

! CHAPTER 4

THE GREATEST NATURAL NUMBER;

! This chapter introduces and develops the concept of the greatest natural number. It is very similar to the previous, given the evident duality between least and natural. There is, however, one major difference: the least natural number pre-supposes only a predicate with non-empty intersection with the natural numbers, while the greatest natural number assumes, in addition to this, that the intersection is finite.

The chapter begins by justifying (P1 and P2) the introduction (P3) of the greatest natural number operator. The most substantial subsequent propositions are late in the chapter: P22 dual of C3.18 (the greatest natural number of a union of predicates equals the larger greatest natural number of the predicates) and P24 dual to C3.21 (the greatest natural number of the finite interval (b _ c) is c).

! 1. i

$\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a])))$ i

P , ! 1 (Prem) i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 2 (Prem) i

$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x])))$
, ! 3 ($\forall E$: V3.75) i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x]))$
, ! 4 ($()E$: 3) i

$\exists x (\omega[x] \ \& \ P[x] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x]))$
, ! 5 ($\Rightarrow E$: 2,4) i

$(\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a]))$
, ! 6 ($\exists E$: 5) i

$\exists a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a]))$
, ! 7 ($\exists I$: 6) i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a]))$
, ! 8 ($\Rightarrow I$: 2,7) i

$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a])))$
, ! 9 ($()I$: 8) i

$\forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \exists a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a])))$
! 10 ($\forall I$: 1,9) i

□

! 2. i

⊢ $\forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \forall a \forall b ((\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a]))$
 $\ \& \ (\omega[b] \ \& \ P[b] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,b]))$
 $\Rightarrow a = b))$ i

P , ! 1 (Prem) i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 2 (Prem) i

a, b , ! 3 (Prem) i

$(\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a]))$
 $\ \& \ (\omega[b] \ \& \ P[b] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,b]))$
, ! 4 (Prem) i

$(\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a]))$
, ! 5 (&E: 4) i

$(\omega[b] \ \& \ P[b] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,b]))$
, ! 6 (&E: 4) i

$\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a])$
, ! 7 (()E: 5) i

$\omega[a] \ \& \ P[a]$, ! 8 (&E: 7) i

$\forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a])$, ! 9 (&E: 7) i

$\omega[b] \ \& \ P[b] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,b])$
, ! 10 (()E: 6) i

$\omega[b] \ \& \ P[b]$, ! 11 (&E: 10) i

$\forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,b])$, ! 12 (&E: 10) i

$(\omega[b] \ \& \ P[b] \Rightarrow \leq[b,a])$, ! 13 (\forall E: 9) i

$\omega[b] \ \& \ P[b] \Rightarrow \leq[b,a]$, ! 14 (()E: 13) i

$\leq[b,a]$, ! 15 (\Rightarrow E: 11,14) i

$(\omega[a] \ \& \ P[a] \Rightarrow \leq[a,b])$, ! 16 (\forall E: 12) i

$\omega[a] \ \& \ P[a] \Rightarrow \leq[a,b]$, ! 17 (()E: 16) i

$\leq[a,b]$, ! 18 (\Rightarrow E: 8,17) i

$\leq[a,b] \ \& \ \leq[b,a]$, ! 19 (&I: 15,18) i

$(\leq[a,b] \ \& \ \leq[b,a] \Rightarrow a = b)$, ! 20 (\forall E: V3.22) i

$\leq[a, b] \ \& \ \leq[b, a] \Rightarrow a = b$,! 21 ((E: 20) i

$a = b$,! 22 (\Rightarrow E: 19,21) i

($\omega[a] \ \& \ P[a] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a])$)
& ($\omega[b] \ \& \ P[b] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, b])$)
 $\Rightarrow a = b$
 ,! 23 (\Rightarrow I: 4,22) i

(($\omega[a] \ \& \ P[a] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a])$)
& ($\omega[b] \ \& \ P[b] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, b])$))
 $\Rightarrow a = b$)
 ,! 24 ((I: 23) i

$\forall a \forall b \ ((\omega[a] \ \& \ P[a] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]))$
& ($\omega[b] \ \& \ P[b] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, b])$))
 $\Rightarrow a = b$)
 ,! 25 (\forall I: 3,24) i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \forall a \forall b \ ((\omega[a] \ \& \ P[a] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]))$
& ($\omega[b] \ \& \ P[b] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, b])$))
 $\Rightarrow a = b$)
 ,! 26 (\Rightarrow I: 2,25) i

($\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$)
 $\Rightarrow \forall a \forall b \ ((\omega[a] \ \& \ P[a] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]))$
& ($\omega[b] \ \& \ P[b] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, b])$))
 $\Rightarrow a = b$))
 ,! 27 ((I: 26) i

$\forall P \ (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \forall a \forall b \ ((\omega[a] \ \& \ P[a] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]))$
& ($\omega[b] \ \& \ P[b] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, b])$))
 $\Rightarrow a = b$))
 ! 28 (\forall I: 1,27) i

□

! 3. i

$\mathbb{T} \ \chi ; (\chi^P) ; \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) ;$
($\omega[a] \ \& \ P[a] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a])$)
 ;! (\mathbb{T} D: P1, P2) i

! 4. i

$\vdash \forall P \ (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \omega[(\chi^P)] \ \& \ P[(\chi^P)] \ \& \ \forall y \ (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi^P)]))$ i

P ,! 1 (Prem) i

$\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P})$,! 2 (Prem) i
 $(\omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})]))$
, ! 3 ($\mathbb{T}\mathbb{I}$: P3,2) i
 $\omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})])$
, ! 4 ((E): 3) i
 $\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P})$
 $\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})])$
, ! 5 ($\Rightarrow\mathbb{I}$: 2,4) i
 $(\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P})$
 $\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})]))$
, ! 6 ((I): 5) i
 $\forall \mathbf{P} (\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P})$
 $\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})]))$
! 7 ($\forall\mathbb{I}$: 1,6) i
□
! 5. i
 $\vdash \forall \mathbf{P} (\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P}) \Rightarrow \omega[(\chi\mathbf{P})])$ i
 \mathbf{P} , ! 1 (Prem) i
 $\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P})$, ! 2 (Prem) i
 $(\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P})$
 $\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})]))$
, ! 3 ($\forall\mathbb{E}$: P4) i
 $\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P})$
 $\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})])$
, ! 4 ((E): 3) i
 $\omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} \ (\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y},(\chi\mathbf{P})])$
, ! 5 ($\Rightarrow\mathbb{E}$: 2,4) i
 $\omega[(\chi\mathbf{P})]$, ! 6 ($\&\mathbb{E}$: 5) i
 $\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P}) \Rightarrow \omega[(\chi\mathbf{P})]$, ! 7 ($\Rightarrow\mathbb{I}$: 2,6) i
 $(\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P}) \Rightarrow \omega[(\chi\mathbf{P})])$
, ! 8 ((I): 7) i
 $\forall \mathbf{P} (\neg (\omega \cap \mathbf{P}) \equiv \phi \ \& \ f \ (\omega \cap \mathbf{P}) \Rightarrow \omega[(\chi\mathbf{P})])$
! 9 ($\forall\mathbb{I}$: 1,8) i
□
! 6. i

$\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow P[(\chi P)])$ i
 P , ! 1 (Prem) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 2 (Prem) i
 $(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (\omega[Y] \ \& \ P[Y] \Rightarrow \leq[Y, (\chi P)]))$, ! 3 ($\forall E$: P4) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (\omega[Y] \ \& \ P[Y] \Rightarrow \leq[Y, (\chi P)])$, ! 4 ($()E$: 3) i
 $\omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (\omega[Y] \ \& \ P[Y] \Rightarrow \leq[Y, (\chi P)])$, ! 5 ($\Rightarrow E$: 2,4) i
 $P[(\chi P)]$, ! 6 ($\&E$: 5) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow P[(\chi P)]$, ! 7 ($\Rightarrow I$: 2,6) i
 $(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow P[(\chi P)])$, ! 8 ($()I$: 7) i
 $\forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow P[(\chi P)])$! 9 ($\forall I$: 1,8) i

□

! 7.

$\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$ i
 P , ! 1 (Prem) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 2 (Prem) i
 $(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (\omega[Y] \ \& \ P[Y] \Rightarrow \leq[Y, (\chi P)]))$, ! 3 ($\forall E$: P4) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (\omega[Y] \ \& \ P[Y] \Rightarrow \leq[Y, (\chi P)])$, ! 4 ($()E$: 3) i
 $\omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (\omega[Y] \ \& \ P[Y] \Rightarrow \leq[Y, (\chi P)])$, ! 5 ($\Rightarrow E$: 2,4) i
 $\omega[(\chi P)] \ \& \ P[(\chi P)]$, ! 6 ($\&I$: 5) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)]$, ! 7 ($\Rightarrow I$: 2,6) i

$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$, ! 8 (()I: 7)	i
$\forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$! 9 (\forall I: 1,8)	i
\square		
! 8.		
$\vdash \forall P \forall y (f (\omega \cap P) \ \& \ \omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$		i
P, y	, ! 1 (Prem)	i
$f (\omega \cap P) \ \& \ \omega[y] \ \& \ P[y]$, ! 2 (Prem)	i
$f (\omega \cap P)$, ! 3 (&E: 2)	i
$\omega[y] \ \& \ P[y]$, ! 4 (&E: 2)	i
$(\omega[y] \ \& \ P[y])$, ! 5 (()I: 4)	i
$\exists x (\omega[x] \ \& \ P[x])$, ! 6 (\exists I: 5)	i
$(\exists x (\omega[x] \ \& \ P[x]) \Rightarrow \neg (\omega \cap P) \equiv \phi)$, ! 7 (\forall E: II5.26)	i
$\exists x (\omega[x] \ \& \ P[x]) \Rightarrow \neg (\omega \cap P) \equiv \phi$, ! 8 (()E: 7)	i
$\neg (\omega \cap P) \equiv \phi$, ! 9 (\Rightarrow E: 6,8)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 10 (&I: 3,9)	i
$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)]))$, ! 11 (\forall E: P4)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$, ! 12 (()E: 11)	i
$\omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$, ! 13 (\Rightarrow E: 10,12)	i
$\forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$, ! 14 (&E: 13)	i
$(\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$, ! 15 (\forall E: 14)	i
$\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)]$, ! 16 (()E: 15)	i
$\leq[y, (\chi P)]$, ! 17 (\Rightarrow E: 4,16)	i
$f (\omega \cap P) \ \& \ \omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)]$, ! 18 (\Rightarrow I: 2,17)	i
$(f (\omega \cap P) \ \& \ \omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$		

	,! 19 ((I: 18)	i
$\forall P \forall y (f (\omega \cap P) \& \omega[y] \& P[y] \Rightarrow \leq[y, (\chi P)])$! 20 ($\forall I$: 1,19)	i
\square		
! 9.		i
$\vdash \forall P \forall y (\langle [(\chi P), y] \rangle \Rightarrow \neg P[y])$		i
P, y	,! 1 (Prem)	i
$\langle [(\chi P), y] \rangle$,! 2 (Prem)	i
$\neg (\omega \cap P) \equiv \phi \& f (\omega \cap P)$,! 3 ($\mathbb{T}E$: P3,2)	i
$f (\omega \cap P)$,! 4 ($\&E$: 3)	i
P[y]	,! 5 (Prem)	i
$f (\omega \cap P) \& P[y]$,! 6 ($\&I$: 4,5)	i
$(\langle [(\chi P), y] \rangle \Rightarrow \omega[y])$,! 7 ($\forall E$: V4.11; (χP): P3,3)	i
$\langle [(\chi P), y] \rangle \Rightarrow \omega[y]$,! 8 ((E): 7)	i
$\omega[y]$,! 9 ($\Rightarrow E$: 2,8)	i
$f (\omega \cap P) \& \omega[y] \& P[y]$,! 10 ($\&I$: 6,9)	i
$(f (\omega \cap P) \& \omega[y] \& P[y] \Rightarrow \leq[y, (\chi P)])$,! 11 ($\forall E$: P8)	i
$f (\omega \cap P) \& \omega[y] \& P[y] \Rightarrow \leq[y, (\chi P)]$,! 12 ((E): 11)	i
$\leq[y, (\chi P)]$,! 13 ($\Rightarrow E$: 10,12)	i
$\langle [(\chi P), y] \rangle \& \leq[y, (\chi P)]$,! 14 ($\&I$: 2,13)	i
$(\langle [(\chi P), y] \rangle \& \leq[y, (\chi P)] \Rightarrow \mathfrak{F})$,! 15 ($\forall E$: V4.19; (χP): P3,3)	i
$\langle [(\chi P), y] \rangle \& \leq[y, (\chi P)] \Rightarrow \mathfrak{F}$,! 16 ((E): 15)	i
\mathfrak{F}	,! 17 ($\Rightarrow E$: 14,16)	i
$P[y] \Rightarrow \mathfrak{F}$,! 18 ($\Rightarrow I$: 5,17)	i
$\neg P[y]$,! 19 ($\neg I$: 18)	i
$\langle [(\chi P), y] \rangle \Rightarrow \neg P[y]$,! 20 ($\Rightarrow I$: 2,19)	i

$(\leq[(\chi P), y] \Rightarrow \neg P[y])$, ! 21 (()I: 20)	i
$\forall P \forall y (\leq[(\chi P), y] \Rightarrow \neg P[y])$! 22 (\forall I: 1, 21)	i
\square		
! P10 to P13 concern the special case where the predicate is contained in the natural numbers.		
! 10.		
$\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P$		
$\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (P[y] \Rightarrow \leq[y, (\chi P)]))$		i
P	, ! 1 (Prem)	i
$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P$, ! 2 (Prem)	i
$(P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P))$, ! 3 (\forall E: IV5.26)	i
$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 4 (()E: 3)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 5 (\Rightarrow E: 2, 4)	i
$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$		
$\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)]))$, ! 6 (\forall E: P4)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$		
$\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$, ! 7 (()E: 6)	i
$\omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$, ! 8 (\Rightarrow E: 5, 7)	i
$\omega[(\chi P)] \ \& \ P[(\chi P)]$, ! 9 (\& E: 8)	i
$\forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, (\chi P)])$, ! 10 (\& E: 8)	i
y	, ! 11 (Prem)	i
$P[y]$, ! 12 (Prem)	i
$P \subseteq \omega$, ! 13 (\& E: 2)	i
$P[y] \ \& \ P \subseteq \omega$, ! 14 (\& I: 12, 13)	i
$(P[y] \ \& \ P \subseteq \omega \Rightarrow \omega[y])$, ! 15 (\forall E: III1.2)	i
$P[y] \ \& \ P \subseteq \omega \Rightarrow \omega[y]$, ! 16 (()E: 15)	i
$\omega[y]$, ! 17 (\Rightarrow E: 14, 16)	i

$\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}]$, ! 18 (&I: 12,17)	i
$(\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})])$, ! 19 (\forall E: 10)	i
$\omega[\mathbf{Y}] \ \& \ \mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})]$, ! 20 ((E): 19)	i
$\leq[\mathbf{Y}, (\chi\mathbf{P})]$, ! 21 (\Rightarrow E: 18,20)	i
$\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})]$, ! 22 (\Rightarrow I: 12,21)	i
$(\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})])$, ! 23 ((I): 22)	i
$\forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})])$, ! 24 (\forall I: 11,23)	i
$\omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})])$, ! 25 (&I: 9,24)	i
$\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P}$		
$\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})])$, ! 26 (\Rightarrow I: 2,25)	i
$(\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P}$		
$\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})]))$, ! 27 ((I): 26)	i
$\forall \mathbf{P} (\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P}$		
$\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})]))$! 28 (\forall I: 1,27)	i
\square		
! 11.		i
$\vdash \forall \mathbf{P} (\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P} \Rightarrow \omega[(\chi\mathbf{P})])$		i
\mathbf{P}	, ! 1 (Prem)	i
$\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P}$, ! 2 (Prem)	i
$(\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P}$		
$\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})]))$, ! 3 (\forall E: P10)	i
$\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P}$		
$\Rightarrow \omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})])$, ! 4 ((E): 3)	i
$\omega[(\chi\mathbf{P})] \ \& \ \mathbf{P}[(\chi\mathbf{P})] \ \& \ \forall \mathbf{Y} (\mathbf{P}[\mathbf{Y}] \Rightarrow \leq[\mathbf{Y}, (\chi\mathbf{P})])$, ! 5 (\Rightarrow E: 2,4)	i
$\omega[(\chi\mathbf{P})]$, ! 6 (&E: 5)	i
$\mathbf{P} \subseteq \omega \ \& \ \neg \mathbf{P} \equiv \phi \ \& \ f \ \mathbf{P} \Rightarrow \omega[(\chi\mathbf{P})]$, ! 7 (\Rightarrow I: 2,6)	i

$(P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)]) \quad ,! \ 8 \ ((I: 7) \quad i$

$\forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)]) \quad ! \ 9 \ (\forall I: 1,8) \quad i$

\square

! 12. i

$\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow P[(\chi P)]) \quad i$

$P \quad ,! \ 1 \ (\text{Prem}) \quad i$

$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \quad ,! \ 2 \ (\text{Prem}) \quad i$

$(P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (P[Y] \Rightarrow \leq[Y, (\chi P)]))$
 $\quad ,! \ 3 \ (\forall E: P10) \quad i$

$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (P[Y] \Rightarrow \leq[Y, (\chi P)])$
 $\quad ,! \ 4 \ ((E: 3) \quad i$

$\omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (P[Y] \Rightarrow \leq[Y, (\chi P)])$
 $\quad ,! \ 5 \ (\Rightarrow E: 2,4) \quad i$

$P[(\chi P)] \quad ,! \ 6 \ (\&E: 5) \quad i$

$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow P[(\chi P)] \quad ,! \ 7 \ (\Rightarrow I: 2,6) \quad i$

$(P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow P[(\chi P)]) \quad ,! \ 8 \ ((I: 7) \quad i$

$\forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow P[(\chi P)]) \quad ! \ 9 \ (\forall I: 1,8) \quad i$

\square

! 13. i

$\vdash \forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)]) \quad i$

$P \quad ,! \ 1 \ (\text{Prem}) \quad i$

$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \quad ,! \ 2 \ (\text{Prem}) \quad i$

$(P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (P[Y] \Rightarrow \leq[Y, (\chi P)]))$
 $\quad ,! \ 3 \ (\forall E: P10) \quad i$

$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P$
 $\Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (P[Y] \Rightarrow \leq[Y, (\chi P)])$
 $\quad ,! \ 4 \ ((E: 3) \quad i$

$\omega[(\chi P)] \ \& \ P[(\chi P)] \ \& \ \forall Y (P[Y] \Rightarrow \leq[Y, (\chi P)])$
 $\quad ,! \ 5 \ (\Rightarrow E: 2,4) \quad i$

$\omega[(\chi P)] \ \& \ P[(\chi P)]$, ! 6 (&E: 5)	i
$P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)]$, ! 7 (\Rightarrow I: 2,6)	i
$(P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$, ! 8 (()I: 7)	i
$\forall P (P \subseteq \omega \ \& \ \neg P \equiv \phi \ \& \ f P \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$! 9 (\forall I: 1,8)	i
\square		
! 14. i		
$\vdash \forall P \forall a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a]) \Rightarrow (\chi P) = a)$		i
P, a	, ! 1 (Prem)	i
$\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a])$, ! 2 (Prem)	i
$\omega[a] \ \& \ P[a]$, ! 3 (&E: 2)	i
$\forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,a])$, ! 4 (&E: 2)	i
$\exists x \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x])$, ! 5 (\exists I: 4)	i
$(\exists x \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x]) \Rightarrow f (\omega \cap P))$, ! 6 (\forall E: C2.64)	i
$\exists x \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y,x]) \Rightarrow f (\omega \cap P)$, ! 7 (()E: 6)	i
$f (\omega \cap P)$, ! 8 (\Rightarrow E: 5,7)	i
$f (\omega \cap P) \ \& \ \omega[a] \ \& \ P[a]$, ! 9 (&I: 3,8)	i
$(f (\omega \cap P) \ \& \ \omega[a] \ \& \ P[a] \Rightarrow \leq[a, (\chi P)])$, ! 10 (\forall E: P8)	i
$f (\omega \cap P) \ \& \ \omega[a] \ \& \ P[a] \Rightarrow \leq[a, (\chi P)]$, ! 11 (()E: 10)	i
$\leq[a, (\chi P)]$, ! 12 (\Rightarrow E: 9,11)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 13 (\mathbb{T} E: P3,12)	i
$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$, ! 14 (\forall E: P7)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)]$, ! 15 (()E: 14)	i

$\omega[(\chi P)] \ \& \ P[(\chi P)]$,! 16 ($\Rightarrow E$: 13,15) i
 $(\omega[(\chi P)] \ \& \ P[(\chi P)] \Rightarrow \leq[(\chi P), a])$,! 17 ($\forall E$: 4;
 (χP) : P3,13) i
 $\omega[(\chi P)] \ \& \ P[(\chi P)] \Rightarrow \leq[(\chi P), a]$,! 18 ($()E$: 17) i
 $\leq[(\chi P), a]$,! 19 ($\Rightarrow E$: 16,18) i
 $\leq[(\chi P), a] \ \& \ \leq[a, (\chi P)]$,! 20 ($\&I$: 12,19) i
 $(\leq[(\chi P), a] \ \& \ \leq[a, (\chi P)] \Rightarrow (\chi P) = a)$,! 21 ($\forall E$: V3.22;
 (χP) : P3,13) i
 $\leq[(\chi P), a] \ \& \ \leq[a, (\chi P)] \Rightarrow (\chi P) = a$,! 22 ($()E$: 21) i
 $(\chi P) = a$,! 23 ($\Rightarrow E$: 20,22) i
 $\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]) \Rightarrow (\chi P) = a$
,! 24 ($\Rightarrow I$: 2,23) i
 $(\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]) \Rightarrow (\chi P) = a)$
,! 25 ($()I$: 24) i
 $\forall P \forall a (\omega[a] \ \& \ P[a] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[y, a]) \Rightarrow (\chi P) = a)$
! 26 ($\forall I$: 1,25) i

□

! 15. i

$\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq[(\chi P), (\chi Q)])$ i
P, Q ,! 1 (Prem) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q$,! 2 (Prem) i
 $\neg (\omega \cap P) \equiv \phi$,! 3 ($\&E$: 2) i
 $f (\omega \cap Q)$,! 4 ($\&E$: 2) i
P \subseteq Q ,! 5 ($\&E$: 2) i
 $(P \subseteq Q \Rightarrow (\omega \cap P) \subseteq (\omega \cap Q))$,! 6 ($\forall E$: II3.32) i
P \subseteq Q \Rightarrow ($\omega \cap P$) \subseteq ($\omega \cap Q$) ,! 7 ($()E$: 6) i
 $(\omega \cap P) \subseteq (\omega \cap Q)$,! 8 ($\Rightarrow E$: 5,7) i
 $f (\omega \cap Q) \ \& \ (\omega \cap P) \subseteq (\omega \cap Q)$,! 9 ($\&I$: 4,8) i
 $(f (\omega \cap Q) \ \& \ (\omega \cap P) \subseteq (\omega \cap Q) \Rightarrow f (\omega \cap P))$
,! 10 ($\forall E$: IV5.12) i

$(\omega \cap P) \subseteq (\omega \cap Q) \ \& \ f(\omega \cap Q) \Rightarrow f(\omega \cap P)$,! 11 ((E: 10)	i
$f(\omega \cap P)$,! 12 (\Rightarrow E: 9,11)	i
$\neg(\omega \cap P) \equiv \phi \ \& \ f(\omega \cap P)$,! 13 (&I: 3,12)	i
$(\neg(\omega \cap P) \equiv \phi \ \& \ f(\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)])$,! 14 (\forall E: P7)	i
$\neg(\omega \cap P) \equiv \phi \ \& \ f(\omega \cap P) \Rightarrow \omega[(\chi P)] \ \& \ P[(\chi P)]$,! 15 ((E: 14)	i
$\omega[(\chi P)] \ \& \ P[(\chi P)]$,! 16 (\Rightarrow E: 13,15)	i
$\omega[(\chi P)]$,! 17 (&E: 16)	i
$P[(\chi P)]$,! 18 (&E: 16)	i
$f(\omega \cap Q) \ \& \ \omega[(\chi P)]$,! 19 (&I: 4,17)	i
$P[(\chi P)] \ \& \ P \subseteq Q$,! 20 (&I: 5,18)	i
$(P[(\chi P)] \ \& \ P \subseteq Q \Rightarrow Q[(\chi P)])$,! 21 (\forall E: III.2; (χP): P3,13)	i
$P[(\chi P)] \ \& \ P \subseteq Q \Rightarrow Q[(\chi P)]$,! 22 ((E: 21)	i
$Q[(\chi P)]$,! 23 (\Rightarrow E: 20,22)	i
$f(\omega \cap Q) \ \& \ \omega[(\chi P)] \ \& \ Q[(\chi P)]$,! 24 (&I: 19,23)	i
$(f(\omega \cap Q) \ \& \ \omega[(\chi P)] \ \& \ Q[(\chi P)] \Rightarrow \leq[(\chi P), (\chi Q)])$,! 25 (\forall E: P8; (χP): P3,13)	i
$f(\omega \cap Q) \ \& \ \omega[(\chi P)] \ \& \ Q[(\chi P)] \Rightarrow \leq[(\chi P), (\chi Q)]$,! 26 ((E: 25)	i
$\leq[(\chi P), (\chi Q)]$,! 27 (\Rightarrow E: 24,26)	i
$\neg(\omega \cap P) \equiv \phi \ \& \ f(\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq[(\chi P), (\chi Q)]$,! 28 (\Rightarrow I: 2,27)	i
$(\neg(\omega \cap P) \equiv \phi \ \& \ f(\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq[(\chi P), (\chi Q)])$,! 29 ((I: 28)	i
$\forall P \forall Q (\neg(\omega \cap P) \equiv \phi \ \& \ f(\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq[(\chi P), (\chi Q)])$! 30 (\forall I: 1,29)	i

□

$\vdash \forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q))$		
P, Q	,! 1 (Prem)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q$,! 2 (Prem)	i
$\neg (\omega \cap P) \equiv \phi$,! 3 (&E)	i
$f (\omega \cap P)$,! 4 (&E)	i
$P \equiv Q$,! 5 (&E)	i
$(P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P)$,! 6 (\forall E: III.1.13)	i
$P \equiv Q \Rightarrow P \subseteq Q \ \& \ Q \subseteq P$,! 7 (()E: 6)	i
$P \subseteq Q \ \& \ Q \subseteq P$,! 8 (\Rightarrow E: 5,7)	i
$P \subseteq Q$,! 9 (&E: 8)	i
$Q \subseteq P$,! 10 (&E: 8)	i
$(P \equiv Q \Rightarrow (\omega \cap P) \equiv (\omega \cap Q))$,! 11 (\forall E: III.3.35)	i
$P \equiv Q \Rightarrow (\omega \cap P) \equiv (\omega \cap Q)$,! 12 (()E: 11)	i
$(\omega \cap P) \equiv (\omega \cap Q)$,! 13 (\Rightarrow E: 5,12)	i
$f (\omega \cap P) \ \& \ (\omega \cap P) \equiv (\omega \cap Q)$,! 14 (&I: 4,13)	i
$(f (\omega \cap P) \ \& \ (\omega \cap P) \equiv (\omega \cap Q) \Rightarrow f (\omega \cap Q))$,! 15 (\forall E: IV.5.5)	i
$f (\omega \cap P) \ \& \ (\omega \cap P) \equiv (\omega \cap Q) \Rightarrow f (\omega \cap Q)$,! 16 (()E: 15)	i
$f (\omega \cap Q)$,! 17 (\Rightarrow E: 14,16)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q)$,! 18 (&I: 3,17)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q$,! 19 (&I: 9,18)	i
$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq [(\chi P), (\chi Q)])$,! 20 (\forall E: P15)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq [(\chi P), (\chi Q)]$,! 21 (()E: 20)	i
$\leq [(\chi P), (\chi Q)]$,! 22 (\Rightarrow E: 19,21)	i
$(\omega \cap P) \equiv (\omega \cap Q) \ \& \ \neg (\omega \cap P) \equiv \phi$,! 23 (&E: 3,13)	i
$((\omega \cap P) \equiv (\omega \cap Q) \ \& \ \neg (\omega \cap P) \equiv \phi \Rightarrow \neg (\omega \cap Q) \equiv \phi)$,! 24 (\forall E: III.1.42)	i

$$\begin{array}{ll}
(\omega \cap P) \equiv (\omega \cap Q) \ \& \ \neg (\omega \cap P) \equiv \phi \Rightarrow \neg (\omega \cap Q) \equiv \phi & ,! \ 25 \ (\ ()E: \ 24) \quad i \\
\neg (\omega \cap Q) \equiv \phi & ,! \ 26 \ (\Rightarrow E: \ 23,25) \quad i \\
\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap P) & ,! \ 27 \ (\&I: \ 4,26) \quad i \\
\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \subseteq P & ,! \ 28 \ (\&I: \ 10,27) \quad i \\
(\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \subseteq P \Rightarrow \leq[(\chi Q), (\chi P)]) & ,! \ 29 \ (\forall E: \ P15) \quad i \\
\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \subseteq P \Rightarrow \leq[(\chi Q), (\chi P)] & ,! \ 30 \ (\ ()E: \ 29) \quad i \\
\leq[(\chi Q), (\chi P)] & ,! \ 31 \ (\Rightarrow E: \ 28,30) \quad i \\
\leq[(\chi P), (\chi Q)] \ \& \ \leq[(\chi Q), (\chi P)] & ,! \ 32 \ (\&I: \ 22,32) \quad i \\
\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) & ,! \ 33 \ (\&I: \ 3,4) \quad i \\
\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap Q) & ,! \ 34 \ (\&I: \ 17,26) \quad i \\
(\leq[(\chi P), (\chi Q)] \ \& \ \leq[(\chi Q), (\chi P)] \Rightarrow (\chi P) = (\chi Q)) & ,! \ 35 \ (\forall E: \ V3.22; \\
& (\chi P): \ P3,33; \\
& (\chi Q): \ P3,34) \quad i \\
\leq[(\chi P), (\chi Q)] \ \& \ \leq[(\chi Q), (\chi P)] \Rightarrow (\chi P) = (\chi Q) & ,! \ 36 \ (\ ()E: \ 35) \quad i \\
(\chi P) = (\chi Q) & ,! \ 37 \ (\Rightarrow E: \ 32,36) \quad i \\
\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q) & ,! \ 38 \ (\Rightarrow I: \ 2,37) \quad i \\
(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q)) & ,! \ 39 \ (\ ()I: \ 38) \quad i \\
\forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q)) & ! \ 40 \ (\forall I: \ 1,39) \quad i
\end{array}$$

□

! 17. i

⊢ $\forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \equiv P \Rightarrow (\chi P) = (\chi Q))$ i

P, Q ,! \ 1 \ (Prem) \quad i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \equiv P$,! \ 2 \ (Prem) \quad i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$,! \ 3 \ (\&E: \ 2) \quad i

$Q \equiv P$, ! 4 (&E: 2)	i
$(Q \equiv P \Rightarrow P \equiv Q)$, ! 5 (\forall E: III.1.10)	i
$Q \equiv P \Rightarrow P \equiv Q$, ! 6 (()E: 5)	i
$P \equiv Q$, ! 7 (\Rightarrow E: 4,6)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q$, ! 8 (&I: 3,7)	i
$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q))$, ! 9 (\forall E: P16)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q)$, ! 10 (()E: 9)	i
$(\chi P) = (\chi Q)$, ! 11 (\Rightarrow E: 8,10)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \equiv P \Rightarrow (\chi P) = (\chi Q)$, ! 12 (\Rightarrow I: 2,11)	i
$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \equiv P \Rightarrow (\chi P) = (\chi Q))$, ! 13 (()I: 12)	i
$\forall P \forall Q (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ Q \equiv P \Rightarrow (\chi P) = (\chi Q))$! 14 (\forall I: 13)	i

□

! 18.

$\vdash \forall P \forall Q \forall n ((\chi P) = n \ \& \ P \equiv Q \Rightarrow (\chi Q) = n)$		i
P, Q, n	, ! 1 (Prem)	i
$(\chi P) = n \ \& \ P \equiv Q$, ! 2 (Prem)	i
$(\chi P) = n$, ! 3 (&E: 2)	i
$P \equiv Q$, ! 4 (&E: 2)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$, ! 5 (\mathbb{T} E: P3,3)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q$, ! 6 (&I: 4,5)	i
$(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q))$, ! 7 (\forall E: P16)	i
$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \ \& \ P \equiv Q \Rightarrow (\chi P) = (\chi Q)$, ! 8 (()E: 7)	i
$(\chi P) = (\chi Q)$, ! 9 (\Rightarrow E: 6,8)	i

$(\chi Q) = n$, ! 10 (=E: 3,9)	i
$(\chi P) = n \ \& \ P \equiv Q \Rightarrow (\chi Q) = n$, ! 11 (\Rightarrow I: 2,10)	i
$((\chi P) = n \ \& \ P \equiv Q \Rightarrow (\chi Q) = n)$, ! 12 (()I: 11)	i
$\forall P \forall Q \forall n ((\chi P) = n \ \& \ P \equiv Q \Rightarrow (\chi Q) = n)$! 13 (\forall I: 1,12)	i
\square		

! 19.

$\vdash \forall P \forall Q \forall n ((\chi P) = n \ \& \ Q \equiv P \Rightarrow (\chi Q) = n)$		i
P, Q, n	, ! 1 (Prem)	i
$(\chi P) = n \ \& \ Q \equiv P$, ! 2 (Prem)	i
$(\chi P) = n$, ! 3 (&E: 2)	i
$Q \equiv P$, ! 4 (&E: 2)	i
$(Q \equiv P \Rightarrow P \equiv Q)$, ! 5 (\forall E: III.1.10)	i
$Q \equiv P \Rightarrow P \equiv Q$, ! 6 (()E: 5)	i
$P \equiv Q$, ! 7 (\Rightarrow E: 4,6)	i
$(\chi P) = n \ \& \ P \equiv Q$, ! 8 (&I: 3,7)	i
$((\chi P) = n \ \& \ P \equiv Q \Rightarrow (\chi Q) = n)$, ! 9 (\forall E: P18)	i
$(\chi P) = n \ \& \ P \equiv Q \Rightarrow (\chi Q) = n$, ! 10 (()E: 9)	i
$(\chi Q) = n$, ! 11 (\Rightarrow E: 8,10)	i
$(\chi P) = n \ \& \ Q \equiv P \Rightarrow (\chi Q) = n$, ! 12 (\Rightarrow I: 2,11)	i
$((\chi P) = n \ \& \ Q \equiv P \Rightarrow (\chi Q) = n)$, ! 13 (()I: 12)	i
$\forall P \forall Q \forall n ((\chi P) = n \ \& \ Q \equiv P \Rightarrow (\chi Q) = n)$! 14 (\forall I: 1,13)	i
\square		

! 20.

$\vdash \forall P \forall Q (P \subseteq Q \ \& \ P[(\chi Q)] \Rightarrow (\chi P) = (\chi Q))$		i
P, Q	, ! 1 (Prem)	i
$P \subseteq Q \ \& \ P[(\chi Q)]$, ! 2 (Prem)	i
$P \subseteq Q$, ! 3 (&E: 2)	i
$P[(\chi Q)]$, ! 4 (&E: 2)	i

$\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap Q)$,! 5 (**TE**: P3,4) i
 $f (\omega \cap Q)$,! 6 (**&E**: 5) i
 $(P \subseteq Q \Rightarrow (\omega \cap P) \subseteq (\omega \cap Q))$,! 7 (**$\forall E$** : II3.32) i
 $P \subseteq Q \Rightarrow (\omega \cap P) \subseteq (\omega \cap Q)$,! 8 (**()E**: 7) i
 $(\omega \cap P) \subseteq (\omega \cap Q)$,! 9 (**$\Rightarrow E$** : 3,8) i
 $f (\omega \cap Q) \ \& \ (\omega \cap P) \subseteq (\omega \cap Q)$,! 10 (**&I**: 6,9) i
 $(f (\omega \cap Q) \ \& \ (\omega \cap P) \subseteq (\omega \cap Q) \Rightarrow f (\omega \cap P))$
,! 11 (**$\forall E$** : II3.12) i
 $f (\omega \cap Q) \ \& \ (\omega \cap P) \subseteq (\omega \cap Q) \Rightarrow f (\omega \cap P)$
,! 12 (**()E**: 11) i
 $f (\omega \cap P)$,! 13 (**$\Rightarrow E$** : 10,12) i
 $f (\omega \cap P) \ \& \ P[(\chi Q)]$,! 14 (**&I**: 4,13) i
 $(\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap Q) \Rightarrow \omega[(\chi Q)])$
,! 15 (**$\forall E$** : P5) i
 $\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap Q) \Rightarrow \omega[(\chi Q)]$,! 16 (**()E**: 15) i
 $\omega[(\chi Q)]$,! 17 (**$\Rightarrow E$** : 5,16) i
 $f (\omega \cap P) \ \& \ \omega[(\chi Q)] \ \& \ P[(\chi Q)]$,! 18 (**&I**: 14,17) i
 $(f (\omega \cap P) \ \& \ \omega[(\chi Q)] \ \& \ P[(\chi Q)] \Rightarrow \leq[(\chi Q), (\chi P)])$
,! 19 (**$\forall E$** : P8;
 (χQ) : P3,5) i
 $f (\omega \cap P) \ \& \ \omega[(\chi Q)] \ \& \ P[(\chi Q)] \Rightarrow \leq[(\chi Q), (\chi P)]$
,! 20 (**()E**: 19) i
 $\leq[(\chi Q), (\chi P)]$,! 21 (**$\Rightarrow E$** : 18,20) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$,! 22 (**TE**: P3,21) i
 $\neg (\omega \cap P) \equiv \phi$,! 23 (**&E**: 22) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q)$,! 24 (**&I**: 6,23) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q$,! 25 (**&I**: 3,24) i
 $(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq[(\chi P), (\chi Q)])$
,! 26 (**$\forall E$** : P15) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap Q) \ \& \ P \subseteq Q \Rightarrow \leq[(\chi P), (\chi Q)]$

	,!	27	((E: 26)	i
$\leq[(\chi^P), (\chi^Q)]$,!	28	(\Rightarrow E: 25,27)	i
$\leq[(\chi^P), (\chi^Q)] \ \& \ \leq[(\chi^Q), (\chi^P)]$,!	29	(&I: 21,28)	i
$(\leq[(\chi^P), (\chi^Q)] \ \& \ \leq[(\chi^Q), (\chi^P)] \Rightarrow (\chi^P) = (\chi^Q))$,!	30	(\forall E: v3.22; (χ^P): P3,22; (χ^Q): P3,5)	i
$\leq[(\chi^P), (\chi^Q)] \ \& \ \leq[(\chi^Q), (\chi^P)] \Rightarrow (\chi^P) = (\chi^Q)$,!	31	((E: 30)	i
$(\chi^P) = (\chi^Q)$,!	32	(\Rightarrow E: 29,31)	i
$P \subseteq Q \ \& \ P[(\chi^Q)] \Rightarrow (\chi^P) = (\chi^Q)$,!	33	(\Rightarrow I: 2,32)	i
$(P \subseteq Q \ \& \ P[(\chi^Q)] \Rightarrow (\chi^P) = (\chi^Q))$,!	34	((I: 33)	i
$\forall P \forall Q (P \subseteq Q \ \& \ P[(\chi^Q)] \Rightarrow (\chi^P) = (\chi^Q))$!	35	(\forall I: 1,34)	i
\square				
! 21.				i
$\vdash \forall P \forall Q \forall n (P \subseteq Q \ \& \ P[n] \ \& \ (\chi^Q) = n \Rightarrow (\chi^P) = n)$				i
P, Q, n	,!	1	(Prem)	i
$P \subseteq Q \ \& \ P[n] \ \& \ (\chi^Q) = n$,!	2	(Prem)	i
$P \subseteq Q$,!	3	(&E: 2)	i
$P[n]$,!	4	(&E: 2)	i
$(\chi^Q) = n$,!	5	(&E: 2)	i
$P[(\chi^Q)]$,!	6	(=E: 4,5)	i
$P \subseteq Q \ \& \ P[(\chi^Q)]$,!	7	(&I: 3,6)	i
$(P \subseteq Q \ \& \ P[(\chi^Q)] \Rightarrow (\chi^P) = (\chi^Q))$,!	8	(\forall E: P20)	i
$P \subseteq Q \ \& \ P[(\chi^Q)] \Rightarrow (\chi^P) = (\chi^Q)$,!	9	((E: 8)	i
$(\chi^P) = (\chi^Q)$,!	10	(\Rightarrow E: 7,9)	i
$P \subseteq Q \ \& \ P[n] \ \& \ (\chi^Q) = n \Rightarrow (\chi^P) = (\chi^Q)$,!	11	(\Rightarrow I: 2,10)	i
$(P \subseteq Q \ \& \ P[n] \ \& \ (\chi^Q) = n \Rightarrow (\chi^P) = (\chi^Q))$,!	12	((I: 11)	i
$\forall P \forall Q \forall n (P \subseteq Q \ \& \ P[n] \ \& \ (\chi^Q) = n \Rightarrow (\chi^P) = n)$!	13	(\forall I: 1,12)	i

□

! 22.

⊢ $\forall P \forall Q (\leq[(\chi P), (\chi Q)] \Rightarrow (\chi(P \cup Q)) = (\chi Q))$

P, Q ,! 1 (Prem) i

$\leq[(\chi P), (\chi Q)]$,! 2 (Prem) i

$\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$,! 3 (TE: P3,2) i

$\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap Q)$,! 4 (TE: P3,2) i

$(\omega[(\chi Q)] \ \& \ (P \cup Q)[(\chi Q)]$
 $\ \& \ \forall Y (\omega[Y] \ \& \ (P \cup Q)[Y] \Rightarrow \leq[Y, (\chi Q)])$
 $\Rightarrow (\chi(P \cup Q)) = (\chi Q))$

,! 5 ($\forall E$: P14;
 (χQ) : P3,4) i

$\omega[(\chi Q)] \ \& \ (P \cup Q)[(\chi Q)] \ \& \ \forall Y (\omega[Y] \ \& \ (P \cup Q)[Y] \Rightarrow \leq[Y, (\chi Q)])$
 $\Rightarrow (\chi(P \cup Q)) = (\chi Q)$

,! 6 ($(\)E$: 5) i

! To show: $\omega[(\chi Q)]$ i

$(\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap Q)$
 $\Rightarrow \omega[(\chi Q)] \ \& \ Q[(\chi Q)] \ \& \ \forall Y (\omega[Y] \ \& \ Q[Y] \Rightarrow \leq[Y, (\chi Q)]))$

,! 7 ($\forall E$: P4) i

$\neg (\omega \cap Q) \equiv \phi \ \& \ f (\omega \cap Q)$

$\Rightarrow \omega[(\chi Q)] \ \& \ Q[(\chi Q)] \ \& \ \forall Y (\omega[Y] \ \& \ Q[Y] \Rightarrow \leq[Y, (\chi Q)])$

,! 8 ($(\)E$: 7) i

$\omega[(\chi Q)] \ \& \ Q[(\chi Q)] \ \& \ \forall Y (\omega[Y] \ \& \ Q[Y] \Rightarrow \leq[Y, (\chi Q)])$

,! 9 ($\Rightarrow E$: 4,8) i

$\omega[(\chi Q)]$

,! 10 ($\&E$: 9) i

! To show: $(P \cup Q)[(\chi Q)]$ i

$Q[(\chi Q)]$

,! 11 ($\&E$: 9) i

$(Q[(\chi Q)] \Rightarrow (P \cup Q)[(\chi Q)])$

,! 12 ($\forall E$: II2.6;
 (χQ) : P3,4) i

$Q[(\chi Q)] \Rightarrow (P \cup Q)[(\chi Q)]$

,! 13 ($(\)E$: 12) i

$(P \cup Q)[(\chi Q)]$

,! 14 ($\Rightarrow E$: 11,13) i

! To show: $\forall Y (\omega[Y] \ \& \ (P \cup Q)[Y] \Rightarrow \leq[Y, (\chi Q)])$ i

\mathbf{y}	,! 15 (Prem)	i
$\omega[\mathbf{y}] \ \& \ (\mathbf{P} \cup \mathbf{Q})[\mathbf{y}]$,! 16 (Prem)	i
$\omega[\mathbf{y}]$,! 17 (&E: 16)	i
$(\mathbf{P} \cup \mathbf{Q})[\mathbf{y}]$,! 18 (&E: 16)	i
$((\mathbf{P} \cup \mathbf{Q})[\mathbf{y}] \Rightarrow \mathbf{P}[\mathbf{y}] \vee \mathbf{Q}[\mathbf{y}])$,! 19 (\forall E: II2.3)	i
$(\mathbf{P} \cup \mathbf{Q})[\mathbf{y}] \Rightarrow \mathbf{P}[\mathbf{y}] \vee \mathbf{Q}[\mathbf{y}]$,! 20 (()E: 19)	i
$\mathbf{P}[\mathbf{y}] \vee \mathbf{Q}[\mathbf{y}]$,! 21 (\Rightarrow E: 18,20)	i
$\mathbf{P}[\mathbf{y}]$,! 22 (Prem)	i
$f (\omega \cap \mathbf{P})$,! 23 (&E: 3)	i
$f (\omega \cap \mathbf{P}) \ \& \ \mathbf{P}[\mathbf{y}]$,! 24 (&I: 22,23)	i
$f (\omega \cap \mathbf{P}) \ \& \ \omega[\mathbf{y}] \ \& \ \mathbf{P}[\mathbf{y}]$,! 25 (&I: 17,24)	i
$(f (\omega \cap \mathbf{P}) \ \& \ \omega[\mathbf{y}] \ \& \ \mathbf{P}[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{P})])$,! 26 (\forall E: P8)	i
$f (\omega \cap \mathbf{P}) \ \& \ \omega[\mathbf{y}] \ \& \ \mathbf{P}[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{P})]$,! 27 (()E: 26)	i
$\leq[\mathbf{y}, (\chi\mathbf{P})]$,! 28 (\Rightarrow E: 25,27)	i
$\leq[\mathbf{y}, (\chi\mathbf{P})] \ \& \ \leq[(\chi\mathbf{P}), (\chi\mathbf{Q})]$,! 29 (&I: 2,28)	i
$(\leq[\mathbf{y}, (\chi\mathbf{P})] \ \& \ \leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})])$,! 30 (\forall E: V3.20; ($\chi\mathbf{P}$): P3,3; ($\chi\mathbf{Q}$): P3,4)	i
$\leq[\mathbf{y}, (\chi\mathbf{P})] \ \& \ \leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})]$,! 31 (()E: 30)	i
$\leq[\mathbf{y}, (\chi\mathbf{Q})]$,! 32 (\Rightarrow E: 29,31)	i
$\mathbf{P}[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})]$,! 33 (\Rightarrow I: 22,32)	i
$\mathbf{Q}[\mathbf{y}]$,! 34 (Prem)	i
$\omega[\mathbf{y}] \ \& \ \mathbf{Q}[\mathbf{y}]$,! 35 (&I: 17,34)	i
$\forall \mathbf{y} (\omega[\mathbf{y}] \ \& \ \mathbf{Q}[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})])$,! 36 (&E: 9)	i
$(\omega[\mathbf{y}] \ \& \ \mathbf{Q}[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})])$,! 37 (\forall E: 36)	i
$\omega[\mathbf{y}] \ \& \ \mathbf{Q}[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})]$,! 38 (()E: 37)	i

$\leq[\mathbf{y}, (\chi\mathbf{Q})]$,!	39 ($\Rightarrow\text{E}$: 35,38)	i
$\mathbf{Q}[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})]$,!	40 ($\Rightarrow\text{I}$: 34,39)	i
$\leq[\mathbf{y}, (\chi\mathbf{Q})]$,!	41 ($\vee\text{E}$: 21,33,40)	i
$\omega[\mathbf{y}] \ \& \ (\mathbf{P} \cup \mathbf{Q})[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})]$,!	42 ($\Rightarrow\text{I}$: 16,41)	i
$(\omega[\mathbf{y}] \ \& \ (\mathbf{P} \cup \mathbf{Q})[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})])$,!	43 ($(\)\text{I}$: 42)	i
$\forall\mathbf{y} (\omega[\mathbf{y}] \ \& \ (\mathbf{P} \cup \mathbf{Q})[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})])$,!	44 ($\forall\text{I}$: 15,41)	i
! Conclusion.			i
$\omega[(\chi\mathbf{Q})] \ \& \ (\mathbf{P} \cup \mathbf{Q})[(\chi\mathbf{Q})]$,!	45 ($\&\text{I}$: 10,14)	i
$\omega[(\chi\mathbf{Q})] \ \& \ (\mathbf{P} \cup \mathbf{Q})[(\chi\mathbf{Q})] \ \& \ \forall\mathbf{y} (\omega[\mathbf{y}] \ \& \ (\mathbf{P} \cup \mathbf{Q})[\mathbf{y}] \Rightarrow \leq[\mathbf{y}, (\chi\mathbf{Q})])$,!	46 ($\&\text{I}$: 44,45)	i
$(\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q})$,!	47 ($\Rightarrow\text{E}$: 6,46)	i
$\leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow (\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q})$,!	48 ($\Rightarrow\text{I}$: 2,47)	i
$(\leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow (\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q}))$,!	49 ($(\)\text{I}$: 48)	i
$\forall\mathbf{P}\forall\mathbf{Q} (\leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow (\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q}))$!	50 ($\forall\text{I}$: 1,49)	i
\square			
! 23.			i
$\vdash \forall\mathbf{P}\forall\mathbf{Q} (\leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow (\chi(\mathbf{Q} \cup \mathbf{P})) = (\chi\mathbf{Q}))$			i
\mathbf{P}, \mathbf{Q}	,!	1 (Prem)	i
$\leq[(\chi\mathbf{P}), (\chi\mathbf{Q})]$,!	2 (Prem)	i
$(\leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow (\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q}))$,!	3 ($\forall\text{E}$: P22)	i
$\leq[(\chi\mathbf{P}), (\chi\mathbf{Q})] \Rightarrow (\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q})$,!	4 ($(\)\text{E}$: 3)	i
$(\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q})$,!	5 ($\Rightarrow\text{E}$: 2,4)	i
$(\mathbf{P} \cup \mathbf{Q}) \equiv (\mathbf{Q} \cup \mathbf{P})$,!	6 ($\forall\text{E}$: II2.16)	i
$(\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q}) \ \& \ (\mathbf{P} \cup \mathbf{Q}) \equiv (\mathbf{Q} \cup \mathbf{P})$,!	7 ($\&\text{I}$: 5,6)	i
$\neg (\omega \cap \mathbf{Q}) \equiv \phi \ \& \ f (\omega \cap \mathbf{Q})$,!	8 (TE : P3,2)	i
$(\chi(\mathbf{P} \cup \mathbf{Q})) = (\chi\mathbf{Q}) \ \& \ (\mathbf{P} \cup \mathbf{Q}) \equiv (\mathbf{Q} \cup \mathbf{P})$ $\Rightarrow (\chi(\mathbf{Q} \cup \mathbf{P})) = (\chi\mathbf{Q})$			

,! 9 ($\forall E$: P18;
 (χQ) : P3,8) i

$(\chi(P \cup Q)) = (\chi Q) \ \& \ (P \cup Q) \equiv (Q \cup P)$
 $\Rightarrow (\chi(Q \cup P)) = (\chi Q)$,! 10 ($(\)E$: 9) i

$(\chi(Q \cup P)) = (\chi Q)$,! 11 ($\Rightarrow E$: 7,10) i

$\leq[(\chi P), (\chi Q)] \Rightarrow (\chi(Q \cup P)) = (\chi Q)$,! 12 ($\Rightarrow I$: 2,11) i

$(\leq[(\chi P), (\chi Q)] \Rightarrow (\chi(Q \cup P)) = (\chi Q))$,! 13 ($(\)I$: 12) i

$\forall P \forall Q (\leq[(\chi P), (\chi Q)] \Rightarrow (\chi(Q \cup P)) = (\chi Q))$
 ! 14 ($\forall I$: 1,13) i

□

! 24. i

$\vdash \forall b \forall c (\leq[b, c] \Rightarrow (\chi(b _ c)) = c)$ i

b, c ,! 1 (Prem) i

$\leq[b, c]$,! 2 (Prem) i

$(\leq[b, c] \Rightarrow (b _ c)[c])$,! 3 ($\forall E$: C2.8) i

$\leq[b, c] \Rightarrow (b _ c)[c]$,! 4 ($(\)E$: 3) i

$(b _ c)[c]$,! 5 ($\Rightarrow E$: 2,4) i

$(\leq[b, c] \Rightarrow \omega[c])$,! 6 ($\forall E$: V3.7) i

$\leq[b, c] \Rightarrow \omega[c]$,! 7 ($(\)E$: 6) i

$\omega[c]$,! 8 ($\Rightarrow E$: 2,7) i

$\omega[c] \ \& \ (b _ c)[c]$,! 9 ($\&I$: 5,8) i

y ,! 10 (Prem) i

$\omega[y] \ \& \ (b _ c)[y]$,! 11 (Prem) i

$(b _ c)[y]$,! 12 ($\&E$: 11) i

$((b _ c)[y] \Rightarrow \leq[y, c])$,! 13 ($\forall E$: C2.6) i

$(b _ c)[y] \Rightarrow \leq[y, c]$,! 14 ($(\)E$: 13) i

$\leq[y, c]$,! 15 ($\Rightarrow E$: 12,14) i

$\omega[y] \ \& \ (b _ c)[y] \Rightarrow \leq[y, c]$,! 16 ($\Rightarrow I$: 11,15) i

$(\omega[y] \ \& \ (\mathbf{b} _ \mathbf{c})[y] \Rightarrow \leq[y, \mathbf{c}])$,! 17 ((I: 16)	i
$\forall y (\omega[y] \ \& \ (\mathbf{b} _ \mathbf{c})[y] \Rightarrow \leq[y, \mathbf{c}])$,! 18 (\forall I: 10,17)	i
$\omega[\mathbf{c}] \ \& \ (\mathbf{b} _ \mathbf{c})[\mathbf{c}] \ \& \ \forall y (\omega[y] \ \& \ (\mathbf{b} _ \mathbf{c})[y] \Rightarrow \leq[y, \mathbf{c}])$,! 19 ($\&$ I: 9,18)	i
$(\ \omega[\mathbf{c}] \ \& \ (\mathbf{b} _ \mathbf{c})[\mathbf{c}] \ \& \ \forall y (\omega[y] \ \& \ (\mathbf{b} _ \mathbf{c})[y] \Rightarrow \leq[y, \mathbf{c}])$ $\Rightarrow (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c})$,! 20 (\forall E: P14)	i
$\omega[\mathbf{c}] \ \& \ (\mathbf{b} _ \mathbf{c})[\mathbf{c}] \ \& \ \forall y (\omega[y] \ \& \ (\mathbf{b} _ \mathbf{c})[y] \Rightarrow \leq[y, \mathbf{c}])$ $\Rightarrow (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c}$,! 21 ((E: 20)	i
$(\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c}$,! 22 (\Rightarrow E: 19,21)	i
$\leq[\mathbf{b}, \mathbf{c}] \Rightarrow (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c}$,! 23 (\Rightarrow I: 2,22)	i
$(\ \leq[\mathbf{b}, \mathbf{c}] \Rightarrow (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c})$,! 24 ((I: 23)	i
$\forall b \forall c (\ \leq[b, c] \Rightarrow (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c})$! 25 (\forall I: 1,24)	i
\square		
! 25.		i
$\vdash \forall P \forall b \forall c (\ P \subseteq (\mathbf{b} _ \mathbf{c}) \ \& \ P[\mathbf{c}] \Rightarrow (\chi P) = \mathbf{c})$		i
P, b, c	,! 1 (Prem)	i
P \subseteq (b _ c) & P[c]	,! 2 (Prem)	i
P[c]	,! 3 ($\&$ E: 2)	i
P \subseteq (b _ c)	,! 4 ($\&$ E: 2)	i
P[c] & P \subseteq (b _ c)	,! 5 ($\&$ I: 3,4)	i
(P[c] & P \subseteq (b _ c) \Rightarrow (b _ c)[c])	,! 6 (\forall E: II1.2)	i
P[c] & P \subseteq (b _ c) \Rightarrow (b _ c)[c]	,! 7 ((E: 6)	i
(b _ c)[c]	,! 8 (\Rightarrow E: 5,7)	i
((b _ c)[c] \Rightarrow \leq[b, c])	,! 9 (\forall E: C2.5)	i
(b _ c)[c] \Rightarrow \leq[b, c]	,! 10 ((E: 9)	i
\leq[b, c]	,! 11 (\Rightarrow E: 8,10)	i
(\leq[b, c] \Rightarrow $(\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c}$)	,! 12 (\forall E: P24)	i
\leq[b, c] \Rightarrow $(\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c}$,! 13 ((E: 12)	i

$(\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c}$,! 14 (\Rightarrow E: 11,13) i
 $\mathbf{P} \subseteq (\mathbf{b} _ \mathbf{c}) \ \& \ \mathbf{P}[\mathbf{c}] \ \& \ (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c}$,! 15 ($\&$ I: 2,14) i
 $(\mathbf{P} \subseteq (\mathbf{b} _ \mathbf{c}) \ \& \ \mathbf{P}[\mathbf{c}] \ \& \ (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c} \Rightarrow (\chi\mathbf{P}) = \mathbf{c})$
, ! 16 (\forall E: P21) i
 $\mathbf{P} \subseteq (\mathbf{b} _ \mathbf{c}) \ \& \ \mathbf{P}[\mathbf{c}] \ \& \ (\chi(\mathbf{b} _ \mathbf{c})) = \mathbf{c} \Rightarrow (\chi\mathbf{P}) = \mathbf{c}$
, ! 17 ($()$ E: 16) i
 $(\chi\mathbf{P}) = \mathbf{c}$,! 18 (\Rightarrow E: 15,17) i
 $\mathbf{P} \subseteq (\mathbf{b} _ \mathbf{c}) \ \& \ \mathbf{P}[\mathbf{c}] \Rightarrow (\chi\mathbf{P}) = \mathbf{c}$,! 19 (\Rightarrow I: 2,18) i
 $(\mathbf{P} \subseteq (\mathbf{b} _ \mathbf{c}) \ \& \ \mathbf{P}[\mathbf{c}] \Rightarrow (\chi\mathbf{P}) = \mathbf{c})$,! 20 ($()$ I: 19) i
 $\forall \mathbf{P} \forall \mathbf{b} \forall \mathbf{c} (\mathbf{P} \subseteq (\mathbf{b} _ \mathbf{c}) \ \& \ \mathbf{P}[\mathbf{c}] \Rightarrow (\chi\mathbf{P}) = \mathbf{c})$! 21 (\forall I: 1,20) i

□

! 26. i

$\vdash \forall \mathbf{a} (\omega[\mathbf{a}] \Rightarrow (\chi(\mathbf{a}^\bullet)) = \mathbf{a})$ i
 \mathbf{a} ,! 1 (Prem) i
 $\omega[\mathbf{a}]$,! 2 (Prem) i
 $(\omega[\mathbf{a}] \Rightarrow (\mathbf{a} _ \mathbf{a}) \equiv (\mathbf{a}^\bullet))$,! 3 (\forall E: C2.30) i
 $\omega[\mathbf{a}] \Rightarrow (\mathbf{a} _ \mathbf{a}) \equiv (\mathbf{a}^\bullet)$,! 4 ($()$ E: 3) i
 $(\mathbf{a} _ \mathbf{a}) \equiv (\mathbf{a}^\bullet)$,! 5 (\Rightarrow E: 2,4) i
 $(\mathbf{a} _ \mathbf{a}) \equiv (\mathbf{a}^\bullet) \Rightarrow (\mathbf{a}^\bullet) \subseteq (\mathbf{a} _ \mathbf{a})$,! 6 (\forall E: II1.12) i
 $(\mathbf{a} _ \mathbf{a}) \equiv (\mathbf{a}^\bullet) \Rightarrow (\mathbf{a}^\bullet) \subseteq (\mathbf{a} _ \mathbf{a})$,! 7 ($()$ I: 6) i
 $(\mathbf{a}^\bullet) \subseteq (\mathbf{a} _ \mathbf{a})$,! 8 (\Rightarrow I: 5,7) i
 $(\mathbf{a}^\bullet)[\mathbf{a}]$,! 9 (\forall E: II8.5) i
 $(\mathbf{a}^\bullet) \subseteq (\mathbf{a} _ \mathbf{a}) \ \& \ (\mathbf{a}^\bullet)[\mathbf{a}]$,! 10 ($\&$ I: 8,9) i
 $(\mathbf{a}^\bullet) \subseteq (\mathbf{a} _ \mathbf{a}) \ \& \ (\mathbf{a}^\bullet)[\mathbf{a}] \Rightarrow (\chi(\mathbf{a}^\bullet)) = \mathbf{a}$
, ! 11 (\forall E: P25) i
 $(\mathbf{a}^\bullet) \subseteq (\mathbf{a} _ \mathbf{a}) \ \& \ (\mathbf{a}^\bullet)[\mathbf{a}] \Rightarrow (\chi(\mathbf{a}^\bullet)) = \mathbf{a}$
, ! 12 ($()$ E: 11) i
 $(\chi(\mathbf{a}^\bullet)) = \mathbf{a}$,! 13 (\Rightarrow E: 10,12) i
 $\omega[\mathbf{a}] \Rightarrow (\chi(\mathbf{a}^\bullet)) = \mathbf{a}$,! 14 (\Rightarrow I: 2,13) i

$(\omega[a] \Rightarrow (\chi(a^\bullet)) = a)$,! 15 ((I: 14) i
 $\forall a (\omega[a] \Rightarrow (\chi(a^\bullet)) = a)$! 16 (\forall I: 1,15) i
 \square
! 27. i
 $\vdash \forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \leq[(\mu P), (\chi P)])$ i
P ,! 1 (Prem) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P)$,! 2 (Prem) i
 $\neg (\omega \cap P) \equiv \phi$,! 3 ($\&$ E: 2) i
 $f (\omega \cap P)$,! 4 ($\&$ E: 2) i
 $(\neg (\omega \cap P) \equiv \phi \Rightarrow \omega[(\mu P)] \ \& \ P[(\mu P)])$,! 5 (\forall E: C3.7) i
 $\neg (\omega \cap P) \equiv \phi \Rightarrow \omega[(\mu P)] \ \& \ P[(\mu P)]$,! 6 ((E: 5) i
 $\omega[(\mu P)] \ \& \ P[(\mu P)]$,! 7 (\Rightarrow E: 3,6) i
 $f (\omega \cap P) \ \& \ \omega[(\mu P)] \ \& \ P[(\mu P)]$,! 8 ($\&$ I: 4,7) i
 $(f (\omega \cap P) \ \& \ \omega[(\mu P)] \ \& \ P[(\mu P)] \Rightarrow \leq[(\mu P), (\chi P)])$,! 9 (\forall E: P8;
 $(\mu P): C3.3,3)$ i
 $f (\omega \cap P) \ \& \ \omega[(\mu P)] \ \& \ P[(\mu P)] \Rightarrow \leq[(\mu P), (\chi P)]$,! 10 ((E: 9) i
 $\leq[(\mu P), (\chi P)]$,! 11 (\Rightarrow E: 8,10) i
 $\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \leq[(\mu P), (\chi P)]$,! 12 (\Rightarrow I: 2,11) i
 $(\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \leq[(\mu P), (\chi P)])$,! 13 ((I: 12) i
 $\forall P (\neg (\omega \cap P) \equiv \phi \ \& \ f (\omega \cap P) \Rightarrow \leq[(\mu P), (\chi P)])$! 14 (\forall I: 1,13) i
 \square