forall n \forall m (\leq_{[n,m]} \Rightarrow \exists A \exists B (\mathcal{N}_{[m,B]} \ \& \ \mathcal{N}_{[(m-n), A]} \ \& \ A \subseteq B))$
! 19 ($\forall I$: 1,18) i

□