SYSTEMS FOR A FOUNDATION OF ARITHMETIC

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Introduction

A foundation for arithmetic, as it is normally considered, should be, if it is not as well other things, a fundamental set of axioms which can prove Peano's Axioms. It is now well-known that Hume's Principle in second-order arithmetic, along with certain definitions of zero, succession, and the natural numbers, can prove Peano's Axioms and thus lay claim to being a foundation. Originally proven by Frege ([7], [8]), it is now called “Frege’s Theorem”. There has been much important investigation in this area in recent years, in particular by Crispin Wright ([12]), George Boolos ([1]), Richard Heck ([9], [10]), and Neil Tennant ([11]), among many others, some of which will be commented below. The import of Frege's Theorem has been controversial, in large part because of Hume's Principle. There are nonetheless at least three other criticisms which should be made. First, the definition of succession has ontological implications which seem untenable. Secondly, Frege Arithmetic makes impredicative assumptions. Thirdly, there is a *more fundamental* (and *predicative*) axiom set which can be used to prove Peano's Axioms. A foundation for arithmetic, therefore, seems elsewhere.

Abbreviations, Peano's and “Frege's” Axioms

Abbreviations:

$P \equiv Q$ for $\forall x\ (Px \Leftrightarrow Qx)$
$P \subseteq Q$ for $\forall x\ (Px \Rightarrow Qx)$
$P \subset Q$ for $\forall x\ (Px \Rightarrow Qx)\ \&\ \neg\, P \equiv Q$
$P \cup Q$ for $\{z : Pz \vee Qz\}$ or $\{y,z : Py,z \vee Qy,z\}$
$P \setminus Q$ for $\{z : Pz\ \&\ \neg\, Qz\}$ or $\{y,z : Py,z\ \&\ \neg\, Qy,z\}$
$P \sim Q$ for $\exists R$ (R is a one-to-one function from P onto Q)
$x \leq_M y$ for $Nx\ \&\ Ny\ \&\ \exists P \exists Q\ (P \subseteq Q\ \&\ Mx,P\ \&\ My,Q)$
$x <_M y$ for $x \leq_M y\ \&\ \neg\, x = y$
ϕ for $\{z : \neg\, z = z\}$ or $\{y,z : \neg\, z = z\}$
$\mathbb{U}$ for $\{z : z = z\}$
$\{a\}$ for $\{z : z = a\}$
$\{(a,b)\}$ for $\{y,z : y = a\ \&\ z = b\}$
$x \in P$ for Px

(R^D) for {x : ∃y Rx,y}

(R^I) for {y : ∃x Rx,y}

(R α a b) for {x,y : Rx,(y α a b) }

where (c α a b)

= c if (¬ c = a & ¬ c = b),

= a if c = b, and

= b if c = a

In practice, as we shall see, M will be fixed, and so the subscript from "$\leq_M$" and "$<_M$" will always be dropped.

Remark that these are *abbreviations* and no decision about what logic we are working in, needs yet to be made. E.g. the interpretation for {a} will be determined by the interpretation of {z : z = a}, whatever thing or entity it might be, in accordance with the underlying logic which is then in use. In fact, we will be ambiguous about the underlying logic whenever possible; in particular, the F axioms given below prove the Peano Axioms considered either as a system in first- *or* second-order logic.

Peano's Axioms pre-addition (PA) will be taken as:

(PA1) N0

(PA2) ∀n∀m (Nn & σn,m ⇒ Nm)

(PA3) ∀n (Nn ⇒ ∃m σn,m)

(PA4) ∀n∀m∀m' (Nn & σn,m & σn,m' ⇒ m = m')

(PA5) ∀n∀m∀n' (Nn & Nn' & σn,m & σn',m ⇒ n = n')

(PA6) ∀n (Nn ⇒ ¬ σn,0)

(PA7) Induction

"Frege's" Axioms are the axioms used to prove Frege's Theorem. It is often said that there is simply one axiom, namely Hume's Principle.

(HP) ∀P∀Q (P ~ Q ⇔ #P = #Q)

where '#' means "the number of", which is a function on big-letters.

Of course, the definitions of N, σ, and 0--the non-logical symbolism in PA--are crucial in any proof. Normally these definitions are presented as abbreviations--which would imply that they cannot be challenged, as they are stipulative. But this is a remarkable thesis, because the building of arithmetic is pre-existing, and there are prior meanings of 0, succession, and finite number. Any stipulation of these symbols, could give them different meanings than they ordinarily have. And doing so, one would risk providing not a support of what arithmetic actually is, but of something else entirely--something which *looks* like arithmetic but is *not*. So, if only for transparency's sake, it is better to take these definitions, not as stipulative, but as axioms in their full right, as assertions which can be challenged and even denied. (And we will promptly question one of them, which surely discounts its right to be a "stipulation"!) In any case, doing so, the axioms of Frege Arithmetic become, in second-order logic with full comprehension (following Boolos "The Standard of Equality of Numbers", [1]):

(FA1) $\forall P \forall Q \, (P \sim Q \Leftrightarrow \#P = \#Q)$

(FA2) $\forall P \, (\#P = 0 \Leftrightarrow \neg \exists x \, Px)$

(FA3) $\forall n \forall m \, (\sigma n,m \Leftrightarrow \exists P \exists a \, (\neg Pa \ \& \ \#P = n \ \& \ \#(P \cup \{a\}) = m))$

(FA4) $\forall n \, (Nn \Leftrightarrow n = 0 \vee \forall P \, [\forall a \forall b ([(a = 0 \vee Pa) \ \& \ \sigma a,b] \Rightarrow Pb) \Rightarrow Pn])$

Now the use of the number operator makes implicitly two additional assumptions--uniqueness and an ontological one--which appear "outside" the axioms, and so are to an extent hidden. In the aim of being completely upfront, though at the risk of being verbose, we therefore split (FA1) aka Hume's Principle into three separate assertions:

(FA1a) $\forall P \forall Q \, (Mn,P \Rightarrow (P \sim Q \Leftrightarrow Mn,Q))$

(FA1b) $\forall P \exists n \, Mn,P$

(FA1c) $\forall P \, (Mn,P \ \& \ Mm,P \Rightarrow n = m)$

Here M is a *third-order* predicate symbol, and Mn,P means that the things satisfying P are n in number. M is not, in and of itself, assumed to be functional or total; that of course is the role of (FA1b) and (FA1c).

Remark that (FA1a) here is the philosophical heart of Hume's Principle, and says that P and Q are equinumerous if they can be put in a one-to-one correspondence. It has, however, evidently been shorn of any ontological (and uniqueness) assumptions, and in particular it is neutral about whether there *are* any numbers.

(FA2) and (FA3) should accordingly be replaced by:

(FA2*) $\forall P\, (M0,P \Leftrightarrow \neg \exists x\, Px)$
(FA3*) $\forall n \forall m\, (\sigma n,m \Leftrightarrow \exists P \exists a\, (\neg Pa\ \&\ Mn,P\ \ \&\ Mm,(P \cup \{a\})))$

Remark that (FA2*) in fact says a little less than (FA2), since it makes no assertion that P's number is unique. Of course, since (FA1c) makes this assertion in its full generality, it is not necessary to repeat it. Uniqueness has also been eliminated from (FA3*). Since it is a biconditional, there is one implication which is stronger, another which is weaker, than its counterpart in (FA3).

Now one criticism of these axioms, which is not usually made but which was alluded to in the introduction, is that (FA3) makes large ontological presumptions. That is, if σm,n holds, then it follows that there exist n things. At the least, this does not seem indisputable, logical, or part of the essential meaning of the word "successor". It just does not seem that, from the fact that 100 follows 99, there should necessarily be 100 things.

Of course, we know that, if 100 follows 99, then there are indeed 100 things, namely the numbers from 0 to 99, or from 1 to 100. However, this knowledge is based on the following facts: that numbers have successors (PA3), that these successors are natural numbers (PA2), that 0 is a natural number (PA1), that these successors are unique (PA4), and that 0 is never a successor (PA6). So, in order to recognize this definition of 'successor' as correct, one needs to accept almost all the Peano Axioms. This presumably diminishes the interest of such an axiom in a foundation, which is meant to justify as well as prove the Peano Axioms. It cannot justify them, since it is justified *by* them.

Indeed, together only with induction and (FA2*) (i.e. FA1a, b or c are not needed), (FA3*) implies quite easily one weak form of "potential infinity" (a stronger form will be discussed in much more detail below), namely that for every natural number, there are at least that many things, i.e.

$\forall n\ (\ Nn \Rightarrow \exists P\ Mn,P\ )$ (*WEAKPOTINF*)

For, by (FA2), $M0,\{x : \neg\, x = x\}$. And, if $(a = 0 \vee Pa)\ \&\ \sigma a,b$, then $\exists Q\ \ Mb,Q$ by the second conjunct and (FA3*).

That this apparently huge assertion falls straight out of induction and a "definition" of succession, is hardly comforting. So, while HP has often been taken to task for *its* ontological entailments, it is not the only culprit among Frege's Axioms. (FA3) carries its own responsibility as well.

There are, as well, other criticisms of FA; most notably there is controversy about Hume's Principle. Boolos ("Is Hume's Principle Analytic?" in [1]) questions its logical status, and Heck ([9]) claims to be a "militant agnostic" about its truth. In fact, Heck replaces (HP) with a principle restricted to *finite* predicates:

(FHP) $\forall P \forall Q\ (\ \mathrm{Finite}(P) \vee \mathrm{Finite}(Q) \Rightarrow (P \sim Q \Leftrightarrow \#P = \#Q)\ )$

He goes on to show that (FHP), modulo the additional axioms (FA2), (FA3), and (FA4), in second-order logic with full (impredicative) comprehension, can prove PA. As with (HP), (FHP) in fact bundles together three separate assertions. The ontological commitment is now much less than (HP)'s: the only predicates guaranteed to have a number are those which are *finite*. Indeed, on his way to present a system in terms of the concept "just as many," Heck ([10]) points out that the following axioms can prove PA:

(H1) $\forall P\ (\#P = 0 \Leftrightarrow \neg \exists x\ Px)$
(H2) $\forall P \forall Q \forall a \forall b\ (\ \#P = \#Q\ \&\ \neg\, Pa\ \&\ \neg\, Qb \Rightarrow \#(P \cup \{a\}) = \#(Q \cup \{b\})\ )$
(H3) $\forall P \forall Q \forall a \forall b\ (\ \#P = \#Q\ \&\ Pa\ \&\ Qb \Rightarrow \#(P \setminus \{a\}) = \#(Q \setminus \{b\})\ )$
(H4) $\forall n \forall m\ (\ \sigma n,m \Leftrightarrow \exists P \exists a\ (\neg\, Pa\ \&\ \#P = n\ \ \&\ \#(P \cup \{a\}) = m\ )\ )$
(H5) $\forall n\ (\ Nn \Leftrightarrow n = 0 \vee \forall P\ [\forall a \forall b([(a = 0 \vee Pa)\ \&\ \sigma a,b] \Rightarrow Pb) \Rightarrow Pn]\ )$

While the "just as many" system is undoubtedly of interest, it has less in common with our own subsequent axiomatizations, and so will not be considered here. The H axioms, however, have certain points in common, and so warrant a review.

Remark first that (FHP) falls out as a theorem in H. As before, break the number operator into its separate parts. Then, the following suffice to prove PA:

(H1a) $\forall P\ (M0,P \Leftrightarrow \neg \exists x\ Px)$

(H1b) $UNIQUE_0$

(H2a) $\forall n\forall m\forall P\forall Q\forall a\forall b\ (Mn,P \ \&\ Mn,Q \ \&\ \neg Pa \ \&\ \neg Qb \ \&\ Mm,(P \cup \{a\}) \Rightarrow Mm,(Q \cup \{b\}))$

(H2b) $\forall n\forall P\forall a\ (Mn,P \ \&\ \neg Pa \Rightarrow \exists m\ Mm,(P \cup \{a\}))$

(H2c) $\forall n\forall m\forall P\forall a\ (Mn,P \ \&\ \neg Pa \ \&\ \sigma n,m \ \&\ UNIQUE_n (P) \Rightarrow UNIQUE_m (P \cup \{a\}))$

(H3a) $\forall n\forall m\forall P\forall Q\forall a\forall b\ (Mn,P \ \&\ Mn,Q \ \&\ Pa \ \&\ Qb \ \&\ Mm,(P \setminus \{a\}) \Rightarrow Mm,(Q \setminus \{b\}))$

(H4) $\forall n\forall m\ (\sigma n,m \Leftrightarrow \exists P\exists a\ (\neg Pa \ \&\ Mn,P \ \&\ Mm,(P \cup \{a\})))$

(H5) $\forall n\ (Nn \Leftrightarrow n = 0 \vee \forall P\ [\forall a\forall b([(a = 0 \vee Pa) \ \&\ \sigma a,b] \Rightarrow Pb) \Rightarrow Pn])$

Here $UNIQUE_n (P)$ abbreviates

$\forall k\ (Mn,P \ \&\ Mk,P \Rightarrow k = n)$

and $UNIQUE_n$ abbreviates

$\forall P\ UNIQUE_n (P)$

Remark that what would have been (H3b) and (H3c) have been omitted, since they are not needed to prove PA. Also, (H2c) has been slightly weakened from what it should have been.

Of course, now that (HP) is no longer an axiom, much less provable, it can hardly present controversy. However, (FA3) aka (H4) still makes an appearance.

Tennant's Axioms

In his *Anti-Realism and Logic* ([11]), Neil Tennant introduces an axiom system which also proves PA. He works in intuitionistic relevant logic and states his assumptions as introduction and elimination rules of deduction. In terms of the present notation, with existence and uniqueness claims separate from the others, with unnecessary axioms removed (as with the H system), they are:

(T1a) UNIQUE_0
(T1b) $\forall n \forall m \forall P \forall Q \forall a\ (\ \sigma n,m\ \&\ Qa\ \&\ \mathit{UNIQUE}_n\ (P)\ \&\ (Q \setminus \{a\}) \sim P$
$\Rightarrow \mathit{UNIQUE}_m\ (P)\ )$
(T2) $\forall P\ (\ M0,P \Leftrightarrow \neg \exists x\ Px\ )$
(T3) $\forall n \forall m \forall P \forall Q \forall a\ (\ \sigma n,m\ \&\ Qa\ \&\ (Q \setminus \{a\}) \sim P$
$\Rightarrow (Mn,P \Leftrightarrow Mm,Q)\ )$
(T4) $\forall n\ (\ Nn \Leftrightarrow n = 0 \vee \forall P\ [\forall a \forall b([(a = 0 \vee Pa)\ \&\ \sigma a,b] \Rightarrow Pb) \Rightarrow Pn]\ )$
(T5) $\forall n \forall P \forall Q \forall a\ (\ Mn,P\ \&\ (Q \setminus \{a\}) \sim P \Rightarrow \exists m\ Mm,Q\ )$

It should be remarked that Tenant's original system, unlike the Wright-Boolos-Heck tradition, which we have seen introduces an # operator which implicits ontology (and uniqueness), is more explicit about the assertion of the existence of numbers and of objects generally; there is no (FA3). Tennant, comparing his system to Frege Arithmetic, can rightfully claim an advantage in assuming only the existence of the finite numbers. (Heck's systems will have this same advantage--and were developed independently--but come some ten years later.) His system is in the Fregean tradition, in that it grounds number on one-to-one correlation. Although Tennant does not need to assert Finite Hume's Principle to prove PA, it apparently would be a justification for some of his axioms, in particular (T3). When assuming (T3), one should, at the back of one's mind, know that one-to-one correlation implies equinumerosity, and that $(Q \setminus \{a\}) \sim P$ means that $(Q \setminus \{a\})$ and P are equinumerous. Indeed, Tennant even explicitly states in *Anti-Realism and Logic* rules of introduction and elimination for number which effectively assumes Finite Hume's Principle (in the form of (FA1)), although they have not been included in the present axiomatization because they are not needed in the proof of PA.

Finally, although formulated in a weakish logic, Tennant, like Heck, makes use of impredicativity.

Towards a Foundation

Consider now the following axioms:

(F1) $\forall n\forall m\forall P\ (Nn \ \&\ Mn,P \ \&\ Mm,P \Rightarrow n = m)$
(F2) $\forall P\ (M0,P \Leftrightarrow \neg \exists x\, Px)$
(F3) $\forall n\forall m\forall P\forall Q\forall a\ (Nn \ \&\ \sigma n,m \ \&\ \neg Pa \ \&\ \forall x(Qx \Leftrightarrow Px \vee x = a) \Rightarrow (Mn,P \Leftrightarrow Mm,Q))$
(F4) (*Induction*.) Let ϕ be a well-formed formula (with no appearance of m). Use $\phi[x\backslash y]$ to mean x replaces all (free) instances of y. Suppose $\phi[0\backslash n]$ and $\forall n\forall m\ (Nn \ \&\ \sigma n,m \ \&\ \phi \Rightarrow \phi[m\backslash n])$. Then $\forall n\ (Nn \Rightarrow \phi)$.
(F5) $N0$
(F6) (*Ad infinitum*) $\forall n\forall P\forall a\ (Nn \ \&\ Mn,P \ \&\ \neg Pa \Rightarrow \exists m\ (Nm \ \&\ Mm,(P \cup \{a\})))$

It will be shown that these six axioms (actually axiom schema, in the case of (F4)), in second-order logic with *only predicative comprehension*, suffice to prove PA.

Of course, the biconditional of (F3) can be rendered into two implications:

(F3a) $\forall n\forall m\forall P\forall Q\forall a\ (Nn \ \&\ \sigma n,m \ \&\ \neg Pa \ \&\ \forall x(Qx \Leftrightarrow Px \vee x = a) \ \&\ Mn,P \Rightarrow Mm,Q)$
(F3b) $\forall n\forall m\forall P\forall Q\forall a\ (Nn \ \&\ \sigma n,m \ \&\ \neg Pa \ \&\ \forall x(Qx \Leftrightarrow Px \vee x = a) \ \&\ Mm,Q \Rightarrow Mn,P)$

The F axioms are essentially the T axioms, except that one-to-correlation has been replaced with equivalence. Their content is therefore considerably weaker. As in the H system, (FHP) is proved and not, either implicitly or explicitly, assumed. Uniqueness is stated in a general way in the form of (F1), rather than based on a ratchet principle--(T1a) and (T1b). However, (F1)'s formulation is not essential, and axioms comparable to (T1a) and (T1b), or indeed (T1a) and (T1b) with equivalence replacing one-to-one correlation, could serve.

Now compare the F axioms with the H ones given above.

(F1) does the work of (H1b) and (H2c). As commented, (F1) can be replaced by equivalents to (H1b) and (H2c).
(F2) and (H1) are the same.
(F3) does the work of (H2a), (H3a), and (H4).
(F4) is induction, so like (H5).

(F5) is part of (H5).
(F6) is comparable to (H2b).

The greatest difference, therefore, is between (F3) and the corresponding trio (H2a), (H3a), and (H4). (H2a) and (H3a) correspond roughly to (F3a) and (F3b) and can also be combined together using a biconditional. Nonetheless, it is clear by comparison with (F3), that there is double work going on in the three H axioms. (F3) is, apparently, a simpler way of putting things.

Significantly, (F3) makes *no* ontological commitments, unlike (H4). The only F axiom which asserts the existence of objects (other than the use of the constant 0), is (F6). And, without (F5) or (F6), there need not be *any* natural numbers. The sub-system of (F1) to (F4) is, in its own right, important. Firstly, many useful propositions, including a form of (FHP), are provable within it. And secondly, because of its minimal ontology, it presumably would be acceptable to an ultrafinitist.

Remark that nowhere is there an axiom which has succession as consequent, again in contrast to (H4) which is a biconditional. That is, succession appears as the antecedent in (F3), and it says what obtains when m is the successor of n. But nowhere is it stated what must obtain for m to be the successor of n. As we shall see, this is not needed and will indeed be provable.

Even though (H4) has existential implications, it is still so to speak very weak. Because of induction, what is important, is to know what m succeeding n implies. (H4) makes a very specific assertion: that there exist certain predicates of size m and n. It therefore lacks generality, by not saying what happens for *all* predicates of size m and n. To make this up, the H system requires (H2a) and (H3a). (F3), in contrast, is fully general, and does not need any help from other axioms.

More positively, both the F and H systems eliminate one-to-one correlation from their assumptions, which is an important liberation from the logical orthodoxy in place since Cantor-Frege, which says that one-to-one correlation is or even must be the basis of a theory of number.

Note that (F1) *cannot* be replaced by

$$\forall n \forall m \forall P\ (\ Nn\ \&\ Nm\ \&\ Mn,P\ \&\ Mm,P\ \Rightarrow n = m\ ).$$

For in this case, it would be possible that both 1 and Julius Caesar succeed 0, so long as Julius Caesar is not a natural number but numbered every predicate satisfied by precisely one thing. But then PA2 would not be true.

(F1) *can*, however, be replaced by the weaker pair:

(F1a) $\forall n \forall P\ (\ Mn,P\ \&\ \neg \exists x\ Px \Rightarrow n = 0\ )$
(F1b) $\forall n \forall m \forall P\ (\ Nn\ \&\ \exists p\ (Np\ \&\ \sigma p,m)\ \&\ Mn,P\ \&\ Mm,P\ \Rightarrow n = m\ )$.

Now (F1) is not equivalent to (F1a) & (F1b). For consider the case of the standard model, with an additional Augustus Caesar, who is not a natural number and has no predecessor yet numbers the predicate $\{x : x = 0\}$. Then all of (F1a),(F1b), (F2), (F3), (F4), (F5), and (F6) are satisfied, but not (F1).

Nonetheless, (F1) clearly implies both (F1a) and (F1b). Remark that (F1b), while weaker than (F1), is stronger than

$$\forall n \forall m \forall P\ (\ Nn\ \&\ Nm\ \&\ \neg\ m = 0\ \&\ Mn,P\ \&\ Mm,P\ \Rightarrow n = m\ ),$$

since the situation with the aforementioned Julius Caesar satisfies this, but it does not satisfy (F1b). The trick is that $\exists p\ (Np\ \&\ \sigma p,m)$ is strictly weaker than $Nm\ \&\ \neg\ m = 0$. (Of course, by an easy induction, it is possible to prove that $Nm\ \&\ \neg\ m = 0 \Rightarrow \exists p\ (Np\ \&\ \sigma p,m)$.)

In the subsequent, we will use only (F1a) and (F1b), and not (F1).

It is easily seen that (F4) is equivalent to

(F4*) Suppose $N0 \Rightarrow \phi\,[0\backslash n]$ and $\forall n \forall m\ (\ Nn\ \&\ Nm\ \&\ \sigma n,m\ \&\ \neg\ m = 0\ \&\ \phi \Rightarrow \phi\,[m\backslash n]\ )$. Then $\forall n\ (\ Nn \Rightarrow \phi\ )$.

Finally, remark that the F axioms can be considered belonging either to a second-order (with one third-order predicate) *or* a first-order system, since only predicative comprehension is needed or allowed. That is, although big- and little-letters are used in the formulation of the axioms, which gives them the appearance of a second-order system (or, according to the beholder, a two-type first-order system), in fact both big- and little-letters may be considered the same type, and so the system would be (one-type) first-order. There is no contradiction because comprehension is predicative. That is, define the *predicate place* of wffs “Px”, “Px,y”, etc. to be the left-hand argument, i.e. “P.” E.g. “y” is in the predicate place of “yu,v” (which in a first-order system, would a wff, just as much as “Pu,v”). In a (one-type) first-order system the rule for predicative comprehension is (for one-place predicates):

$$\forall x\, (\, \{x :\ \phi\, \}x \Leftrightarrow \phi\,)$$

where ϕ does not contain either “x” or any quantified variable in the predicate place.

Indeed, one could also use a hybrid system (which, historically, is the system in which the author developed it), where big-letters can be substituted for universally quantified small-lettered variables, but not the reverse (enacting the thesis that every thing, including a predicate, is a thing, but not every thing is a predicate).

The axioms therefore have an applicability across many different logical systems.

Potential Infinity

Recall the weak form already introduced above. The strong form of “potential infinity” asserts that, for every finite number, there are more than that many things; formally,

$$\forall n\ (\ Nn \Rightarrow \exists P \exists a\ (Mn,P\ \&\ \neg\ Pa)\ )\quad (\mathit{POTINF}\,)$$

A glance at the axioms shows that *POTINF* and

$$\exists P \exists a\ (Mn,P\ \&\ \neg\ Pa)\quad (\mathit{POTINF}_n\,)$$

play an important role in the F system. Both (F6) and one half of (F3)’s biconditional require, in effect, $POTINF_n$ in order for their conclusions to be drawn.

A proof of $POTINF$ plays a central role in the demonstration that the F axioms prove PA. Historically, there are two approaches, one dating from Dedekind, the other from Frege. Dedekind ([2]) used a *hierarchal* approach. Constructing a chain of logical or pseudo-logical things (in Dedekind's case, thoughts, but we would substitute sets or predicates), one shows that the next member of the hierarchy cannot belong to the hitherto existing part. The hierarchal approach cannot be formulated, at least naturally, in second-order logic, which after all has only two levels, things and predicates. It requires a first-order or first-orderish theory.

Frege proceeded with what might be called the *bootstrap* approach, which uses the natural numbers themselves. 0 is one thing, 0 and 1 are two things, and in general the numbers less than n are n in number. $POTINF$ follows once one establishes as well that n is not itself less than n.

It is interesting that in fact $POTINF$ can be demonstrated using only (F1) to (F4), i.e. that there *do not need to be any natural numbers at all* to prove an assertion which “says” that there are an infinite number of things. (Indeed, if there were no natural numbers, then $POTINF$ is trivially true.) Since among (F1) to (F6), an ultrafinitist presumably only rejects (F6), the “ad infinitum” premise, he is therefore committed to asserting an infinity of things.

There is evidently a Skolemish angle here which is worthy of emphasis. “Inside” the theory there is an infinitude of numbers, while seen from the “outside” there is not. It is remarkable that Frege's approach goes through and underlines its ontological stinginess. This has, apparently, hitherto not been widely commented, because the use of the number operator in Frege Arithmetic made it seem, falsely, that an ontological commitment was necessary in the proof. Dedekind's approach is more demanding, in the sense that it requires the existence of a pseudo-logical hierarchy of a first-order-type logic.

Lemmas

We first prove seven lemmas, using only (F1)-(F4); in particular no appeal is made to either (F5) or (F6).

L1. Suppose $\exists n\, Nn$. Then $N0$.
Pf: It suffices to show $\forall n\, (\, Nn \Rightarrow N0\,)$. Proceed by induction (F4*), with ϕ as $N0$.

L2. $\forall P \forall n\, (\, Mn,P \,\&\, \neg\, n = 0 \Rightarrow \exists x\, Px\,)$
Pf: Assume $Mn,P \,\&\, \neg\, n = 0$. Suppose $\neg\, \exists x\, Px$. Then $M0,P$ by (F2). So by (F1a), $n = 0$.

L3. $\forall P \forall Q\, (\, M0,P \Rightarrow (P \sim Q \Leftrightarrow M0,Q)\,)$
Pf: Assume $M0,P$. By (F2), $\neg\, \exists x\, Px$. Suppose $P \sim Q$. Then evidently $\neg\, \exists x\, Qx$. By (F2), $M0,Q$. On the other hand, suppose $\neg\, \exists x\, Qx$. By (F2), $M0,Q$. So $P \sim Q$.

L4. $\forall P \forall Q\, (\, Nn \,\&\, Mn,P \Rightarrow (P \sim Q \Leftrightarrow Mn,Q)\,)$
Remark: This is a form of Finite Hume’s Principle. Remark that the ensuing proof does not use (F1b), i.e. it only requires (F1a), (F2), (F3), and (F4).
Pf: By induction, (F4*). L3 proves the case $n = 0$. Assume $Nn \,\&\, Nm \,\&\, \sigma n,m \,\&\, \neg\, m = 0 \,\&\, \forall P \forall Q\, (\, Mn,P \Rightarrow (P \sim Q \Leftrightarrow Mn,Q)\,)$. And suppose Mm,P. By L2, Pa for some a. Consider $P' = P \setminus \{a\}$. By (F3), Mn,P'.

Suppose $P \sim Q$. Let R be a one-to-one relationship from P onto Q. Then Ra,b for some b such that Qb. Set $Q' = Q \setminus \{b\}$. Evidently, $R \setminus \{(a,b)\}$ is one-to-one from P' onto Q', so $P' \sim Q'$. By the induction hypothesis, Mn,Q'. By (F3), Mm,Q.

Now suppose Mm,Q. Since $\neg\, m = 0$, Qb for some b, by L2. Consider again $Q' = Q \setminus \{b\}$. By (F3), Mn,Q'. By the induction hypothesis, $P' \sim Q'$. Let R' be a one-to-one relationship from P' onto Q'. Then $R' \cup \{(a,b)\}$ is one-to-one from P onto Q. So $P \sim Q$.

We have already mentioned this lemma, but we state it here so to use it in the proofs of the next two lemmas:
L5 $\forall n\, (\, Nn \,\&\, \neg\, n = 0 \Rightarrow \exists p\, (Np \,\&\, \sigma p,n)\,)$
Pf: Proceed by induction with ϕ as $\neg\, n = 0 \Rightarrow \exists p\, (Np \,\&\, \sigma p,n)$. $n = 0$ is trivial. Evidently, $Nn \,\&\, \sigma n,m \,\&\, \phi \Rightarrow \exists p\, (Np \,\&\, \sigma p,m)$.

L6 $\forall n \forall m \forall P\,(\, Nn \,\&\, Nm \,\&\, Mn,P \,\&\, Mm,P \;\Rightarrow\; n = m \,)$
Pf: Suppose $Nn \,\&\, Nm \,\&\, Mn,P \,\&\, Mm,P$. If $m = 0$, then apply (F1a). If $\neg\, m = 0$, then $Np \,\&\, \sigma p,m$ for some p, by L5. Apply (F1b).

The next proposition is striking by the appearance of succession as the consequent.

L7. $\forall n \forall m \forall P \forall a\,(\, Nn \,\&\, Nm \,\&\, Mn,P \,\&\, Mm,(P \cup \{a\}) \,\&\, \neg\, Pa \;\Rightarrow\; \sigma n,m \,)$
Pf: Assume $Nn \,\&\, Nm \,\&\, Mn,P \,\&\, Mm,(P \cup \{a\}) \,\&\, \neg\, Pa$. By (F2), $\neg\, m = 0$, so by L5, $Np \,\&\, \sigma p,m$ for some p. By (F3), Mp,P. By L6, $n = p$.

Assuming *POTINF*, the F Axioms imply PA

Now turn to the proof of PA, where *POTINF* is assumed. Of course, the proof appeals to (F5) and (F6), and not merely (F1)-(F4).

PA1 is just (F5), and PA7 is just (F4). There remains to prove PA2 through PA6. We assume *POTINF*.

PA2. $\forall n\,(\, Nn \,\&\, \sigma n,m \;\Rightarrow\; Nm \,)$
Pf:
Suppose $Nn \,\&\, \sigma n,m$. By *POTINF*, $Mn,P \,\&\, \neg\, Pa$ for some P,a. By (F3), $Mm,(P \cup \{a\})$. By (F6), $Nm' \,\&\, Mm',(P \cup \{a\})$ for some m'. By (F1b), $m = m'$.

PA3. $\forall n\,(\, Nn \Rightarrow \exists m\, \sigma n,m \,)$
Pf:
Suppose Nn. By *POTINF*, $Mn,P \,\&\, \neg\, Pa$ for some P,a. By (F6), $Nm \,\&\, Mm,(P \cup \{a\})$ for some m. By L7, $\sigma n,m$.

PA4+PA5. $\forall n \forall m \forall n' \forall m'(\, Nn \,\&\, Nn' \,\&\, \sigma n,m \,\&\, \sigma n',m' \;\Rightarrow\; (n = n \Leftrightarrow m = m')\,)$
Pf:
Assume $Nn \,\&\, Nn' \,\&\, \sigma n,m \,\&\, \sigma n',m'$.

Suppose $n = n'$. By *POTINF* and (F3), $Mn,P \,\&\, \neg\, Pa \,\&\, Mm,(P \cup \{a\}) \,\&\, Mm',(P \cup \{a\})$ for some P,a. By PA2, Nm. Now apply (F1b).

Suppose $m = m'$. By *POTINF* and (F3), $Mn,P \,\&\, \neg\, Pa \,\&\, Mm,(P \cup \{a\}) \,\&\, Mn',P' \,\&\, \neg\, P'a' \,\&\, Mm,(P' \cup \{a'\})$ for some P,P',a,a'. By PA2, Nm. So $(P \cup \{a\}) \sim (P' \cup \{a'\})$ by L4, hence evidently $P \sim P'$. By L4 again, Mn,P'. By

L6, $n = n'$.

(PA6) $\forall n\,(Nn \Rightarrow \neg\, \sigma n,0)$
Pf:
Suppose Nn but $\sigma n,0$. Then by *POTINF* and (F3), $M0,(P \cup \{a\})$ for some P,a. But this contradicts (F2).

Two Proofs of *POTINF*.

Herein are provided both "bootstrap" and "hierarchal" proofs for *POTINF*. As already mentioned, both proofs only use (F1)-(F4); no appeal to (F5) or (F6) is needed.

First, the "bootstrap" approach. Remark that any use of comprehension is predicative.

B1.
a. If Nn & Mn,P, then $0 \leq n$ and $n \leq n$.
b. If $\exists x\, Nx$, then both $0 \leq 0$ and $((0 \leq z \;\&\; z \leq 0) \Leftrightarrow z = 0)$.
Pf:
a. Suppose Nn & Mn,P. By (F2), $M0,\phi$. By L1, $N0$. Thus $N0$ & Nn & $\phi \subseteq P$ & $M0,\phi$ & Mn,P. So $0 \leq n$. Also, Nn & Nn & $P \subseteq P$ & Mn,P & Mn,P. So $n \leq n$.
b. Assume $\exists x\, Nx$. By L1, $N0$. Of course $M0,\phi$ by (F2). So by part a, $0 \leq 0$. Now suppose $0 \leq z$ & $z \leq 0$. By the second inequality, Nz & $N0$ & $\exists P \exists Q\,(P \subseteq Q \;\&\; Mz,P \;\&\; M0,Q)$. By (F2), $Q \equiv \phi$. Hence $P \equiv \phi$, so by L2, $z = 0$.

B2. $\forall n \forall P \forall Q\,(Nn \;\&\; Mn,P \;\&\; Mn,Q \;\&\; P \subseteq Q \Rightarrow P \equiv Q)$
Pf: By induction (F4*) on n, with ϕ as $\forall P \forall Q\,(Nn \;\&\; Mn,P \;\&\; Mn,Q \;\&\; P \subseteq Q \Rightarrow P \equiv Q)$.
Obvious for $n = 0$, using (F2). Assume then that Nn & Nm & $\sigma n,m$ & $\neg\, m = 0$ & ϕ, and suppose Mm,P & Mm,Q & $P \subseteq Q$. By L2, Px for some x. Set $P' = P \setminus \{x\}$ and $Q' = Q \setminus \{x\}$. By (F3), Mn,P' & Mn,Q'. Clearly $P' \subseteq Q'$, so by the induction hypothesis, $P' \equiv Q'$. Hence $P \equiv Q$.

B3. Suppose Nn & Mn,P & Mn,Q. If $\neg\, P \equiv \mathbb{U}$, then $\neg\, Q \equiv \mathbb{U}$.
Pf: Suppose $\neg\, P \equiv \mathbb{U}$ but $Q \equiv \mathbb{U}$. $P \equiv Q$ follows immediately from B2, a contradiction.

B4. Let $n < m$. Then $Nn \mathbin{\&} Nm \mathbin{\&} \exists P \exists Q\, (P \subset Q \mathbin{\&} Mn,P \mathbin{\&} Mm,Q)$.
Pf: $n \leq m$, so $Nn \mathbin{\&} Nm \mathbin{\&} P \subseteq Q \mathbin{\&} Mn,P \mathbin{\&} Mm,Q$, for some P and Q. Suppose $P \equiv Q$. Evidently $P \sim Q$, so by L4, Mm,P. By L6, $n = m$, contradicting the definition of $n < m$.

B5. Assume $Nn \mathbin{\&} \sigma n,m \mathbin{\&} \neg\, m = 0 \mathbin{\&} \mathit{POTINF}_m$. Then POTINF_n.
Pf: Mm,P for some P, since POTINF_m. By L2, Pa for some a. Then by (F3), $Mn,(P\backslash\{a\})$. Evidently, $\neg\, (P\backslash\{a\})a$. Hence POTINF_n.

B6. Assume $Nn \mathbin{\&} \sigma n,m$. If $k < m$, then $k \leq n$.
Pf: Let $k < m$. By B4, $Nk \mathbin{\&} Nm \mathbin{\&} P \subset Q \mathbin{\&} Mk,P \mathbin{\&} Mm,Q$, for some P and Q. Then $\exists a\, (Qa \mathbin{\&} \neg\, Pa)$. By (F3), $Mn,(Q\backslash\{a\})$. Evidently, $P \subseteq Q\backslash\{a\}$. So $k \leq n$.

B7. Assume $Nm \mathbin{\&} \sigma n,m \mathbin{\&} \mathit{POTINF}_n$. If $k \leq n$, then $k < m$.
Pf: Let $k \leq n$. $Nk \mathbin{\&} Nn \mathbin{\&} P \subseteq Q \mathbin{\&} Mk,P \mathbin{\&} Mn,Q$, for some P and Q. By POTINF_n, there are Q' and a' such that $\neg\, Q'a' \mathbin{\&} Mn,Q'$. So $Q \sim Q'$ by L4. Evidently $\neg\, Q' \equiv \mathbb{U}$, and thus by B3, $\neg\, Q \equiv \mathbb{U}$. Hence $\neg\, Qa$ for some a. By (F3), $Mm,(Q \cup \{a\})$. Of course $P \subseteq (Q \cup \{a\})$, so $k \leq m$. If $k = m$, then $P \equiv (Q \cup \{a\})$ by B2, which contradicts $P \subseteq Q \mathbin{\&} \neg\, Qa$.

B8. Assume $Nn \mathbin{\&} Nm \mathbin{\&} \sigma n,m \mathbin{\&} \mathit{POTINF}_n$. Then $(0 \leq z \mathbin{\&} z \leq m) \Leftrightarrow (0 \leq z \mathbin{\&} z \leq n) \vee z = m$. Also, $\neg\, (0 \leq m \mathbin{\&} m \leq n)$.
Pf: Suppose $0 \leq z \mathbin{\&} z \leq m$. By B6, either $z = m$ or $z \leq n$.

Now suppose $0 \leq z \mathbin{\&} z \leq n$. By B7, $z < m$ so $z \leq m$.

Suppose $z = m$. By POTINF_n and (F3), Mm,Q for some Q. By B1a, $m \leq m$.

Finally, suppose, to the contrary, $(0 \leq m \mathbin{\&} m \leq n)$. By B7, $m < m$, a contradiction.

B9. $Nn \mathbin{\&} \mathit{POTINF}_n \Rightarrow \exists P\, \forall z\, (0 \leq z \mathbin{\&} z \leq n \Leftrightarrow Pz)$
Pf: Proceed by induction, (F4*), with ϕ as $(\mathit{POTINF}_n \Rightarrow \exists P\, \forall z\, (0 \leq z \mathbin{\&} z \leq n \Leftrightarrow Pz))$. Suppose $N0 \mathbin{\&} \mathit{POTINF}_0$. By B1b, $0 \leq z \mathbin{\&} z \leq n \Leftrightarrow z = 0 \Leftrightarrow z \in \{x : x = 0\}$.

Assume $Nn \mathbin{\&} \sigma n,m \mathbin{\&} \neg\, m = 0 \mathbin{\&} \phi$. And suppose POTINF_m. By B5,

$POTINF_n$, so by the induction hypothesis, for some P, $\forall z\ (0 \le z \ \&\ z \le n \Leftrightarrow Pz))$. Now by B8, $(0 \le z \ \&\ z \le m) \Leftrightarrow ((0 \le z \ \&\ z \le n) \vee z = m) \Leftrightarrow z \in \{x : Px \vee z = m\}$.

When Nn & $POTINF_n$, we will use $\{0 \_ n\}$ to represent a predicate P (clearly unique up to equivalence) promised in the proposition. Remark that it is now a corollary of B8 that

B10. Assume Nn & Nm & $\sigma n,m$ & $POTINF_n$ & $POTINF_m$. Then $\{0 \_ m\} \equiv \{0 \_ n\} \cup \{m\}$, where in fact $\neg\ m \in \{0 \_ n\}$.

B11. $\forall n\ (Nn \ \&\ POTINF_n \Rightarrow \forall k \in \{0 \_ n\}\ \forall P\ (\{0 \_ n\} \subseteq P \Rightarrow \neg\ Mk,P))$.
Pf: By induction, (F4*) with ϕ as

$$POTINF_n \Rightarrow \forall k \in \{0 \_ n\}\ \forall P\ (\{0 \_ n\} \subseteq P \Rightarrow \neg\ Mk,P).$$

Suppose N0 & $POTINF_0$, and let $k \in \{0 \_ 0\}$. Then by Prop B1b, $k = 0$. If $\{0 \_ 0\} \subseteq P$, then P0. So by (F2), $\neg$ M0,P.

Now assume Nn & Nm & $\sigma n,m$ & $\neg\ m = 0$ & ϕ . Let $POTINF_m$, and suppose that for some $p \in \{0 \_ m\}$ and some P, $\{0 \_ m\} \subseteq P$ and Mp,P.

By B5, $POTINF_n$, and so

$$\forall k \in \{0 \_ n\}\ \forall P\ (\{0 \_ n\} \subseteq P \Rightarrow \neg\ Mk,P) \quad (*)$$

from the induction hypothesis.

By B10, $\{0 \_ m\} \equiv \{0 \_ n\} \cup \{m\}$, but $\neg\ m \in \{0 \_ n\}$. Then either $p \in \{0 \_ n\}$ or $p = m$.

$p \in \{0 \_ n\}$ contradicts (*), since $\{0 \_ n\} \subseteq \{0 \_ m\} \subseteq P$.

So $p = m$. Since $\{m\} \subseteq \{0 \_ m\}$, we have Pm. By (F3), $Mn,(P \backslash \{m\})$. Evidently, $\{0 \_ n\} \subseteq P \backslash \{m\}$. By B1a, $n \in \{0 \_ n\}$. But this again contradicts (*).

B12. $\forall n\ (Nn \ \&\ POTINF_n \Rightarrow \forall k \in \{0 \_ n\}\ \neg\ Mk,\{0 \_ n\})$.
Pf: Use B11.

B13. $\forall n \forall p\ (Nn \ \&\ \sigma n,p \ \&\ POTINF_n \Rightarrow Mp,\{0 \_ n\})$.

Pf: Proceed by induction, (F4*), with ϕ as ∀p (σn,p & POTINF_n ⇒ Mp,{0 _ n}).

Suppose N0 & σ0,p & POTINF_0. By B1b, {0 _ n} ≡ {0}. By (F2) and (F3), Mp,{0}.

Now assume Nn & Nm & σn,m & ¬ m = 0 & ϕ . And suppose σm,p & POTINF_m . By B5, POTINF_n . By the induction hypothesis, Mm,{0 _ n}. By B10, {0 _ m} ≡ {0 _ n} ∪ {m}, but ¬ m ∈ {0 _ n}. So by (F3), Mp,{0 _ m}.

B14 (*POTINF*). ∀n (Nn ⇒ POTINF_n)

Pf: Proceed by induction, (F4), with ϕ as POTINF_n .

POTINF_0 , since M0,ϕ and ¬ 0 ∈ ϕ.

Suppose Nn & σn,m & POTINF_n . By B13, Mm,{0 _ n}. By B13, ∀k∈{0 _ n} ¬ Mk,{0 _ n}. So ¬ m∈{0 _ n}.

Next, Dedekind’s proof. It appeals only to (F2), (F3), and (F4). Again, the proof cannot be conducted in second-order logic, but instead either in a first-order theory (big-letters are just like small-letters), or in a theory where big-letters may be substituted for little-letters but not the reverse. That is, from ∀x (x = x), one can conclude that P = P. But from ∀P (P ⊆ P), one cannot conclude that x ⊆ x. Contradiction (e.g. via an equivalent of Russell’s Paradox) is avoided, since comprehension is predicative, i.e. (for unary predicates)

∀x ({x : ϕ }x ⇔ ϕ)

provided ϕ does not contain in a predicate place either x or a bound variable.

Use ⌘ P to abbreviate ∀x (Px ⇒ x ⊂ P)

D1. ∀P (⌘ P ⇒ ¬ PP)

Pf: Suppose ⌘ P. Then PP ⇒ P ⊂ P.

D2. ⌘ ϕ

Pf: ¬ ∃x ϕx. So ∀x (Px ⇒ x ⊂ P) vacuously.

Use (P+) to abbreviate {x : Px ∨ x = P}. Remark this is predicative, since the only variable in a predicate position, namely “P”, is free. So, by comprehension, D3. $\forall P \forall x\,(\,(P+)x \Leftrightarrow Px \vee x = P)$

D4. $\forall P\,(\,\mathfrak{H}\,P \Rightarrow P \subset (P+)\,)$
Pf: Assume $\mathfrak{H}\,P$. By D3, $P \subseteq (P+)$. Suppose $P \equiv (P+)$. $(P+)P$ by D3, so PP. This contradicts D1.

D5. $\forall P\,(\,\mathfrak{H}\,P \Rightarrow \mathfrak{H}\,(P+)\,)$
Pf: Assume $\mathfrak{H}\,P$. Suppose $(P+)x$. Then $Px \vee x = P$ by D3. If Px, then $x \subset P$ by $\mathfrak{H}\,P$; so $x \subset (P+)$ since $P \subseteq (P+)$. If $x = P$, then again $x \subset (P+)$ by D4.

D6. $\forall n\,(\,Nn \Rightarrow \exists P\,(\mathfrak{H}\,P \;\&\; Mn,P)\,)$
Pf: By induction, with ϕ as $\exists P\,(\mathfrak{H}\,P \;\&\; Mn,P)$. By D2 and (F2), $\mathfrak{H}\,\phi \;\&\; M0,\phi$, so true for $n = 0$. Suppose $Nn \;\&\; \sigma n,m \;\&\; \mathfrak{H}\,P \;\&\; Mn,P$, for some P. Then $\mathfrak{H}\,(P+)$ by D5. $\neg\, PP$ by D1, so $Mm,(P+)$ by D3 and (F3).

D7. (*POTINF*). $\forall n\,(\,Nn \Rightarrow \exists P \exists a\,(Mn,P \;\&\; \neg\, Pa)\,)$
Pf: Assume Nn. By D6, $\mathfrak{H}\,P \;\&\; Mn,P$, for some P. $\neg\, PP$ by D1.

Finally, for its philosophical interest, define *ACTINF* as:

$$\exists P \forall n\,(\,Nn \Rightarrow \neg\, Mn,P\,).$$

Then:

ACTINF.
Pf:
Suppose $Nn \;\&\; Mn,\mathbb{U}$, for some n. By *POTINF*, $Mn,P \;\&\; \neg\, Pa$, for some P,a. Evidently $P \subseteq \mathbb{U}$. By B2, $P \equiv \mathbb{U}$, a contradiction.

Evidently, *ACTINF* also follows from F1,F2,F3, and F4.

Counting and the C Axioms

The non-logical constants in F axioms are evidently meant to be interpreted as follows:

- ‘Nn’ as “n is a finite number”,
- ‘σn,m’ as “the number m follows the number n”,
- ‘Mn,P’ as “P numbers n”,

‘0’ as “zero”.
Given these meanings, are the F axioms fundamental?

(F1) can reasonably be held to be so: our concept of numbering carries with it the idea that number is unique. If something has a number, well, then that’s the only number it has.

(F2) and (F5) appear fundamental, and while there is always a question about induction (F4), put it on the fundamental side as well.

Which leaves (F3) and (F6).

When one reads (F3), it does have the aura of analyticity, in some way capturing an intrinsic relationship between succession and numbering. On closer reflection, though, it seems more like a fact about succession and numbering that we *just know*. Consider, for instance, this half of (F3):

(F3a) $Nn \,\&\, \sigma n,m \,\&\, \neg\, Pa \,\&\, \forall x(Qx \Leftrightarrow Px \vee x = a) \,\&\, Mn,P \Rightarrow Mm,Q$

Now, if numbering carries with it the notion of uniqueness--and we have said it does--then it does not seem immediate that the implication holds. That is, suppose we know that P numbers n, that Q has precisely one additional thing, and the number m follows the number n. Then clearly *some* count of Q should yield m, namely the count beginning with the Ps (which must yield n) and then finishing with the additional thing (which yields m). But maybe there is *another* way to count it, which is somehow different. And, if there were, then Q (because numbering is unique) would not have a number. The same kind of remark puts in question the other direction of (F3)’s biconditional as well. Of course, we *know* counting is unique, so we *know* (F3) as well. But the justification for (F3) seems to be other than that it just follows immediately from the meaning of its terms: one needs to know in addition that counting is unique. Either this is empirical, something we learn at a very early age, or a *derived* truth, from other, more fundamental axioms, so something which we can prove. The “or” here is not meant to be exclusive.

Exactly the same sort of doubt applies to (F6). Why, after all, must $P \cup \{a\}$ have a *unique* number, just because P does? Of course, there will be some counting of $P \cup \{a\}$ which yields m, but maybe another counting, which *doesn't* pass through P, yields a different number.

One can fault Frege's analysis of number ([6]) for precisely this reason. Just because the tomatoes and the oranges in my refrigerator are in one-to-one correspondence, they do not necessarily have *the* same number. One can conclude from the existence of the correspondence *only* that they share every number that they have. But both may have more than one number (if one does not insist that number is a unique attribute) or no number at all (if numbers are unique).

There is a tension, then, between (F1) and both (F3) and (F6). If one accepts that numbering must be unique, (F3) and (F6) now need justifications if not proofs. Fortunately, our discussion gives hints how to produce them. Indeed, "P numbers n" can be analyzed as:

1) *Some* counting of P yields n
2) *All* countings of P yield n

It would seem, then, that an axiomatization with counting (and not numbering or one-to-one correlation) as primitive, is called for. Therefore, let 'X' be a two-place predicate, where 'Xn,C' is meant to say that C is a relationship which counts something as n in number. For instance, if $C = \{(1,\{a\}),(2,\{b\})\}$, where $\neg\, a = b$, then C counts $\{a,b\}$ as 2.

Consider the following axioms:

(C0) $\forall n \forall C \forall i \forall j \forall x\, (Xn,C \;\&\; Ci,x \;\&\; Cj,x \Rightarrow i = j)$

(C1a) $\forall n \forall C\, (Xn,C \;\&\; \neg\, m = 0 \Rightarrow \exists a\, Cn,a)$

(C1b) $\forall n \forall C \forall m \forall k \forall a\, (Nn \;\&\; Nm \;\&\; \exists p(Np \;\&\; \sigma p,k) \;\&\; Xn,C \;\&\; \neg\, a \in (C^{I}) \;\&\; Xm,(C \cup \{(m,a)\}) \;\&\; Xk,(C \cup \{(k,a)\}) \Rightarrow m = k)$

(C2) $\forall C\, (X0,C \Leftrightarrow \neg \exists x \exists y\, Cx,y)$

(C3) $\forall n \forall m \forall C \forall D \forall a\, (Nn \;\&\; \sigma n,m \;\&\; \neg\, a \in (C^{I}) \;\&\; \forall x \forall y\, (Dx,y \Leftrightarrow Cx,y \vee (x = m \;\&\; y = a)) \Rightarrow (Xn,C \Leftrightarrow Xm,D))$

(C4) Induction. Let ϕ be a well-formed formula (with no appearance of m). Use $\phi[x\backslash y]$ to mean x replaces all (free) instances of y. Suppose $\phi[0\backslash n]$ and $\forall n \forall m\, (Nn \;\&\; \sigma n,m \;\&\; \phi \Rightarrow \phi[m\backslash n])$. Then $\forall n\, (Nn \Rightarrow \phi)$.

(C5) $N0$

(C6) $\forall C \forall a \forall n$ (Nn & Xn,C & $\neg$ a $\in$ (C^I) $\Rightarrow \exists$m (Nm & Xm,(C $\cup$ {(m,a)})))

In comparison with the F system, (C0) is a new axiom. It says that countings must be one-to-one. That is, if one counts the same thing twice, one is not counting properly.

(C1a) implies of course as a contrapositive

$$\forall n \forall C \text{ (Xn,C \& C} \equiv \phi \Rightarrow n = 0 \text{)}$$

which is akin to (F1a). Nonetheless, it of course says more: that, if one has a counting to n, then one must have counted something as n.

(C1b) resembles (F1b). Counting only makes sense if, for a given thing to be counted, the next count is unique. That is, if a needs to be counted, then the next number which can be used to count it, has to be unique.

(C2) is just the normal axiom about 0.

(C3) says: suppose m follows n, and a has not been counted by C, and D counts like C does but counts a as m. Then C is a counting to n if and only if D is a counting to m. Again, this seems to be basic to what counting is.

(C4) and (C5) are just (F4) and (F5), respectively.

(C6) is akin to (F6a) and says that if D is a counting to n, and a has not yet been counted, then there is a number m and a counting to m of all the things which are P and a. Again, the ability to count one additional thing, appears to be a fundamental rule of counting.

Remark that the C system is divisible, like the F system before it, into those axioms which make no ontological assumptions about the natural numbers, (C0) through (C4), and those which do, (C5) and (C6). As expected, (C0) through (C4) are the axioms which imply (F1) through (F4).

Use Mn,P to abbreviate $\exists$C (Xn,C & (C^I) $\equiv$ P). We prove the axioms of the F system. Remark that the uniqueness of numberings does not just fall out of the definition of 'M', so the content of (F1) is substantial and important: for any P, there is only one number that can count P.

First, we assert a few lemmas.

K1. $\forall n\, (Nn \,\&\, \neg\, n = 0 \Rightarrow \exists p(Np \,\&\, \sigma p,n))$

K2. $\forall n \forall C\, (Xn,C \,\&\, C \equiv \varphi \Rightarrow n = 0)$

K3. $\forall n \forall C \forall D \forall a \forall b\, (Nn \,\&\, Xn,C \,\&\, D \equiv (C\ \alpha\ a\ b) \Rightarrow Xn,D)$.

Pf: By induction, with ϕ as

$$\forall C \forall D \forall a \forall b\, (Xn,C \,\&\, D \equiv (C\ \alpha\ a\ b) \Rightarrow Xn,D).$$

Suppose $N0 \,\&\, X0,C \,\&\, D \equiv (C\ \alpha\ a\ b)$. Then $C \equiv \varphi$ by (C2). So $D \equiv \varphi$, and by (C2) again, $X0,D$.

Now assume $Nn \,\&\, Nm \,\&\, \sigma n,m \,\&\, \neg\, m = 0 \,\&\, \phi$. And suppose $Nm \,\&\, Xm,C \,\&\, D \equiv (C\ \alpha\ a\ b)$. By (C1a) Cm,c for some c. Define $C_1 = C \setminus \{(m,c)\}$. C is one-to-one by (C0), so $\neg\, c \in (C_1{}^I)$. By (C3) Xn,C_1. By the induction hypothesis, $Xn,(C_1\ \alpha\ a\ b)$. Now $P \equiv Q \cup R$ if and only if $(P\ \alpha\ a\ b) \equiv (Q\ \alpha\ a\ b) \cup (R\ \alpha\ a\ b)$. So $D \equiv (C\ \alpha\ a\ b) \equiv (C_1\ \alpha\ a\ b) \cup (\{(m,c)\}\ \alpha\ a\ b)$. And $\neg\, c \in (C_1{}^I)$ implies $\neg\, (c\ \alpha\ a\ b) \in ((C_1\ \alpha\ a\ b)^I)$. So by (C3), Xm,D.

Remark Tenant has defined (RSab) on pp. 280-1 of [11] so that it effectively matches $(R\ \alpha\ a\ b)$. Also remark that $(C\ \alpha\ a\ a) \equiv C$ for any a, so K3 implies that

$$\forall n \forall C \forall D\, (Nn \,\&\, Xn,C \,\&\, D \equiv C \Rightarrow Xn,D).$$

(F1a) $\forall n \forall P\, (Mn,P \,\&\, \neg \exists x\, Px \Rightarrow n = 0)$

Pf: Suppose $Mn,P \,\&\, \neg \exists x\, Px$. Then $Xn,C \,\&\, (C^I) \equiv P$. $\neg \exists x,y\, Cx,y$ since $\neg \exists x\, Px$, so $C \equiv \varphi$. By (C1a), $n = 0$.

(F1b) $\forall n \forall m \forall P\, (Nn \,\&\, \exists p\, (Np \,\&\, \sigma p,m) \,\&\, Mn,P \,\&\, Mm,P \Rightarrow n = m)$.

Pf: By induction, with ϕ as

$$\forall k \forall P\, (\exists p\, (Np \,\&\, \sigma p,k) \,\&\, Mn,P \,\&\, Mk,P \Rightarrow n = k).$$

Suppose $N0 \,\&\, \exists p\, (Np \,\&\, \sigma p,k) \,\&\, M0,P \,\&\, Mk,P$, so $X0,C \,\&\, (C^I) \equiv P \,\&\, Xk,D \,\&$

$(D^{I}) \equiv P$, for some C,D. By (C2), $C \equiv \phi$, so $P \equiv \phi$, so $D \equiv \phi$. So by (C1a), $k = 0$.

Assume Nn & Nm & σn,m & ¬ m = 0 & φ . And suppose ∃p (Np & σp,k) & Mm,P & Mk,P. So Xm,C & $(C^{I}) \equiv P$ & Xk,D & $(D^{I}) \equiv P$, for some C,D. If k = 0, then as above, m = 0. So suppose ¬ k = 0. By (C1a), Cm,c & Dk,d for some c,d. By assumption, Nj & σj,k, for some j. Since $(C^{I}) \equiv (D^{I})$, $\exists i \in (C^{I})$ s.t. Ci,d. Set C" = (C α c d). Then by K3, Xm,C". C" is one-to-one, by (C0). Of course, C"m,d. Consider C' = C"\{(m,d)} and D' = D\{(k,d)}. Now C" and D are one-to-one, D by (C0), so $\neg\, d \in (C'^{I})$ and $\neg\, d \in (D'^{I})$. Set $Q = (C'^{I})$. Remark $(D'^{I}) \equiv Q$. By (C3), Xn,C' and Xj,D'. So Mn,Q & Mj,Q. If j = 0, then $Q \equiv \phi$ by (C2). By K2, n = 0, and so n = j. On the other hand, suppose ¬ j = 0. Then ∃p (Np & σp,j). By the induction hypothesis, n = j. That is, in either case, n = j. By (C3), Xm,(D' ∪ {(m,d)}) and Xk,(D' ∪ {(k,d)}) By (C1b), m = k.

(F2) ∀P (M0,P ⇔ ¬ ∃x Px)

Pf: Suppose M0,P. Then X0,C & $(C^{I}) \equiv P$, for some C. By (C2), ¬ ∃x∃y Cx,y. So $C \equiv \phi$, so $P \equiv \phi$, so ¬ ∃x Px.

Now suppose ¬ ∃x Px. Then $P \equiv \phi$. Then $(\phi^{I}) \equiv P$ and, by (C2), X0,ϕ. Hence M0,P.

(F3) ∀n∀m∀P∀Q∀a (Nn & σn,m & ¬ Pa & ∀x(Qx ⇔ Px ∨ x = a) ⇒ (Mn,P ⇔ Mm,Q)).

Pf: Assume Nn & σn,m & ¬ Pa & ∀x(Qx ⇔ Px ∨ x = a).

Suppose Mn,P. Then Xn,C & $(C^{I}) \equiv P$, for some C. Then $\neg\, a \in (C^{I})$. By (C3), Xm,(C ∪ {(m,a)}). Evidently, $((C \cup \{(m,a)\})^{I}) \equiv Q$. So Mm,Q.

Now suppose Mm,Q. Then Xm,D & $(D^{I}) \equiv Q$, for some D. ¬ m = 0, since Qa, by (F1a) and (F2), which have already been proved. By (C1a), Dm,b for some b. Set D' = D \ {(m,b)}. D is one-to-one by (C0), so $\neg\, b \in (D'^{I})$. By (C3), Xn,D'. Set C' = (D' α a b). By K3, Xn,C'. Evidently, $(C'^{I}) \equiv ((D'^{I}) \cup \{b\}) \setminus \{a\} \equiv (D^{I}) \setminus \{a\} \equiv Q \setminus \{a\} \equiv P$. Hence, Mn,P.

(F4) and (F5) are just (C4) (in a more restricted language) and (C5).

(F6) $\forall P \forall a \forall n \,(Nn \,\&\, Mn,P \,\&\, \neg Pa \Rightarrow \exists m\, (Nm \,\&\, Mm,(P \cup \{a\})))$

Pf: Assume $Nn \,\&\, Mn,P \,\&\, \neg Pa$. Then $Xn,C \,\&\, (C^{I}) \equiv P$, for some C. By (C6), $Nm \,\&\, Xm,(C \cup \{(m,a)\})$, for some m. Evidently, $((C \cup \{(m,a)\})^{I}) \equiv P \cup \{a\}$. So $Mm,(P \cup \{a\})$.

Without making too much of this point, remark that the proof of (F1b) uses every axiom except (C5) and (C6). That is to say, what one needs to prove uniqueness of counting, suffices to prove (F1) through (F4) as well. Or to put it yet another way, if you know enough to say that counting is unique, you can figure out the rest.

Conclusion

The F axioms are a simple way of proving the Peano Axioms. What is more, there are various subsystems of F which have interesting properties. For instance, the system composed of the axioms F1,F2,F3, and F4 (with predicative comprehension) should be acceptable to an ultrafinitist, yet is capable of providing definitions for addition, multiplication, and exponentiation, and proving many facts, such as versions of the Commutative, Associative, and Distributive Laws, as well as the Euclidean Algorithm and Unique Prime Factorization (aka the “Fundamental Theorem of Arithmetic”). It is not, however, capable of proving the Chinese Remainder Theorem or the infinity of primes, that is the assertion $\neg \exists x\, (Nx \,\&\, Mx,\pi)$, where π represents the set of all primes. This will be a subject of a subsequent paper.

The C axioms have a good claim to being a foundation of elementary arithmetic. They can prove the Peano Axioms, because they prove the F axioms. And they appear to be fundamental, in that they come directly from our notion of counting.

A certain amount of caution, however, must be exercised before accepting this claim, since no analysis has been presented for the concept “fundamental.” Nonetheless, at the least, they appear *more* fundamental (on an intuitive level) than the F system, as well as the Frege-like systems, since the concept of numbering does seem to depend on the belief that counting the same things in different ways always yields the same number. Counting would make sense even if this counting were not unique, but numbering does not.

In any case, the F system has pushed deep, and the C system has pushed deeper still, in the long search to establish a foundation of arithmetic.

Acknowledgments

The author would like to acknowledge assistance from Richard Heck, Neil Tennant, Solomon Feferman, and Fernando Ferreira. Any errors are of course the author's own.

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