

! CHAPTER 4

PRIME NUMBERS AND THEIR INFINITUDE;

! This last chapter introduces the notion of prime number, and ends with the proof that there is an infinitude of prime numbers.

! 1. π is satisfied by those things which are prime numbers, i.e. numbers which are not 1 and divisible only by 1 and themselves.

$\mathbb{D} \pi ; \pi ; ; \{ a : \omega[a] \ \& \ \neg a = 1$
 $\quad \quad \quad \& \ \forall n (n | a \Rightarrow n = 1 \vee n = a) \}$

! 2.

$\vdash \forall x (\pi[x] \Leftrightarrow \omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n | x \Rightarrow n = 1 \vee n = x)) ;$

$\forall x (\{ a : \omega[a] \ \& \ \neg a = 1 \ \& \ \forall n (n | a \Rightarrow n = 1 \vee n = a) \}[x]$
 $\quad \quad \quad \Leftrightarrow \omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n | x \Rightarrow n = 1 \vee n = x))$
, ! 1 (Pred) i

$\forall x (\pi[x] \Leftrightarrow \omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n | x \Rightarrow n = 1 \vee n = x))$
! 2 ($\mathbb{D}I$: P1,1) i

□

! 3.

$\vdash \forall x (\pi[x] \Rightarrow \omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n | x \Rightarrow n = 1 \vee n = x)) ;$

\mathbf{x} , ! 1 (Prem) i

$(\pi[\mathbf{x}] \Leftrightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}))$
, ! 2 ($\forall E$: P2) i

$\pi[\mathbf{x}] \Leftrightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$
, ! 3 ($()E$: 2) i

$\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$
, ! 4 ($\Leftrightarrow E$: 3) i

$(\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}))$
, ! 5 ($()I$: 4) i

$\forall x (\pi[x] \Rightarrow \omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n | x \Rightarrow n = 1 \vee n = x))$
! 6 ($\forall I$: 1,5) i

□

! 4.

$\vdash \forall x (\omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n | x \Rightarrow n = 1 \vee n = x) \Rightarrow \pi[x]) ;$

\mathbf{x} , ! 1 (Prem) i

$(\pi[\mathbf{x}] \Leftrightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}))$
, ! 2 ($\forall E$: P2) i

$\pi[\mathbf{x}] \Leftrightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$
, ! 3 ($(\)E$: 2) i

$\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}) \Rightarrow \pi[\mathbf{x}]$
, ! 4 ($\Leftrightarrow E$: 3) i

$(\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}) \Rightarrow \pi[\mathbf{x}])$
, ! 5 ($(\)I$: 4) i

$\forall x (\omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n | x \Rightarrow n = 1 \vee n = x) \Rightarrow \pi[x])$
! 6 ($\forall I$: 1,5) i

□

! 5. i

$\vdash \forall x (\pi[x] \Rightarrow \omega[x])$ i

\mathbf{x} , ! 1 (Prem) i

$\pi[\mathbf{x}]$, ! 2 (Prem) i

$(\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}))$
, ! 3 ($\forall E$: P3) i

$\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$
, ! 4 ($(\)E$: 3) i

$\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n | \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$
, ! 5 ($\Rightarrow E$: 2,4) i

$\omega[\mathbf{x}]$, ! 6 ($\&E$: 5) i

$\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}]$, ! 7 ($\Rightarrow I$: 2,6) i

$(\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}])$, ! 8 ($(\)I$: 7) i

$\forall x (\pi[x] \Rightarrow \omega[x])$! 9 ($\forall I$: 1,8) i

□

! 6. i

$\vdash \pi \subseteq \omega$ i

$\pi \subseteq \omega$! 1 ($\S I$: III.1.1, P5) i

□

! 7. 1 is not a prime number. i

$\vdash \forall x (\pi[x] \Rightarrow \neg x = 1)$ i

\mathbf{x}	,! 1 (Prem)	i
$\pi[\mathbf{x}]$,! 2 (Prem)	i
$(\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}))$,! 3 ($\forall E$: P3)	i
$\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$,! 4 ($()E$: 3)	i
$\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$,! 5 ($\Rightarrow E$: 2,4)	i
$\neg \mathbf{x} = 1$,! 6 ($\&E$: 5)	i
$\pi[\mathbf{x}] \Rightarrow \neg \mathbf{x} = 1$,! 7 ($\Rightarrow I$: 2,6)	i
$(\pi[\mathbf{x}] \Rightarrow \neg \mathbf{x} = 1)$,! 8 ($()I$: 7)	i
$\forall \mathbf{x} (\pi[\mathbf{x}] \Rightarrow \neg \mathbf{x} = 1)$! 9 ($\forall I$: 1,8)	i

□

! 8. The only numbers dividing a prime number are 1 and itself. i

$\vdash \forall \mathbf{x} \forall n (\pi[\mathbf{x}] \ \& \ n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$		i
\mathbf{x}, n	,! 1 (Prem)	i
$\pi[\mathbf{x}] \ \& \ n \mathbf{x}$,! 2 (Prem)	i
$\pi[\mathbf{x}]$,! 3 ($\&E$: 2)	i
$n \mathbf{x}$,! 4 ($\&E$: 2)	i
$(\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}))$,! 5 ($\forall E$: P3)	i
$\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$,! 6 ($()E$: 5)	i
$\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$,! 7 ($\Rightarrow E$: 3,6)	i
$\forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$,! 8 ($\&E$: 7)	i
$(n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$,! 9 ($\forall E$: 8)	i
$n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}$,! 10 ($()E$: 9)	i
$n = 1 \vee n = \mathbf{x}$,! 11 ($\Rightarrow E$: 4,10)	i
$\pi[\mathbf{x}] \ \& \ n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}$,! 12 ($\Rightarrow I$: 2,11)	i
$(\pi[\mathbf{x}] \ \& \ n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x})$,! 13 ($()I$: 12)	i

$\forall x \forall n (\pi[x] \ \& \ n \mid x \Rightarrow n = 1 \vee n = x)$! 14 ($\forall I$: 1,13) i

□

! 9. If a number is divisible by something other than 1 or itself, then it is not prime. i

$\vdash \forall x \forall n (n \mid x \ \& \ \neg n = 1 \ \& \ \neg n = x \Rightarrow \neg \pi[x])$ i

x, n ,! 1 (Prem) i

n | x & ¬ n = 1 & ¬ n = x ,! 2 (Prem) i

n | x ,! 3 (&E: 2) i

¬ n = 1 & ¬ n = x ,! 4 (&E: 2) i

(**¬ n = 1 & ¬ n = x ⇒ ¬ (n = 1 ∨ n = x)**) ,! 5 ($\forall E$: I3.14) i

¬ n = 1 & ¬ n = x ⇒ ¬ (n = 1 ∨ n = x) ,! 6 (())E: 5) i

¬ (n = 1 ∨ n = x) ,! 7 ($\Rightarrow E$: 4,6) i

π[x] ,! 8 (Prem) i

π[x] & n | x ,! 9 (&I: 3,8) i

(**π[x] & n | x ⇒ n = 1 ∨ n = x**) ,! 10 ($\forall E$: P8) i

π[x] & n | x ⇒ n = 1 ∨ n = x ,! 11 (())E: 10) i

n = 1 ∨ n = x ,! 12 ($\Rightarrow E$: 9,11) i

(**n = 1 ∨ n = x**) ,! 13 (())I: 12) i

⊥ ,! 14 ($\mathcal{F}I$: 7,13) i

π[x] ⇒ ⊥ ,! 15 ($\Rightarrow I$: 8,14) i

¬ π[x] ,! 16 ($\neg I$: 15) i

n | x & ¬ n = 1 & ¬ n = x ⇒ ¬ π[x] ,! 17 ($\Rightarrow I$: 2,16) i

(**n | x & ¬ n = 1 & ¬ n = x ⇒ ¬ π[x]**) ,! 18 (())I: 17) i

$\forall x \forall n (n \mid x \ \& \ \neg n = 1 \ \& \ \neg n = x \Rightarrow \neg \pi[x])$! 19 ($\forall I$: 1,18) i

□

! 10. A non-prime number (different from 1) is divisible by a number other than 1 and itself. i

$\vdash \forall x (\omega[x] \ \& \ \neg \pi[x] \ \& \ \neg x = 1$

	$\Rightarrow \exists m (\neg m = 1 \ \& \ \neg m = x \ \& \ m \mid x)$		i
x		,! 1 (Prem)	i
$\omega[x] \ \& \ \neg \pi[x] \ \& \ \neg x = 1$,! 2 (Prem)	i
$\omega[x]$,! 3 (&E: 2)	i
$\neg \pi[x]$,! 4 (&E: 2)	i
$\neg x = 1$,! 5 (&E: 2)	i
$\neg \exists m (\neg m = 1 \ \& \ \neg m = x \ \& \ m \mid x)$,! 6 (Prem)	i
n		,! 7 (Prem)	i
n x		,! 8 (Prem)	i
$\neg (n = 1 \ \vee \ n = x)$,! 9 (Prem)	i
$(\neg (n = 1 \ \vee \ n = x) \Rightarrow \neg n = 1 \ \& \ \neg n = x)$,! 10 (\forall E: I3.13)	i
$\neg (n = 1 \ \vee \ n = x) \Rightarrow \neg n = 1 \ \& \ \neg n = x$,! 11 (())E: 10)	i
$\neg n = 1 \ \& \ \neg n = x$,! 12 (\Rightarrow E: 9,11)	i
$\neg n = 1 \ \& \ \neg n = x \ \& \ n \mid x$,! 13 (&I: 8,12)	i
$(\neg n = 1 \ \& \ \neg n = x \ \& \ n \mid x)$,! 14 (())I: 13)	i
$\exists m (\neg m = 1 \ \& \ \neg m = x \ \& \ m \mid x)$,! 15 (\exists I: 14)	i
\mathfrak{F}		,! 16 (\mathfrak{F} I: 6,15)	i
$\neg (n = 1 \ \vee \ n = x) \Rightarrow \mathfrak{F}$,! 17 (\Rightarrow I: 9,16)	i
$\neg\neg (n = 1 \ \vee \ n = x)$,! 18 (\neg I: 17)	i
$(n = 1 \ \vee \ n = x)$,! 19 (\neg E: 18)	i
$n = 1 \ \vee \ n = x$,! 20 (())E: 19)	i
n x $\Rightarrow n = 1 \ \vee \ n = x$,! 21 (\Rightarrow I: 8,20)	i
$(n \mid x \Rightarrow n = 1 \ \vee \ n = x)$,! 22 (())I: 21)	i
$\forall n (n \mid x \Rightarrow n = 1 \ \vee \ n = x)$,! 23 (\forall I: 7,22)	i
$\omega[x] \ \& \ \neg x = 1$,! 24 (&I: 3,5)	i
$\omega[x] \ \& \ \neg x = 1 \ \& \ \forall n (n \mid x \Rightarrow n = 1 \ \vee \ n = x)$,! 25 (&I: 23,24)	i

$(\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}) \Rightarrow \pi[\mathbf{x}])$, ! 26 ($\forall E$: P4)	i
$\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \ \& \ \forall n (n \mathbf{x} \Rightarrow n = 1 \vee n = \mathbf{x}) \Rightarrow \pi[\mathbf{x}]$, ! 27 ($(\)E$: 26)	i
$\pi[\mathbf{x}]$, ! 28 ($\Rightarrow E$: 25, 27)	i
\mathfrak{F}	, ! 29 ($\mathfrak{F}I$: 4, 28)	i
$\neg \exists m (\neg m = 1 \ \& \ \neg m = \mathbf{x} \ \& \ m \mathbf{x}) \Rightarrow \mathfrak{F}$, ! 30 ($\Rightarrow I$: 6, 29)	i
$\neg \neg \exists m (\neg m = 1 \ \& \ \neg m = \mathbf{x} \ \& \ m \mathbf{x})$, ! 31 ($\neg I$: 30)	i
$\exists m (\neg m = 1 \ \& \ \neg m = \mathbf{x} \ \& \ m \mathbf{x})$, ! 32 ($\neg E$: 31)	i
$\omega[\mathbf{x}] \ \& \ \neg \pi[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \Rightarrow \exists m (\neg m = 1 \ \& \ \neg m = \mathbf{x} \ \& \ m \mathbf{x})$, ! 33 ($\Rightarrow I$: 2, 32)	i
$(\omega[\mathbf{x}] \ \& \ \neg \pi[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \Rightarrow \exists m (\neg m = 1 \ \& \ \neg m = \mathbf{x} \ \& \ m \mathbf{x}))$, ! 34 ($(\)I$: 33)	i
$\forall \mathbf{x} (\omega[\mathbf{x}] \ \& \ \neg \pi[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1$		
$\Rightarrow \exists m (\neg m = 1 \ \& \ \neg m = \mathbf{x} \ \& \ m \mathbf{x}))$! 35 ($\forall I$: 1, 34)	i
\square		
! 11. 0 is not a prime number.		i
$\vdash \neg \pi[0]$		i
$2 0 \ \& \ \neg 2 = 1$, ! 1 ($\&I$: IV9.18, VI1.18)	i
$2 0 \ \& \ \neg 2 = 1 \ \& \ \neg 2 = 0$, ! 2 ($\&I$: IV9.17, 1)	i
$(2 0 \ \& \ \neg 2 = 1 \ \& \ \neg 2 = 0 \Rightarrow \neg \pi[0])$, ! 3 ($\forall E$: P9)	i
$2 0 \ \& \ \neg 2 = 1 \ \& \ \neg 2 = 0 \Rightarrow \neg \pi[0]$, ! 4 ($(\)E$: 3)	i
$\neg \pi[0]$! 5 ($\Rightarrow E$: 1, 4)	i
\square		
! 12. 1 is not a prime number.		i
$\vdash \neg \pi[1]$		i
$\pi[1]$, ! 1 (Prem)	i
$(\pi[1] \Rightarrow \neg 1 = 1)$, ! 2 ($\forall E$: P7)	i
$\pi[1] \Rightarrow \neg 1 = 1$, ! 3 ($(\)E$: 2)	i

$\neg 1 = 1$,! 4 (\Rightarrow E: 1,3)	i
$1 = 1$,! 5 (=I)	i
\mathfrak{F}	,! 6 (\mathfrak{F} I: 4,5)	i
$\pi[1] \Rightarrow \mathfrak{F}$,! 7 (\Rightarrow I: 1,6)	i
$\neg \pi[1]$! 8 (\neg I: 7)	i
\square		
! 13. 2 is a prime number.		i
$\vdash \pi[2]$		i
$(\omega[2] \ \& \ \neg 2 = 1 \ \& \ \forall n (n \mid 2 \Rightarrow n = 1 \vee n = 2) \Rightarrow \pi[2])$,! 1 (\forall E: P4)	i
$\omega[2] \ \& \ \neg 2 = 1$,! 2 (&I: IV9.11, IV9.18)	i
$\omega[2] \ \& \ \neg 2 = 1 \ \& \ \forall n (n \mid 2 \Rightarrow n = 1 \vee n = 2)$,! 3 (&I: VI1.48)	i
$\omega[2] \ \& \ \neg 2 = 1 \ \& \ \forall n (n \mid 2 \Rightarrow n = 1 \vee n = 2) \Rightarrow \pi[2]$,! 4 ((\Rightarrow)E: 3)	i
$\pi[2]$! 5 (\Rightarrow E: 3,4)	i
\square		
! 14. All prime numbers are non-zero.		i
$\vdash \forall x (\pi[x] \Rightarrow \neg x = 0)$		i
x	,! 1 (Prem)	i
$\pi[x]$,! 2 (Prem)	i
$x = 0$,! 3 (Prem)	i
$\pi[0]$,! 4 (=E: 2,3)	i
\mathfrak{F}	,! 5 (\mathfrak{F} I: P11,4)	i
$x = 0 \Rightarrow \mathfrak{F}$,! 6 (\Rightarrow I: 3,5)	i
$\neg x = 0$,! 7 (\neg I: 6)	i
$\pi[x] \Rightarrow \neg x = 0$,! 8 (\Rightarrow I: 2,7)	i
$(\pi[x] \Rightarrow \neg x = 0)$,! 9 ((\Rightarrow)I: 8)	i
$\forall x (\pi[x] \Rightarrow \neg x = 0)$! 10 (\forall I: 1,9)	i
\square		

! 15. If x is a prime number and n is a finite number, then the pre-supposition of the gcd (VI5.28) holds. i

$\vdash \forall x \forall n (\pi[x] \ \& \ \omega[n] \Rightarrow \omega[x] \ \& \ \omega[n] \ \& \ (\neg x = 0 \vee \neg n = 0))$ i

$\mathbf{x, n}$, ! 1 (Prem) i

$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}]$, ! 2 (Prem) i

$\pi[\mathbf{x}]$, ! 3 (&E: 2) i

$\omega[\mathbf{n}]$, ! 4 (&E: 2) i

$(\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}])$, ! 5 (\forall E: P5) i

$\pi[\mathbf{x}] \Rightarrow \omega[\mathbf{x}]$, ! 6 (()E: 5) i

$\omega[\mathbf{x}]$, ! 7 (\Rightarrow E: 3,6) i

$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}]$, ! 8 (&I: 4,7) i

$(\pi[\mathbf{x}] \Rightarrow \neg \mathbf{x} = 0)$, ! 9 (\forall E: P14) i

$\pi[\mathbf{x}] \Rightarrow \neg \mathbf{x} = 0$, ! 10 (()E: 9) i

$\neg \mathbf{x} = 0$, ! 11 (\Rightarrow E: 3,10) i

$\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0$, ! 12 (\vee I: 11) i

$(\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0)$, ! 13 (()I: 12) i

$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0)$, ! 14 (&I: 8,13) i

$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0)$
, ! 15 (\Rightarrow I: 2,14) i

$(\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0))$
, ! 16 (()I: 15) i

$\forall x \forall n (\pi[x] \ \& \ \omega[n] \Rightarrow \omega[x] \ \& \ \omega[n] \ \& \ (\neg x = 0 \vee \neg n = 0))$
! 17 (\forall I: 1,16) i

□

! 16. The gcd of a prime number and another number is either 1 or the prime number. i

$\vdash \forall x \forall n (\pi[x] \ \& \ \omega[n] \Rightarrow (x \ \Delta \ n) = 1 \vee (x \ \Delta \ n) = x)$ i

$\mathbf{x, n}$, ! 1 (Prem) i

$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}]$, ! 2 (Prem) i

$\pi[\mathbf{x}]$, ! 3 (&E: 2) i

$(\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0))$,! 4 ($\forall E$: P15)	i
$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow \omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0)$,! 5 ($()E$: 4)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0)$,! 6 ($\Rightarrow E$: 2,5)	i
$(\omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0) \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) \mid \mathbf{x})$,! 7 ($\forall E$: VI5.31)	i
$\omega[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0) \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) \mid \mathbf{x}$,! 8 ($()E$: 7)	i
$(\mathbf{x} \ \Delta \ \mathbf{n}) \mid \mathbf{x}$,! 9 ($\Rightarrow E$: 6,8)	i
$\pi[\mathbf{x}] \ \& \ (\mathbf{x} \ \Delta \ \mathbf{n}) \mid \mathbf{x}$,! 10 ($\&I$: 3,9)	i
$(\pi[\mathbf{x}] \ \& \ (\mathbf{x} \ \Delta \ \mathbf{n}) \mid \mathbf{x} \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) = 1 \vee (\mathbf{x} \ \Delta \ \mathbf{n}) = \mathbf{x})$,! 11 ($\forall E$: P8; ($\mathbf{x} \ \Delta \ \mathbf{n}$): VI5.28,6)	i
$\pi[\mathbf{x}] \ \& \ (\mathbf{x} \ \Delta \ \mathbf{n}) \mid \mathbf{x} \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) = 1 \vee (\mathbf{x} \ \Delta \ \mathbf{n}) = \mathbf{x}$,! 12 ($()E$: 11)	i
$(\mathbf{x} \ \Delta \ \mathbf{n}) = 1 \vee (\mathbf{x} \ \Delta \ \mathbf{n}) = \mathbf{x}$,! 13 ($\Rightarrow E$: 10,12)	i
$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) = 1 \vee (\mathbf{x} \ \Delta \ \mathbf{n}) = \mathbf{x}$,! 14 ($\Rightarrow I$: 2,13)	i
$(\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) = 1 \vee (\mathbf{x} \ \Delta \ \mathbf{n}) = \mathbf{x})$,! 15 ($()I$: 14)	i
$\forall x \forall n (\pi[x] \ \& \ \omega[n] \Rightarrow (x \ \Delta \ n) = 1 \vee (x \ \Delta \ n) = x)$! 16 ($\forall I$: 1,15)	i
\square		
! 17. P17 is a contrapositive of P16.		
$\vdash \forall x \forall n (\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \ \Delta \ n) = 1 \Rightarrow (x \ \Delta \ n) = x)$		i
\mathbf{x}, \mathbf{n}	,! 1 (Prem)	i
$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \ \& \ \neg (\mathbf{x} \ \Delta \ \mathbf{n}) = 1$,! 2 (Prem)	i
$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}]$,! 3 ($\&E$: 2)	i
$\neg (\mathbf{x} \ \Delta \ \mathbf{n}) = 1$,! 4 ($\&E$: 2)	i
$(\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) = 1 \vee (\mathbf{x} \ \Delta \ \mathbf{n}) = \mathbf{x})$,! 5 ($\forall E$: P16)	i
$\pi[\mathbf{x}] \ \& \ \omega[\mathbf{n}] \Rightarrow (\mathbf{x} \ \Delta \ \mathbf{n}) = 1 \vee (\mathbf{x} \ \Delta \ \mathbf{n}) = \mathbf{x}$		

	,! 6 (())E: 5)	i
$(\mathbf{x} \Delta \mathbf{n}) = 1 \vee (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}$,! 7 (\Rightarrow E: 3,6)	i
$((\mathbf{x} \Delta \mathbf{n}) = 1 \vee (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x})$,! 8 (())I: 7)	i
$((\mathbf{x} \Delta \mathbf{n}) = 1 \vee (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}) \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1$,! 9 (&I: 4,8)	i
$(\pi[\mathbf{x}] \& \omega[\mathbf{n}] \Rightarrow \omega[\mathbf{x}] \& \omega[\mathbf{n}] \& (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0))$,! 10 (\forall E: P15)	i
$\pi[\mathbf{x}] \& \omega[\mathbf{n}] \Rightarrow \omega[\mathbf{x}] \& \omega[\mathbf{n}] \& (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0)$,! 11 (())E: 10)	i
$\omega[\mathbf{x}] \& \omega[\mathbf{n}] \& (\neg \mathbf{x} = 0 \vee \neg \mathbf{n} = 0)$,! 12 (\Rightarrow E: 3,11)	i
$(((\mathbf{x} \Delta \mathbf{n}) = 1 \vee (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}) \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1$ $\Rightarrow (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x})$,! 13 (\forall E: I3.5; ($\mathbf{x} \Delta \mathbf{n}$): VI5.28,12)	i
$((\mathbf{x} \Delta \mathbf{n}) = 1 \vee (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}) \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1$ $\Rightarrow (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}$,! 14 (())E: 13)	i
$(\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}$,! 15 (\Rightarrow E: 9,14)	i
$\pi[\mathbf{x}] \& \omega[\mathbf{n}] \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1 \Rightarrow (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}$,! 16 (\Rightarrow I: 2,15)	i
$(\pi[\mathbf{x}] \& \omega[\mathbf{n}] \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1 \Rightarrow (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x})$,! 17 (())I: 16)	i
$\forall \mathbf{x} \forall \mathbf{n} (\pi[\mathbf{x}] \& \omega[\mathbf{n}] \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1 \Rightarrow (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x})$! 18 (\forall I: 1,17)	i

□

! 18. If the gcd of a prime number and another number is not 1, then the prime number must divide the other number. i

$\vdash \forall \mathbf{x} \forall \mathbf{n} (\pi[\mathbf{x}] \& \omega[\mathbf{n}] \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1 \Rightarrow \mathbf{x} \mathbf{n})$		i
\mathbf{x}, \mathbf{n}	,! 1 (Prem)	i
$\pi[\mathbf{x}] \& \omega[\mathbf{n}] \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1$,! 2 (Prem)	i
$(\pi[\mathbf{x}] \& \omega[\mathbf{n}] \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1 \Rightarrow (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x})$,! 3 (\forall E: P17)	i
$\pi[\mathbf{x}] \& \omega[\mathbf{n}] \& \neg (\mathbf{x} \Delta \mathbf{n}) = 1 \Rightarrow (\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}$,! 4 (())E: 3)	i
$(\mathbf{x} \Delta \mathbf{n}) = \mathbf{x}$,! 5 (\Rightarrow E: 2,4)	i

$((x \Delta n) = x \Rightarrow x n)$,! 6 ($\forall E$: VI5.34)	i
$(x \Delta n) = x \Rightarrow x n$,! 7 ($()E$: 6)	i
$x n$,! 8 ($\Rightarrow E$: 5,7)	i
$\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \Delta n) = 1 \Rightarrow x n$,! 9 ($\Rightarrow I$: 2,8)	i
$(\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \Delta n) = 1 \Rightarrow x n)$,! 10 ($()I$: 9)	i
$\forall x \forall n (\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \Delta n) = 1 \Rightarrow x n)$! 11 ($\forall I$: 1,10)	i

□

! 19. A prime number which divides a product must divide one of the factors. i

$\vdash \forall x \forall n \forall m (\pi[x] \ \& \ x (n \times m) \Rightarrow x n \vee x m)$		i
x, n, m	,! 1 (Prem)	i
$\pi[x] \ \& \ x (n \times m)$,! 2 (Prem)	i
$\pi[x]$,! 3 ($\&E$: 2)	i
$x (n \times m)$,! 4 ($\&E$: 2)	i
$\exists x (x \times x) = (n \times m)$,! 5 ($\E: VI1.1,4)	i
$(x \times a) = (n \times m)$,! 6 ($\exists E$: 5)	i
$\omega[n] \ \& \ \omega[m]$,! 7 ($\mathbb{T}E$: V7.9,5)	i
$\omega[n]$,! 8 ($\&E$: 7)	i
$\pi[x] \ \& \ \omega[n]$,! 9 ($\&I$: 3,8)	i
$(\pi[x] \ \& \ \omega[n] \Rightarrow \omega[x] \ \& \ \omega[n] \ \& \ (\neg x = 0 \vee \neg n = 0))$,! 10 ($\forall E$: P15)	i
$\pi[x] \ \& \ \omega[n] \Rightarrow \omega[x] \ \& \ \omega[n] \ \& \ (\neg x = 0 \vee \neg n = 0)$,! 11 ($()E$: 10)	i
$\omega[x] \ \& \ \omega[n] \ \& \ (\neg x = 0 \vee \neg n = 0)$,! 12 ($\Rightarrow E$: 9,11)	i
$((x \Delta n) = 1 \vee \neg (x \Delta n) = 1)$,! 13 ($\forall E$: I3.4; $(x \Delta n)$: VI5.28,12)	i
$(x \Delta n) = 1 \vee \neg (x \Delta n) = 1$,! 14 ($()E$: 13)	i
$(x \Delta n) = 1$,! 15 (Prem)	i
$x (n \times m) \ \& \ (x \Delta n) = 1$,! 16 ($\&I$: 4,15)	i

$(x \mid (n \times m) \ \& \ (x \Delta n) = 1 \Rightarrow x \mid m)$,! 17 ($\forall E$: VI5.55) ;
 $x \mid (n \times m) \ \& \ (x \Delta n) = 1 \Rightarrow x \mid m$,! 18 ($(\)E$: 17) ;
 $x \mid m$,! 19 ($\Rightarrow E$: 16,18) ;
 $x \mid n \vee x \mid m$,! 20 ($\vee I$: 19) ;
 $(x \Delta n) = 1 \Rightarrow x \mid n \vee x \mid m$,! 21 ($\Rightarrow I$: 15,20) ;
 $\neg (x \Delta n) = 1$,! 22 (Prem) ;
 $\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \Delta n) = 1$,! 23 ($\& I$: 9,22) ;
 $(\ \pi[x] \ \& \ \omega[n] \ \& \ \neg (x \Delta n) = 1 \Rightarrow x \mid n)$,! 24 ($\forall E$: P18) ;
 $\pi[x] \ \& \ \omega[n] \ \& \ \neg (x \Delta n) = 1 \Rightarrow x \mid n$,! 25 ($(\)E$: 24) ;
 $x \mid n$,! 26 ($\Rightarrow E$: 23,25) ;
 $x \mid n \vee x \mid m$,! 27 ($\vee I$: 26) ;
 $\neg (x \Delta n) = 1 \Rightarrow x \mid n \vee x \mid m$,! 28 ($\Rightarrow I$: 22,27) ;
 $x \mid n \vee x \mid m$,! 29 ($\vee E$: 14,21,28) ;
 $\pi[x] \ \& \ x \mid (n \times m) \Rightarrow x \mid n \vee x \mid m$,! 30 ($\Rightarrow I$: 2,29) ;
 $(\ \pi[x] \ \& \ x \mid (n \times m) \Rightarrow x \mid n \vee x \mid m)$,! 31 ($(\)I$: 30) ;
 $\forall x \forall n \forall m (\ \pi[x] \ \& \ x \mid (n \times m) \Rightarrow x \mid n \vee x \mid m)$! 32 ($\forall I$: 1,31) ;

□

! 20. A number which is not 1 is divisible by a prime number. ;

$\vdash \forall n (\ \omega[n] \ \& \ \neg n = 1 \Rightarrow \exists m (\ \pi[m] \ \& \ m \mid n)$) ;
 n ,! 1 (Prem) ;
 $\omega[n] \ \& \ \neg n = 1$,! 2 (Prem) ;
 $\omega[n]$,! 3 ($\& E$: 2) ;
 $\neg n = 1$,! 4 ($\& E$: 2) ;
 $\forall x (\ \{a : \neg a = 1 \ \& \ \neg \exists m (\ \pi[m] \ \& \ m \mid a)$ }[x]
 $\Leftrightarrow \neg x = 1 \ \& \ \neg \exists m (\ \pi[m] \ \& \ m \mid x)$)) ,! 5 (Pred) ;

! For perspicuity we introduce **P**, via lines 6 and 7, to replace
 $\{a : \neg a = 1 \ \& \ \neg \exists m (\ \pi[m] \ \& \ m \mid a)$ } ;

$$\exists P \forall x (P[x] \leftrightarrow \neg x = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m | x))$$

, ! 6 ($\exists I$: 5) i

$$\forall x (P[x] \leftrightarrow \neg x = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m | x))$$

, ! 7 ($\exists E$: 6) i

$$(\neg \exists m (\mu P) = m \Rightarrow \forall n (\omega[n] \Rightarrow \neg P[n]))$$

, ! 8 ($\forall E$: VI3.24) i

$$\neg \exists m (\mu P) = m \Rightarrow \forall n (\omega[n] \Rightarrow \neg P[n])$$

, ! 9 ($()E$: 8) i

! Proceed by contradiction to prove: $\exists m (\mu P) = m$ i

$$\exists m (\mu P) = m$$

, ! 10 (Prem) i

$$(\mu P) = m$$

, ! 11 ($\exists E$: 10) i

$$\neg (\omega \cap P) \equiv \phi$$

, ! 12 ($\mathbb{T}E$: VI3.3, 11) i

$$(\neg (\omega \cap P) \equiv \phi$$

$$\Rightarrow \omega[(\mu P)] \ \& \ P[(\mu P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[(\mu P), y]))$$

, ! 13 ($\forall E$: VI3.4) i

$$\neg (\omega \cap P) \equiv \phi$$

$$\Rightarrow \omega[(\mu P)] \ \& \ P[(\mu P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[(\mu P), y])$$

, ! 14 ($()E$: 13) i

$$\omega[(\mu P)] \ \& \ P[(\mu P)] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[(\mu P), y])$$

, ! 15 ($\Rightarrow E$: 12, 14) i

$$\omega[m] \ \& \ P[m] \ \& \ \forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[m, y])$$

, ! 16 ($=E$: 11, 15) i

$$\omega[m]$$

, ! 17 ($\&E$: 15) i

$$P[m]$$

, ! 18 ($\&E$: 15) i

$$\forall y (\omega[y] \ \& \ P[y] \Rightarrow \leq[m, y])$$

, ! 19 ($\&E$: 15) i

$$(P[m] \leftrightarrow \neg m = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m | m))$$

, ! 20 ($\forall E$: 7) i

$$P[m] \leftrightarrow \neg m = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m | m)$$

, ! 21 ($()E$: 20) i

$$P[m] \Rightarrow \neg m = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m | m)$$

, ! 22 ($\Leftrightarrow E$: 21) i

$$\neg m = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m | m)$$

, ! 23 ($\Rightarrow E$: 18, 22) i

$$\neg m = 1$$

, ! 24 ($\&E$: 23) i

$\neg \exists m (\pi[m] \ \& \ m \mid \mathbf{m})$,! 25 (&E: 23)	i
$\omega[\mathbf{m}] \ \& \ \neg \mathbf{m} = 1$,! 26 (&I: 17,24)	i
$\pi[\mathbf{m}]$,! 27 (Prem)	i
$(\omega[\mathbf{m}] \Rightarrow \mathbf{m} \mid \mathbf{m})$,! 28 (\forall E: VI1.10)	i
$\omega[\mathbf{m}] \Rightarrow \mathbf{m} \mid \mathbf{m}$,! 29 ($(\)$ E: 28)	i
$\mathbf{m} \mid \mathbf{m}$,! 30 ($(\)$ E: 29)	i
$\pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{m}$,! 31 (&E: 27,30)	i
$(\pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{m})$,! 32 ($(\)$ I: 31)	i
$\exists m (\pi[m] \ \& \ m \mid \mathbf{m})$,! 33 (\exists I: 32)	i
\mathfrak{F}	,! 34 (\mathfrak{F} I: 25,33)	i
$\pi[\mathbf{m}] \Rightarrow \mathfrak{F}$,! 35 (\Rightarrow I: 27,34)	i
$\neg \pi[\mathbf{m}]$,! 36 (\neg I: 35)	i
$\omega[\mathbf{m}] \ \& \ \neg \pi[\mathbf{m}] \ \& \ \neg \mathbf{m} = 1$,! 37 (&I: 26,36)	i
$(\omega[\mathbf{m}] \ \& \ \neg \pi[\mathbf{m}] \ \& \ \neg \mathbf{m} = 1$ $\Rightarrow \exists m (\neg m = 1 \ \& \ \neg m = \mathbf{m} \ \& \ m \mid \mathbf{m}))$,! 38 (\forall E: P10)	i
$\omega[\mathbf{m}] \ \& \ \neg \pi[\mathbf{m}] \ \& \ \neg \mathbf{m} = 1$ $\Rightarrow \exists m (\neg m = 1 \ \& \ \neg m = \mathbf{m} \ \& \ m \mid \mathbf{m})$,! 39 ($(\)$ E: 38)	i
$\exists m (\neg m = 1 \ \& \ \neg m = \mathbf{m} \ \& \ m \mid \mathbf{m})$,! 40 (\Rightarrow E: 37,39)	i
$(\neg \mathbf{y} = 1 \ \& \ \neg \mathbf{y} = \mathbf{m} \ \& \ \mathbf{y} \mid \mathbf{m})$,! 41 (\exists E: 40)	i
$\neg \mathbf{y} = 1 \ \& \ \neg \mathbf{y} = \mathbf{m} \ \& \ \mathbf{y} \mid \mathbf{m}$,! 42 ($(\)$ E: 41)	i
$\neg \mathbf{y} = 1$,! 43 (&E: 42)	i
$\neg \mathbf{y} = \mathbf{m}$,! 44 (&E: 42)	i
$\mathbf{y} \mid \mathbf{m}$,! 45 (&E: 42)	i
! It will be shown that $\mathbf{y} = \mathbf{m}$, contradicting 44.		i
$(\mathbf{y} \mid \mathbf{m} \Rightarrow \omega[\mathbf{y}])$,! 46 (\forall E: VI1.3)	i
$\mathbf{y} \mid \mathbf{m} \Rightarrow \omega[\mathbf{y}]$,! 47 ($(\)$ E: 46)	i
$\omega[\mathbf{y}]$,! 48 (\Rightarrow E: 45,47)	i
! To show: $\mathbf{P}[\mathbf{y}]$		i

$(\mathbf{P}[\mathbf{y}] \Leftrightarrow \neg \mathbf{y} = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m \mathbf{y}))$,! 49 ($\forall E$: 7)	i
$\exists m (\pi[m] \ \& \ m \mathbf{y})$,! 50 (Prem)	i
$(\pi[\mathbf{z}] \ \& \ \mathbf{z} \mathbf{y})$,! 51 ($\exists E$: 50)	i
$\pi[\mathbf{z}] \ \& \ \mathbf{z} \mathbf{y}$,! 52 ($(\)E$: 51)	i
$\pi[\mathbf{z}]$,! 53 ($\&E$: 52)	i
$\mathbf{z} \mathbf{y}$,! 54 ($\&E$: 52)	i
$\mathbf{z} \mathbf{y} \ \& \ \mathbf{y} \mathbf{m}$,! 55 ($\&I$: 45,54)	i
$(\mathbf{z} \mathbf{y} \ \& \ \mathbf{y} \mathbf{m} \Rightarrow \mathbf{z} \mathbf{m})$,! 56 ($\forall E$: VI1.24)	i
$\mathbf{z} \mathbf{y} \ \& \ \mathbf{y} \mathbf{m} \Rightarrow \mathbf{z} \mathbf{m}$,! 57 ($(\)E$: 56)	i
$\mathbf{z} \mathbf{m}$,! 58 ($\Rightarrow E$: 55,57)	i
$\pi[\mathbf{z}] \ \& \ \mathbf{z} \mathbf{m}$,! 59 ($\&I$: 53,58)	i
$(\pi[\mathbf{z}] \ \& \ \mathbf{z} \mathbf{m})$,! 60 ($(\)I$: 59)	i
$\exists m (\pi[m] \ \& \ m \mathbf{m})$,! 61 ($\exists I$: 60)	i
\mathfrak{F}	,! 62 ($\mathfrak{F}I$: 25,61)	i
$\exists m (\pi[m] \ \& \ m \mathbf{y}) \Rightarrow \mathfrak{F}$,! 63 ($\Rightarrow I$: 50,62)	i
$\neg \exists m (\pi[m] \ \& \ m \mathbf{y})$,! 64 ($\neg I$: 63)	i
$\neg \mathbf{y} = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m \mathbf{y})$,! 65 ($\&I$: 43,64)	i
$\mathbf{P}[\mathbf{y}] \Leftrightarrow \neg \mathbf{y} = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m \mathbf{y})$,! 66 ($(\)E$: 49)	i
$\neg \mathbf{y} = 1 \ \& \ \neg \exists m (\pi[m] \ \& \ m \mathbf{y}) \Rightarrow \mathbf{P}[\mathbf{y}]$,! 67 ($\Leftrightarrow E$: 66)	i
$\mathbf{P}[\mathbf{y}]$,! 68 ($\Rightarrow E$: 65,67)	i
! To show: $\mathbf{y} = \mathbf{m}$, by showing $\leq[\mathbf{m}, \mathbf{y}]$ and $\leq[\mathbf{y}, \mathbf{m}]$		i
$\omega[\mathbf{y}] \ \& \ \mathbf{P}[\mathbf{y}]$,! 69 ($\&I$: 48,68)	i
$(\omega[\mathbf{y}] \ \& \ \mathbf{P}[\mathbf{y}] \Rightarrow \leq[\mathbf{m}, \mathbf{y}])$,! 70 ($\forall E$: 19)	i
$\omega[\mathbf{y}] \ \& \ \mathbf{P}[\mathbf{y}] \Rightarrow \leq[\mathbf{m}, \mathbf{y}]$,! 71 ($(\)E$: 70)	i
$\leq[\mathbf{m}, \mathbf{y}]$,! 72 ($\Rightarrow E$: 69,71)	i
$\mathbf{m} = 0$,! 73 (Prem)	i
$\pi[2] \ \& \ 2 0$,! 74 ($\&I$: VI1.18, P13)	

			i
$\pi[2] \ \& \ 2 \mid \mathbf{m}$,!	75 (=E: 73,74)	i
$(\ \pi[2] \ \& \ 2 \mid \mathbf{m} \)$,!	76 (()I: 75)	i
$\exists \mathbf{m} (\ \pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{m} \)$,!	77 (\exists I: 76)	i
\mathfrak{F}	,!	78 (\mathfrak{F} I: 25,77)	i
$\mathbf{m} = 0 \Rightarrow \mathfrak{F}$,!	79 (\Rightarrow I: 73,78)	i
$\neg \mathbf{m} = 0$,!	80 (\neg I: 79)	i
$\mathbf{y} \mid \mathbf{m} \ \& \ \neg \mathbf{m} = 0$,!	81 ($\&$ I: 45,80)	i
$(\ \mathbf{y} \mid \mathbf{m} \ \& \ \neg \mathbf{m} = 0 \Rightarrow \leq[\mathbf{y},\mathbf{m}] \)$,!	82 (\forall E: VI1.47)	i
$\mathbf{y} \mid \mathbf{m} \ \& \ \neg \mathbf{m} = 0 \Rightarrow \leq[\mathbf{y},\mathbf{m}]$,!	83 (()E: 82)	i
$\leq[\mathbf{y},\mathbf{m}]$,!	84 (\Rightarrow E: 81,83)	i
$\leq[\mathbf{y},\mathbf{m}] \ \& \ \leq[\mathbf{m},\mathbf{y}]$,!	85 ($\&$ I: 72,84)	i
$(\ \leq[\mathbf{y},\mathbf{m}] \ \& \ \leq[\mathbf{m},\mathbf{y}] \Rightarrow \mathbf{y} = \mathbf{m} \)$,!	86 (\forall E: V3.22)	i
$\leq[\mathbf{y},\mathbf{m}] \ \& \ \leq[\mathbf{m},\mathbf{y}] \Rightarrow \mathbf{y} = \mathbf{m}$,!	87 (()E: 86)	i
$\mathbf{y} = \mathbf{m}$,!	88 (\Rightarrow E: 85,87)	i
\mathfrak{F}	,!	89 (\mathfrak{F} I: 44,88)	i
$\exists \mathbf{m} (\mu \mathbf{P}) = \mathbf{m} \Rightarrow \mathfrak{F}$,!	90 (\Rightarrow I: 10,89)	i
$\neg \exists \mathbf{m} (\mu \mathbf{P}) = \mathbf{m}$,!	91 (\neg I: 90)	i
$\forall \mathbf{n} (\ \omega[\mathbf{n}] \Rightarrow \neg \mathbf{P}[\mathbf{n}] \)$,!	92 (\Rightarrow E: 9,91)	i
$(\ \omega[\mathbf{n}] \Rightarrow \neg \mathbf{P}[\mathbf{n}] \)$,!	93 (\forall E: 92)	i
$\omega[\mathbf{n}] \Rightarrow \neg \mathbf{P}[\mathbf{n}]$,!	94 (()E: 93)	i
$\neg \mathbf{P}[\mathbf{n}]$,!	95 (\Rightarrow E: 3,94)	i
$(\ \mathbf{P}[\mathbf{n}] \Leftrightarrow \neg \mathbf{n} = 1 \ \& \ \neg \exists \mathbf{m} (\ \pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{n} \) \)$,!	96 (\forall E: 7)	i
$\mathbf{P}[\mathbf{n}] \Leftrightarrow \neg \mathbf{n} = 1 \ \& \ \neg \exists \mathbf{m} (\ \pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{n} \)$,!	97 (()E: 96)	i
$\neg \exists \mathbf{m} (\ \pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{n} \)$,!	98 (Prem)	i
$\neg \mathbf{n} = 1 \ \& \ \neg \exists \mathbf{m} (\ \pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{n} \)$,!	99 ($\&$ I: 4,98)	i
$\neg \mathbf{n} = 1 \ \& \ \neg \exists \mathbf{m} (\ \pi[\mathbf{m}] \ \& \ \mathbf{m} \mid \mathbf{n} \) \Rightarrow \mathbf{P}[\mathbf{n}]$,!	100 (\Leftrightarrow E: 97)	i

$P[n]$,! 101 ($\Rightarrow E$: 99,100)	i
\mathfrak{F}	,! 102 ($\mathfrak{F}I$: 95,101)	i
$\neg \exists m (\pi[m] \ \& \ m n) \Rightarrow \mathfrak{F}$,! 103 ($\Rightarrow I$: 98,102)	i
$\neg \neg \exists m (\pi[m] \ \& \ m n)$,! 104 ($\neg I$: 103)	i
$\exists m (\pi[m] \ \& \ m n)$,! 105 ($\neg E$: 104)	i
$\omega[n] \ \& \ \neg n = 1 \Rightarrow \exists m (\pi[m] \ \& \ m n)$,! 106 ($\Rightarrow I$: 2,105)	i
$(\omega[n] \ \& \ \neg n = 1 \Rightarrow \exists m (\pi[m] \ \& \ m n))$,! 107 ($(\)I$: 106)	i
$\forall n (\omega[n] \ \& \ \neg n = 1 \Rightarrow \exists m (\pi[m] \ \& \ m n))$! 108 ($\forall I$: 1,107)	i

□

! 21. There are an infinitude of prime numbers. i

$\vdash \pi$		i
$f \pi$,! 1 (Prem)	i
$f \pi \ \& \ \pi \subseteq \omega$,! 2 ($\&I$: P6,1)	i
$f \pi \ \& \ \pi \subseteq \omega \ \& \ \neg \pi[0]$,! 3 ($\&I$: P11,2)	i
$f \pi \ \& \ \pi \subseteq \omega \ \& \ \neg \pi[0] \ \& \ \neg \pi[1]$,! 4 ($\&I$: P12,3)	i
$(f \pi \ \& \ \pi \subseteq \omega \ \& \ \neg \pi[0] \ \& \ \neg \pi[1]$ $\Rightarrow \exists x (\omega[x] \ \& \ \neg x = 1 \ \& \ \forall y (\pi[y] \Rightarrow \neg y x)))$,! 5 ($\forall E$: VI1.57)	i
$f \pi \ \& \ \pi \subseteq \omega \ \& \ \neg \pi[0] \ \& \ \neg \pi[1]$ $\Rightarrow \exists x (\omega[x] \ \& \ \neg x = 1 \ \& \ \forall y (\pi[y] \Rightarrow \neg y x))$,! 6 ($(\)E$: 5)	i
$\exists x (\omega[x] \ \& \ \neg x = 1 \ \& \ \forall y (\pi[y] \Rightarrow \neg y x))$,! 7 ($\Rightarrow E$: 4,6)	i
$(\omega[x] \ \& \ \neg x = 1 \ \& \ \forall y (\pi[y] \Rightarrow \neg y x))$,! 8 ($\exists E$: 7)	i
$\omega[x] \ \& \ \neg x = 1 \ \& \ \forall y (\pi[y] \Rightarrow \neg y x)$,! 9 ($(\)E$: 8)	i
$\omega[x] \ \& \ \neg x = 1$,! 10 ($\&E$: 9)	i
$\forall y (\pi[y] \Rightarrow \neg y x)$,! 11 ($\&E$: 9)	i

$(\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \Rightarrow \exists m (\pi[m] \ \& \ m \mathbf{x}))$,!	12	($\forall E$: P20)	i
$\omega[\mathbf{x}] \ \& \ \neg \mathbf{x} = 1 \Rightarrow \exists m (\pi[m] \ \& \ m \mathbf{x})$,!	13	($()E$: 12)	i
$\exists m (\pi[m] \ \& \ m \mathbf{x})$,!	14	($\Rightarrow E$: 10,13)	i
$(\pi[m] \ \& \ m \mathbf{x})$,!	15	($\exists E$: 14)	i
$\pi[m] \ \& \ m \mathbf{x}$,!	16	($()E$: 15)	i
$\pi[m]$,!	17	($\&E$: 16)	i
$m \mathbf{x}$,!	18	($\&E$: 16)	i
$(\pi[m] \Rightarrow \neg m \mathbf{x})$,!	19	($\forall E$: 11)	i
$\pi[m] \Rightarrow \neg m \mathbf{x}$,!	20	($()E$: 19)	i
$\neg m \mathbf{x}$,!	21	($\Rightarrow E$: 17,20)	i
\mathfrak{F}	,!	22	($\mathfrak{F}I$: 18,21)	i
$f \ \pi \Rightarrow \mathfrak{F}$,!	23	($\Rightarrow I$: 1,22)	i
$\neg f \ \pi$,!	24	($\neg I$: 23)	i
$\iota \ \pi$!	25	($\mathfrak{S}I$: IV6.1,24)	i
\square				
! QED				i