# General Arithmetic

by Andrew Boucher
created: 24 Feb 2008 modified: 1 Mar 2008

General Arithmetic is the theory consisting of induction on a successor function. Normal arithmetic, say in the system called Peano Arithmetic, makes certain additional demands on the successor function. First, that it be total. Secondly, that it be one-to-one. And thirdly, that there be a first element which is not in its image. General Arithmetic abandons all of these further assumptions, yet is still able to prove many meaningful arithmetic truths, such as, most basically, *Commutativity* and *Associativity of Addition* and *Multiplication*, but also *Lagrange's Four-Square Theorem*. Adding one more axiom, the one-oneness of succession, one can prove many more theorems, such as *Quadratic Reciprocity* and *Fermat's Little Theorem*. By looking at arithmetic in this general setting, one receives a deeper understanding of the underlying structures.

## 0. Abbreviations

$x \in P$ for $Px$
$Dom(R)$ for $\{x : \exists y\ Rx,y\}$
$Im(R)$ for $\{y : \exists x\ Rx,y\}$
$IsFunction(R)$ for $\forall x \forall y \forall z\ (Rx,y\ \&\ Rx,z \Rightarrow y = z)$
$Is1\text{-}1(R)$ for $\forall x \forall y \forall z\ (Rx,y\ \&\ Rz,y \Rightarrow x = z)$
$P \equiv Q$ for $\forall x\ (Px \Leftrightarrow Qx)$ or $\forall x \forall y\ (Px,y \Leftrightarrow Qx,y)$
$P \subseteq Q$ for $\forall x\ (Px \Rightarrow Qx)$ or $\forall x \forall y\ (Px,y \Rightarrow Qx,y)$
$P \cup Q$ for $\{z : Pz \vee Qz\}$ or $\{y,z : Py,z \vee Qy,z\}$ or $\{x,y,z : Px,y,z \vee Qx,y,z\}$
$P \setminus Q$ for $\{z : Pz\ \&\ \neg Qz\}$
$\{a\}$ for $\{z : z = a\}$
$\{(a,b)\}$ for $\{y,z : y = a\ \&\ z = b\}$
$\phi$ for $\{z : \neg z = z\}$ or $\{y,z : \neg z = z\}$

The first half of this paper covers much of the same ground as Henkin in his paper "On Mathematical Induction" [Henkin]. That paper looked at a base system which assumed induction, the totality of successoring (aka the "Successor Axiom"), and successoring's functionality. He proved the existence of addition and multiplication and described the models

## 1. Introduction

We will be working in predicative second-order logic. That is, comprehension is restricted to predicative (arithmetic) predicates. The language will be augmented by a first-order constant 0, a second-order one-place predicate N (representing the natural numbers), and a second-order two-place predicate σ (representing succession). Call this base theory, with no non-logical axioms, but with the specified additions to the language, **SLA**.

If N$y$ & σ$x,y$, then $y$ is called a *successor* of $x$; and if N$x$ & σ$x,y$, then $x$ is called a *predecessor* of $y$.

The theory of Peano Arithmetic **PA** contains these axioms:

(PA1) N0
(PA2) $\forall n$ ( N$n \Rightarrow \exists m$ (N$m$ & $\sigma n,m$) )
(PA3) $\forall n \forall m \forall m'$ ( N$n$ & N$m$ & N$m'$ & $\sigma n,m$ & $\sigma n,m' \Rightarrow m = m'$ )
(PA4) $\forall n \forall m \forall n'$ ( N$n$ & N$m$ & N$n'$ & $\sigma n,m$ & $\sigma n',m \Rightarrow n = n'$ )
(PA5) $\forall n$ ( N$n \Rightarrow \neg\, \sigma n,0$ )
(PA6) Induction schema. Let $\phi$ be a well-formed formula (with no appearance of $m$). Use $\phi\,[x\backslash y]$ to mean $x$ replaces all (free) instances of $y$. Suppose $\phi\,[0\backslash n]$ and $\forall n \forall m$ ( N$n$ & $\sigma n,m$ & $\phi \Rightarrow \phi\,[m\backslash n]$ ). Then $\forall n$ ( N$n \Rightarrow \phi$ ).

Note that one can apply (PA6) using N$n \Rightarrow \phi$, so in fact (PA6) can be replaced with the ostensibly stronger principle:

(PA6') Let $\phi$ be a well-formed formula (with no appearance of $m$). Suppose N0 $\Rightarrow \phi\,[0\backslash n]$ and $\forall n \forall m$ ( N$n$ & N$m$ & $\sigma n,m$ & $\phi \Rightarrow \phi\,[m\backslash n]$ ). Then $\forall n$ ( N$n \Rightarrow \phi$ ).

**FPA** is the sub-theory without axiom (PA2) - called the *Successor Axiom* - and (PA1), which is a trivial axiom of little interest. **FPA** is able to develop much of arithmetic, which is demonstrated in *Arithmetic without the Successor Axiom.* It can, for instance, prove the law of *Quadratic Reciprocity* and *Bertrand's Postulate*.

It is a natural question to wonder how much arithmetic can be developed in weaker sub-theories, which still contain (PA6) induction. Let **IND** be the sub-theory having as axioms all instances of the induction schema, obviously the weakest such theory. Remark that:

$$\mathbf{SLA} \subseteq \mathbf{IND} \subseteq \mathbf{FPA} \subseteq \mathbf{PA}$$

where

**IND** = **SLA** + (PA6)
**FPA** = **PA** \ {(PA1),(PA2)}.

Here is a sampling of what **IND** can prove.

(**IND**) *Prop 1.01.* $\forall n$ (N$n \Rightarrow$ N0)
*Pf:*
By induction (PA6'), with $\phi$ as N0. ∎

The previous proposition removes most interest from (PA1), since if there is any natural number, (PA1) holds; in other words, either the vacuous system where there are no natural numbers, or (PA1).

(**IND**) *Prop 1.02.* Let N0 & $\forall n$ (N$n$ & $\sigma 0,n \Rightarrow n = 0$). Then N $\equiv$ {0}.
*Pf:*
We claim that $\forall n$ (N$n \Rightarrow$ N0). Proceed by induction (PA6'), with $\phi$ as $n = 0$.
Clearly N0 $\Rightarrow \phi\,[0\backslash n]$ holds.

Now let $\mathrm{N}n \,\&\, \mathrm{N}m \,\&\, \sigma n,m \,\&\, \phi$. By the Induction Hypothesis, $n = 0$. By assumption, this forces $m = 0$.

So $\forall n\ (\mathrm{N}n \Rightarrow \mathrm{N}0)$. But N0, so $\mathrm{N} \equiv \{0\}$. □

(**IND**) *Corollary 1.03*. Let $\mathrm{N}0 \,\&\, \forall n\ (\mathrm{N}n \Rightarrow \neg\, \sigma 0,n)$. Then $\mathrm{N} \equiv \{0\}$. □

The next proposition asserts that every non-zero natural number is preceded by a natural number different from it.

(**IND**) *Prop 1.04*. $\forall n\ (\mathrm{N}n \,\&\, \neg\, n = 0 \Rightarrow \exists k\ (\mathrm{N}k \,\&\, \sigma k,n \,\&\, \neg\, k = n)\ )$.

*Pf:*

By induction, with $\phi$ as

$$(\neg\, n = 0 \Rightarrow\ \exists k\ (\mathrm{N}k \,\&\, \sigma k,n \,\&\, \neg\, k = n)\ ).$$

$\phi$ holds trivially when $n = 0$.

Now let $\mathrm{N}n \,\&\, \sigma n,m \,\&\, \phi$. Set $k = n$. Then $\mathrm{N}k \,\&\, \sigma k,m$. If $k = m$, then $m = n$, so by the induction hypothesis, $(\neg\, m = 0 \Rightarrow\ \exists k\ (\mathrm{N}k \,\&\, \sigma k,n \,\&\, \neg\, k = m)\ )$. Otherwise. $\neg\, k = m$, so $\mathrm{N}k \,\&\, \sigma k,m \,\&\ \neg\, k = m$, as desired. □

If any natural number has a successor, then 0 has one, too:

(**IND**) *Prop 1.05*. $\forall n \forall k\ (\mathrm{N}n \,\&\, \mathrm{N}k \,\&\, \sigma n,k \Rightarrow \exists j\ (\mathrm{N}j \,\&\, \sigma 0,j)\ )$.

*Pf:*

By induction, with $\phi$ as

$$(\exists k\ (\mathrm{N}k \,\&\, \sigma n,k) \Rightarrow \exists j\ (\mathrm{N}j \,\&\, \sigma 0,j)\ ).$$

$\phi$ holds trivially when $n = 0$.

Now let $\mathrm{N}n \,\&\, \sigma n,m \,\&\, \phi$. Then $\exists k\ (\mathrm{N}k \,\&\, \sigma n,k)$, so by the induction hypothesis, $\exists j\ (\mathrm{N}j \,\&\, \sigma 0,j)$. □

**IND** also proves various propositions about the ancestral relationship, which will be grouped below in the section where that relationship is introduced.

If addition is introduced, **IND** has more to play with. Suppose + is a 3-ary relationship or formula satisfying:

(ADD0) $\forall k \forall n \forall m\ (\ +(k,n,m) \Rightarrow \mathrm{N}k \,\&\, \mathrm{N}n \,\&\, \mathrm{N}m\ )$
(ADD1) $\forall n\ (\ \mathrm{N}n \Rightarrow +(n,0,n)\ )$
(ADD2) $\forall n \forall m\ (\ +(n,0,m) \Rightarrow n = m\ )$
(ADD3) $\forall k \forall n \forall m \forall n' \forall m'\ (\ \mathrm{N}n' \,\&\, \mathrm{N}m' \,\&\, +(k,n,m) \,\&\, \sigma n,n' \,\&\, \sigma m,m'\ \Rightarrow +(k,n',m')\ )$
(ADD4) $\forall k \forall n \forall n' \forall m'\ (\ \mathrm{N}n \,\&\, +(k,n',m') \,\&\, \sigma n,n' \Rightarrow \exists m\ (\sigma m,m' \,\&\, +(k,n,m))\ )$

And define commutativity as follows:

(+-COMM) $\forall k \forall n \forall m\ (\ +(k,n,m) \Rightarrow +(n,k,m)\ )$

Then it can be shown in **IND** that addition is commutative if and only the successor relationship is a function, i.e. that (+-COMM) if and only if (PA3). Indeed, **SLA** suffices to show one direction:

(**SLA**) *Prop 1.06.* (+-COMM) $\Rightarrow$ (PA3).
*Pf:*

Suppose (+-COMM) & $\neg$ (PA3). Then $\mathrm{N}n$ & $\mathrm{N}m$ & $\mathrm{N}k$ & $\sigma n,m$ & $\sigma n,k$, for some $n,m,k$ where $\neg\ m = k$. Now $+(n,0,n)$ by (ADD1), so $+(0,n,n)$ by (+-COMM). Hence $+(0,m,k)$ by (ADD3), and so $+(m,0,k)$ by (+-COMM), contradicting (ADD2). □

Now let's prove the other direction in four steps.

(**IND**) *Prop 1.07.* $\forall n\ (\ \mathrm{N}n \Rightarrow +(0,n,n)\ )$.
*Pf:*

By induction (PA6'), with $\phi$ as

$$+(0,n,n).$$

$\mathrm{N}0 \Rightarrow \phi$ holds when $n = 0$, by (ADD1).

Now let $\mathrm{N}n$ & $\mathrm{N}m$ & $\sigma n,m$ & $\phi$. So $+(0,n,n)$. Then $+(0,m,m)$ by (ADD3). □

(**SLA**) *Prop 1.08.* $\forall n \forall m \forall k\ (\ \sigma 0,k\ \&\ +(n,k,m) \Rightarrow \sigma n,m\ )$.
*Pf:*

Let $\sigma 0,k$ & $+(n,k,m)$. $+(n,0,c)$, for some $c$ where $\sigma c,m$, by (ADD4). But $n = c$, by (ADD2). □

(**IND**) *Prop 1.09.* $\forall n \forall k \forall n' \forall v\ (\mathrm{N}n\ \&\ +(k,n',v)\ \&\ \sigma n,n' \Rightarrow \exists k'\ (\sigma k,k'\ \&\ +(k',n,v))\ )$.
*Pf:*

By induction (PA6'), with $\phi$ as

$$\forall k \forall n' \forall v\ (+(k,n',v)\ \&\ \sigma n,n' \Rightarrow \exists k'\ (\sigma k,k'\ \&\ +(k',n,v))\ ).$$

Suppose $+(k,n',v)$ & $\sigma 0,n'$. Then $\sigma k,v$ by *Prop 1.08* and $+(v,0,v)$ by (ADD1).

Now let $\mathrm{N}n$ & $\mathrm{N}m$ & $\sigma n,m$ & $\phi$. Assume $+(k,m',v)$ & $\sigma m,m'$. Then there exists $u$ such that $\sigma u,v$ & $+(k,m,u)$, by (ADD4). By the induction hypothesis, $\sigma k,k'$ & $+(k',n,u)$. Then $+(k',m,v)$ by (ADD3). □

(**IND**) *Prop 1.10.* (PA3) $\Rightarrow$ (+-COMM).

*Pf:*

Assume (PA3). Proceed by induction, with $\phi$ as

$$\forall x \forall y\ (\ +(x,n,y) \Rightarrow +(n,x,y)\ ).$$

Suppose $+(x,0,y)$. Then $x = y$ by (ADD2) and N$x$ by (ADD0). So $+(0,x,y)$ by *Prop 1.07*.

Now let N$n$ & $\sigma n,m$ & $\phi$. And suppose $+(x,m,y)$. Then $\sigma x,x'$ & $+(x',n,y)$ for some $x'$, by *Prop 1.09* By the induction hypothesis, $+(n,x',y)$. By *Prop 1.09* again, $\sigma n,z$ & $+(z,x,y)$, for some $z$. By (ADD0), N$m$ & N$z$. Thus $m = z$, by (PA3). Hence $+(m,x,y)$. □

One might question whether other properties of addition hold, such as associativity. Unfortunately, associativity, as it is usually stated, pre-supposes the functionality of the addition relationship, and unlike commutativity, it is less clear exactly what associativity should mean in an environment where this functionality does not occur. This suggests that functionality is, after induction, the first assumption that we would like to have. Whether or not this is really the case, let us proceed as if it were, and limit our investigations henceforth to those systems which include functionality of the successor relationship (which, as we shall see, implies functionality of the addition relationship).

That is, let **GA** (*General Arithmetic*) be the sub-theory of **PA** with (PA3) and all instances of (PA6) as axioms. Note that:

**SLA** ⊆ **IND** ⊆ **GA** ⊆ **FPA** ⊆ **PA**,

where

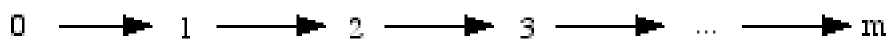
**IND** = **SLA** + (PA6)
**GA** = **IND** + (PA3)
**FPA** = **PA** \ {(PA1),(PA2)} = **GA** + (PA4) + (PA5)

Evidently, *Prop 1.10* shows that in **GA** any relationship or formula satisfying the conditions (ADD0) to (ADD4), is commutative.

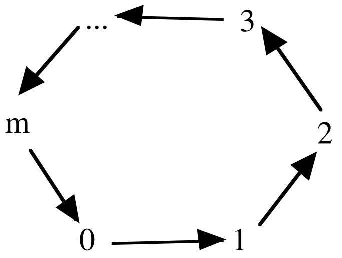
It will be seen that within **GA** the natural numbers have the following kinds of models:

1/ The Standard Model.


2/ The Finite Segments.

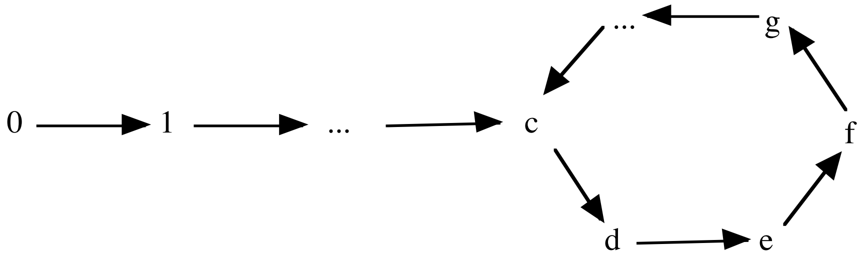



3/ The Pure Cycles.




The Pure Cycles have the same structure as the set of congruent numbers mod *m*+1, although this characterization can't be made in the system itself, since *m*+1 doesn't exist.

4/ The Tadpoles.




5/ The Empty Set. There are no natural numbers, obviously a trivial and mostly uninteresting possibility.

*Examples*. Write a model of the axioms in linear fashion, *a* - *b* - *c* - ..., where "*x* - *y*" appears if and only if σ*x*,*y*. Then the above models can be represented as follows:

1/ 0 - 1 - 2 - 3 - ...

2/ 0 - 1 - 2 -3 - ... - *m*

3/ 0 - 1 - 2 - 3 - ... - *m* - 0

4/ 0 - 1 - ... *c* - *d* - *e* - *f* - *g* - ... - *c*

Specific examples would be:

0 (the trivial finite segment, with only 0 as natural number but not σ0,0,)

0 - 0 (the trivial pure cycle, with only 0 as natural number and with σ0,0)

0 - 1 - 2 - 3 (the finite segment through 3)

0 - 1 - 2 - 3 - 0 (the pure cycle with top number 3)

0 - 1 - 2 - 3 - 1 (a tadpole)

Notice that models of pure cycles and tadpoles can be represented in non-unique ways. For instance,

0 - 1 - 2 - 3 - 0 and
0 - 1 - 2 - 3 - 0 - 1 - 2 - 3

represent the same structure. 0 - 1 - 2 - 3 - 0 is its smallest representation, and

0 - 1 - 2 - 3 - 0 - 1 - 2 - 3 is the smallest representation where $x$ is an ancestor of $y$ if and only if $x$ appears to the left of $y$. □

## 2. The Ancestral

Frege introduced and was the first to study the ancestral.

*Def 2.01*. *Hered*($P$) abbreviates

$\forall a,b(\mathrm{N}a \ \& \ \mathrm{N}b \ \& \ Pa \ \& \ \sigma a,b \Rightarrow Pb)$

$\sigma^{*=}x,y$ abbreviates

$\mathrm{N}x \ \& \ \mathrm{N}y \ \& \ \forall P \ (Px \ \& \ Hered(P) \Rightarrow Py)$

In this case we say that $x$ is an *ancestor* of $y$.

$\sigma^{*}x,y$ abbreviates

$\sigma^{*}x,y \ \& \ \neg \ x = y$ □

*Example*. $\sigma^{*}x,y$ if and only if there is a manner of representing the model where $y$ is the the right of $x$.

$\sigma^{*}0,3$ in 0 - 1 - 2 - 3
$\sigma^{*}0,3$ in 0 - 1 - 2 - 3 - 1
$\sigma^{*}3,2$ in 0 - 1 - 2 - 3 - 1 since this is equvialent to 0 - 1 - 2 - 3 - 1 - 2. □

The first several propositions are all well known and due to Frege, The header of these propositions include "(**SLA**)" or "(**IND**)" to emphasize the fact that they can be proved in these systems (and do not need full **GA**).

(**SLA**) *Prop 2.02*. Let $\mathrm{N}x \ \& \ \mathrm{N}y \ \& \ \sigma x,y$. Then $\sigma^{*=}x,y$.
*Pf:*
Suppose $Px$ & *Hered*($P$). Then $Py$. Hence $\sigma^{*=}x,y$. □

(**SLA**) *Prop 2.03*. (*Transitivity*) Let $\sigma^{*=}x,y \ \& \ \sigma^{*=}y,z$. Then $\sigma^{*=}x,z$.
*Pf:*
Suppose $Px$ & *Hered*($P$). Then $Py$ since $\sigma^{*=}x,y$. So $Pz$ since $\sigma^{*=}y,z$. Thus $\sigma^{*=}x,z$.

□

(**SLA**) *Corollary 2.04*.
(*a*) If N$z$ & $\sigma^{*=}x,y$ & $\sigma y,z$, then $\sigma^{*=}x,z$.
(*b*) If N$z$ & $\sigma x,y$ & $\sigma^{*=}y,z$, then $\sigma^{*=}x,z$.
*Pf:*

Use *Props 2.02* and *2.03*. □

(**SLA**) *Prop 2.05*. (*Reflexivity*) $\forall n$ (N$n \Rightarrow \sigma^{*=}n,n$).
*Pf:*

Let N$n$. Suppose $Pn$ & *Hered*($P$). Then evidently $Pn$. □

The next proposition asserts that if $m$ has a predecessor $n$ and the condition of one-to-oneness holds at $m$, i.e. if $n$ is the unique predecessor of $m$, then $x$ is an ancestor of $m$ if and only if $x$ equals $m$ or $x$ is an ancestor of $n$. Remark that the conclusion need not hold in general, since in the tadpole model 0 - 1 - 2 - 3 - 1, 0 is a predecessor of 1, and 3 is an ancestor of 1, but 3 is not an ancestor of 0.

(**SLA**) *Prop 2.06*. $\forall n \forall m$ (N$n$ & N$m$ & $\sigma n,m$ & $\neg \exists x$ (N$x$ & $\sigma x,m$ & $\neg x = n$) $\Rightarrow$
$\forall x(\sigma^{*=}x,m \Leftrightarrow \sigma^{*=}x,n \vee x = m)$).
*Pf:*

Let N$n$ & N$m$ & $\sigma n,m$ & $\neg \exists x$ (N$x$ & $\sigma x,m$ & $\neg x = n$).

Assume $\sigma^{*=}a,m$ & $\neg a = m$. So N$a$ by *Def 2.01*. Suppose $\neg \sigma^{*=}a,n$. Then $Pa$ & *Hered*($P$) & $\neg Pn$, for some $P$. Consider $Q$ as $P \setminus \{m\}$. Then $Qa$. To see that *Hered*($Q$), let N$u$ & N$v$ & $Qu$ & $\sigma u,v$. Then $Pu$, so $Pv$ since *Hered*($P$). $\neg v = m$, since $\neg u = n$ & $\sigma n,m$ & $\neg \exists x$ (N$x$ & $\sigma x,m$ & $\neg x = n$). Hence $Qv$. So *Hered*($Q$). Thus $Qm$, a contradiction. So $\sigma^{*=}a,n$.

Now assume $\sigma^{*=}a,n$. Then $\sigma^{*=}x,m$ by *Corollary 2.04*(*a*). □

0 is a minimal element (although it may not be the only minimal element).

(**IND**) *Prop 2.07*. $\forall n$ (N$n \Rightarrow \sigma^{*=}0,n$)
*Pf:*

By induction (PA6'), with $\phi$ as

$$\sigma^{*=}0,n.$$

Then N0 $\Rightarrow \sigma^{*=}0,0$ by *Prop 2.05*.
Now let N$n$ & N$m$ & $\sigma n,m$ & $\phi$ . So $\sigma^{*=}0,n$. By *Corollary 2.04*(*a*), $\sigma^{*=}0,m$. □

(**IND**) *Corollary 2.08.* $\forall n\,(\mathrm{N}n \Leftrightarrow \sigma^{*=}0,n)$ ∎

The next proposition is the first which requires (PA3). Henceforth, if nothing is indicated in the header of a proposition, it will be assumed that the theory being used is **GA**.

*Prop 2.09.* $\forall n \forall x\,(\sigma n,n \;\&\; \sigma^{*=}n,x \Rightarrow n = x)$.
*Pf:*

Let $\sigma n,n$ & $\sigma^{*=}n,x$. Consider $P$ as $\{n\}$. Then $Pn$ and, by (PA3), *Hered*($P$). So $Px$, i.e. $n = x$. ∎

*Prop 2.10.* Let $\mathrm{N}y$ & $\sigma x,y$ & $\sigma^{*}x,z$. Then $\sigma^{*=}y,z$.
*Pf:*

Assume $Py$ & *Hered*($P$). Set $Q$ to $P \cup \{x\}$.

To show: *Hered*($Q$). For let $\mathrm{N}a$ & $\mathrm{N}b$ & $Qa$ & $\sigma a,b$. If $Pa$, then $Pb$ since *Hered*($P$), so $Qb$. And if $\neg\, Pa$, then $a = x$, so $b = y$ by (PA3); so $Pb$, and thus $Qb$. So *Hered*($Q$).

Now $Qx$. So $Qz$ & $\neg\, z = x$ since $\sigma^{*}x,z$. Hence $Pz$.

Thus $\sigma^{*=}y,z$. ∎

Two numbers can always be compared using the ancestral.

*Prop 2.11.* $\forall n \forall k\,(\mathrm{N}n \;\&\; \mathrm{N}k \Rightarrow \sigma^{*=}n,k \vee \sigma^{*}k,n)$
*Pf:*

By induction (PA6'), with $\phi$ as

$$\mathrm{N}n \Rightarrow \forall k\,(\mathrm{N}k \Rightarrow \sigma^{*=}n,k \vee \sigma^{*}k,n)$$

If $n = 0$, then $\phi$ since $\forall k\,(\mathrm{N}k \Rightarrow \sigma^{*=}0,k)$ by *Prop 2.07.*

Now let $\mathrm{N}n$ & $\sigma n,m$ & $\phi$ & $\mathrm{N}m$. And let $\mathrm{N}k$. Then $\sigma^{*=}n,k \vee \sigma^{*}k,n$.

*Case 1.* $n = k$. Then $\sigma k,m$, so $\sigma^{*=}k,m$ by *Prop 2.02.*

*Case 2.* $\sigma^{*}n,k$. Then by *Prop 2.10*, $\sigma^{*=}m,k$.

*Case 3.* $\sigma^{*}k,n$. $\sigma^{*=}n,m$ by *Corollary 2.04*(*a*), so $\sigma^{*=}k,m$ by *Prop 2.03.*

Thus $\sigma^{*=}m,k \vee \sigma^{*=}k,m$. But this is equivalent to $\sigma^{*=}m,k \vee \sigma^{*}k,m$. ∎

It should be remarked that the choice presented in the previous proposition is not necessarily mutually exclusive. That is, it may be the case that there exist $n$ and $k$ such that both $\sigma^{*=}n,k$ and $\sigma^{*}k,n$.

*Prop 2.12*. Let $\sigma^* x,y$. Then $\exists z(\mathrm{N}z \ \& \ \sigma x,z \ \& \ \sigma^{*=} z,y)$.
*Pf:*

Suppose $\neg \exists z(\mathrm{N}z \ \& \ \sigma x,z)$. Set $X$ to $\{x\}$. Trivially, *Hered*($X$). But $Xx \ \& \ \neg Xy$ , so $\neg$ *Hered*($X$). Thus $\mathrm{N}z \ \& \ \sigma x,z$, for some $z$.

It is claimed that $\sigma^{*=} z,y$. For let $Pz$ & *Hered*($P$). Set $Q$ to $P \cup \{x\}$. Then $Q$ is hereditary (this uses (PA3)). But $Qx$. So $Qy$. But $\neg\, y = x$, so $Py$. Thus $\sigma^{*=} z,y$. □

*Corollary 2.13*. Let $\sigma^{*=} x,y \ \& \ \mathrm{N}z \ \& \ \sigma y,z$. Then $\exists v(\mathrm{N}v \ \& \ \sigma x,v)$.
*Pf:*

If $x = y$, then the conclusion follows trivially.

Otherwise, $\neg\, x = y$. Then $\sigma^* x,y$, and the conclusion follows by *Prop 2.12*. □

Although 1 is introduced later, the following proposition essentially asserts that 1 is minimal among all the non-zero numbers.

*Prop 2.14*. $\forall n \forall u \ (\mathrm{N}n \ \& \ \mathrm{N}u \ \& \ \neg\, n = 0 \ \& \ \sigma 0,u \Rightarrow \sigma^{*=} u,n)$.
*Pf:*

Suppose $\mathrm{N}u \ \& \ \sigma 0,u$. Proceed by induction (PA6'), with $\phi$ as

$$\neg\, n = 0 \Rightarrow \sigma^{*=} u,n.$$

$\phi$ holds trivially when $n = 0$.

Now let $\mathrm{N}n \ \& \ \mathrm{N}m \ \& \ \sigma n,m \ \& \ \phi$. Suppose $n = 0$. Then $m = u$ by (PA3), so trivially $\sigma^{*=} u,m$.

On the other hand, suppose $\neg\, n = 0$. Then by the induction hypothesis, $\sigma^{*=} u,n$. By *Corollary 2.04*(*a*), $\sigma^{*=} u,m$. □

## 3. Maximality

The standard model doesn't have maximal numbers, but every other type of model (except the vacuous) does, whence their interest and importance.

*Def 3.01*. Use *Max*($x$), read "$x$ is *maximal*," to abbreviate

$$\mathrm{N}x \ \& \ \forall n(\mathrm{N}n \Rightarrow \sigma^{*=} n,x)$$ □

*Examples*. A number is maximal in a model if and only if there is a representation of the model where the number appears farthest on the right.

(*a*) The finite segment 0 - 1 - 2 - 3. The only maximal number is 3.

(*b*) The pure cycle 0 - 1 - 2 - 3 - 0. Then all numbers are maximal, as the model has equivalent representations 0 - 1 - 2 - 3 - 0, 0 - 1 - 2 - 3 - 0 - 1, 0 - 1 - 2 - 3 - 0 - 1 -2, and 0 - 1 - 2 - 3 - 0 - 1 - 2 - 3.

(*c*) The tadpole 0 - 1 - 2 - 3 - 1. Then all numbers but 0 are maximal, as the model has equivalent representations 0 - 1 - 2 - 3 - 1, 0 - 1 - 2 - 3 - 1 - 2, and 0 - 1 - 2 - 3 - 1 - 2 - 4.

(*d*) The trivial tadpole 0 - 1 - 2 - 3 - 4 - 4. The only maximal number is 4. ∎

(**SLA**) *Prop 3.02.* $\forall x \forall y\ (\sigma^{*=}x,y\ \&\ Max(x) \Rightarrow Max(y))$.
*Pf:*

Let $\sigma^{*=}x,y\ \&\ Max(x)$.
Suppose N*n*. Then $\sigma^{*=}n,x$ by *Def 3.01*. By *Transitivity* (*Prop 2.03*), $\sigma^{*=}n,y$.
Thus $Max(y)$. ∎

(**SLA**) *Corollary 3.03.* $\forall x \forall y\ (\mathrm{N}y\ \&\ \sigma x,y\ \&\ Max(x) \Rightarrow Max(y))$.
*Pf.*

$\mathrm{N}x\ \&\ \mathrm{N}y\ \&\ \sigma x,y \Rightarrow \sigma^{*=}x,y$, by *Prop 2.02*. Now apply the proposition. ∎

(**IND**) *Corollary 3.04.* $Max(0) \Rightarrow \forall n\ (\mathrm{N}n \Rightarrow Max(n))$.
*Pf:*

Let $Max(0)$ & N*n*. By *Prop 2.07*, $\sigma^{*=}0,n$. By *Prop 3.02*, $Max(n)$. ∎

If *x* is its own successor, then it is maximal:

*Prop 3.05.* $\forall x\ (\mathrm{N}x\ \&\ \sigma x,x \Rightarrow Max(x))$.
*Pf:*

Assume N*x* & $\sigma x,x$. Let N*n*. By *Prop 2.11*, $\sigma^{*=}n,x \vee \sigma^{*}x,n$. If the latter, by *Prop 2.09*, $n = x$, so still $\sigma^{*=}n,x$. ∎

In fact, if *x* is its own succssor, then *x* is not only maximal, it is the *unique* maximal number:

*Prop 3.06.* $\forall x \forall y\ (\mathrm{N}x\ \&\ \sigma x,x\ \&\ \neg\ y = x \Rightarrow \neg\ Max(y))$.
*Pf:*

Assume N*x* & $\sigma x,x$ & $\neg\ y = x$. Set *P* to {*x*}. Then *Hered*(*P*), by (PA3). But *Px* and $\neg$ *Py*. So $\neg\ \sigma^{*=}x,y$, and thus $\neg\ Max(y)$. ∎

Another condition for unique maximality is when a number does not have a successor. First, we prove maximality:

*Prop 3.07*. Let N$n$ & ¬ ∃$m$ (N$m$ & σ$n$,$m$). Then *Max*($n$).
*Proof:*

Let N$x$. By *Prop 2.11*, $\sigma^{*=}n,x \vee \sigma^{*}x,n$.

Suppose $\sigma^{*=}n,x$. Set $P$ to {$n$}. Then *Hered*($P$) and obviously $Pn$. So $Px$. Thus $x = n$ and so $\sigma^{*=}x,n$. And obviously if $\sigma^{*}x,n$, then $\sigma^{*=}x,n$.

Thus in both cases $\sigma^{*=}x,n$. Hence *Max*($n$). ∎

*Corollary 3.08*. Suppose ¬ ∃$x$ *Max*($x$). Then ∀$x$ (N$x$ ⇒ ∃$z$(N$z$ & σ$x$,$z$)). ∎

*Prop 3.09*. Let *Max*($n$). Then ∀$x$ (N$x$ & ¬ $x = n$ ⇒ ∃$z$(N$z$ & σ$x$,$z$)).
*Proof:*

Let N$x$ & ¬ $x = n$. Then $\sigma^{*}x,n$ by *Def 3.01*. By *Prop 2.12*, N$z$ & σ$x$,$z$ & $\sigma^{*=}z,n$, for some $z$. ∎

*Corollary 3.10*. Let *Max*($n$) & N$m$ & σ$n$,$m$. Then ∀$x$ (N$x$ ⇒ ∃$z$(N$z$ & σ$x$,$z$)).
*Proof:*

Let N$x$. If $x = n$, then σ$x$,$m$. Otherwise, ¬ $x = n$, so by *Prop 3.09*, ∃$z$(N$z$ & σ$x$,$z$). ∎

*Corollary 3.11*. Let *Max*($n$) & *Max*($m$) & σ$n$,$m$. Then ∀$x$ (N$x$ ⇒ ∃$z$(N$z$ & σ$x$,$z$)).
*Proof:*

N$m$ by *Def 3.01*. Now use *Corollary 3.10*. ∎

In summary, if $n$ is a natural number without a successor, then it is the unique maximal number.

*Corollary 3.12*. Let N$n$ and suppose ¬ ∃$m$ (N$m$ & σ$n$,$m$). Then *Max*($n$) & ∀$x$(*Max*($x$) ⇒ $x = n$).
*Pf:*

*Max*($n$) by *Prop 3.07*.

Suppose *Max*($x$). Then $\sigma^{*=}n,x$. If ¬ $x = n$, then there exists $z$ such that N$z$ & σ$n$,$z$ by *Prop 2.12*, a contradiction. Hence $x = n$. ∎

*Corollary 3.13*. Let *Max*($n$) & *Max*($k$) & ¬ $n = k$. Then ∃$m$ (σ$n$,$m$ & *Max*($m$)).

*Proof:*

By *Prop 3.09*, $\exists m\,(\mathrm{N}m \,\&\, \sigma n,m)$. Now use *Corollary 3.03*. □

It has previously been noted that, while either $\sigma^{*=}x,y$ or $\sigma^{*}y,x$ for any natural numbers $x$ and $y$, this choice is not necessarily mutually exclusive. In fact, both alternatives obtain if and only if both $x$ and $y$ are distinct and maximal:

*Prop 3.14*. $\forall x\forall y\,(\sigma^{*}x,y \,\&\, \sigma^{*}y,x \Leftrightarrow Max(x) \,\&\, Max(y) \,\&\, \neg\, x = y)$

*Pf:*

Let $\sigma^{*}x,y \,\&\, \sigma^{*}y,x$. It suffices to prove $Max(x)$. We proceed by induction (PA6'), with $\phi$ as

$$\sigma^{*=}n,x.$$

$\mathrm{N}0 \Rightarrow \sigma^{*=}0,x$ by *Prop 2.07*.

Now let $\mathrm{N}n \,\&\, \sigma n,m \,\&\, \phi \,\&\, \mathrm{N}m$. By the induction hypothesis, $\sigma^{*=}n,x$. It suffices to show that:

$$\forall P\,(Pm \,\&\, Hered(P) \Rightarrow Px).$$

So let $Pm \,\&\, Hered(P)$. Set $Q$ to $P \cup \{n\}$.

$Hered(Q)$. For, suppose $\mathrm{N}a \,\&\, \mathrm{N}b \,\&\, Qa \,\&\, \sigma a,b$. If $Pa$, then $Pb$ since $Hered(P)$, so $Qb$. And if $\neg\, Pa$, then $a = n$, so $b = m$ by (PA3). So $Pb$ since $Hered(P)$, hence $Qb$.

Thus $Qn \,\&\, Hered(Q)$. By the induction hypothesis, $Qx$. If $\neg\, n = x$, then $Px$. On the other hand, suppose $n = x$. Then $\sigma x,m \,\&\, \sigma^{*}x,y$. By *Prop 2.10*, $\sigma^{*=}m,y$. By *Transitivity* (*Prop 2.03*), $\sigma^{*=}m,x$.

So, by induction, $\forall n\,(\mathrm{N}n \Rightarrow \sigma^{*=}n,x)$. Hence $Max(x)$.

Now suppose $Max(x) \,\&\, Max(y) \,\&\, \neg\, x = y$. Then $\sigma^{*=}y,x \,\&\, \sigma^{*=}x,y$ by *Def 3.01*. The result follows since $\neg\, x = y$. □

*Corollary 3.15*. $\forall x\forall y\forall z\,(\mathrm{N}x \,\&\, \mathrm{N}y \,\&\, \mathrm{N}z \,\&\, \sigma y,x \,\&\, \sigma z,x \,\&\, \neg\, Max(x) \Rightarrow y = z)$

*Remark*: The corollary asserts that σ is one-to-one at non-maximal numbers.

*Pf:*

Let $\mathrm{N}x \,\&\, \mathrm{N}y \,\&\, \mathrm{N}z \,\&\, \sigma y,x \,\&\, \sigma z,x \,\&\, \neg\, Max(x)$.

Assume $\neg\, y = z$. By *Prop 2.11*, $\sigma^{*}y,z \vee \sigma^{*}z,y$. WLOG suppose the former. By *Prop 2.10*, $\sigma^{*=}x,z$. But by *Prop 2.02*, $\sigma^{*=}z,x$. $\neg\, Max(x)$, so by *Prop 3.05*, $\neg\, z = x$. So by *Prop* 3.14, $Max(x)$, a contradiction. □

*Corollary 3.16*. $\forall n\forall m\,(\mathrm{N}n \,\&\, \mathrm{N}m \,\&\, \sigma n,m \,\&\, \neg\, Max(m) \Rightarrow$
$\forall x(\sigma^{*=}x,m \Leftrightarrow \sigma^{*=}x,n \vee x = m))$.

*Pf:*

Let $\mathrm{N}n \,\&\, \mathrm{N}m \,\&\, \sigma n,m \,\&\, \neg\, Max(m)$. If $\mathrm{N}x \,\&\, \sigma x,m$, then by *Corollary 3.15*, $x = n$.

Thus $\neg \exists x$ (N$x$ & $\sigma x,m$ & $\neg\, x = n$). So by *Prop 2.06*, $\forall x(\sigma^{*=}x,m \Leftrightarrow \sigma^{*=}x,n \vee x = m)$). ∎

*Prop 3.17*. Let $\exists z$(N$z$ & $\sigma 0,z$). Then *Max*(0) if and only if $\exists z$(N$z$ & $\sigma z,0$).

*Remark*: $\exists z$(N$z$ & $\sigma z,0$) is simply the condition $\neg$ (PA5).

*Pf:*

Let N$m$ & $\sigma 0,m$.

Suppose *Max*(0). And suppose $\neg \exists z$(N$z$ & $\sigma z,0$). If $m = 0$, then evidently $\exists z$(N$z$ & $\sigma z,0$). So let $\neg\, m = 0$. Now $\sigma^{*=}m,0$ by *Def 3.01* Consider $P = \{x : \neg\, x = 0\}$. Then $Pm$ and trivially *Hered*($P$). So $P0$, a contradiction. Thus $\exists z$(N$z$ & $\sigma z,0$).

Now suppose N$z$ & $\sigma z,0$, for some $z$. If $z = 0$, then *Max*(0) by *Prop 3.05*. Otherwise, suppose $\neg\, z = 0$. Then $\sigma^{*}z,0$ by *Prop 2.02* and $\sigma^{*}0,z$ by *Prop 2.07*. Thus by *Prop 3.14*, *Max*(0). ∎

*Corollary 3.18*. Let $\exists z$(N$z$ & $\sigma z,0$). Then *Max*(0).
*Pf:*

By *Prop 1.01*, N0.
By *Prop 3.07*, if $\neg \exists m$ (N$m$ & $\sigma 0,m$), then *Max*(0).
Otherwise, $\exists m$ (N$m$ & $\sigma 0,m$). So by *Prop 3.17*, *Max*(0). ∎

*Prop 3.19*. $\forall n$ (N$n$ & $\neg$ *Max*($n$) $\Rightarrow \exists P\, \forall x(Px \Leftrightarrow \sigma^{*=}x,n)$ ).
*Pf:*

By induction (PA6'), with $\phi$ as

$$\neg\, Max(n) \Rightarrow \exists P\, \forall x(Px \Leftrightarrow \sigma^{*=}x,n).$$

Suppose N0 & $\neg$ *Max*(0). Clearly {0} exists, so it suffices to show that $\sigma^{*=}x,0$ if and only if $x = 0$. The "only if" follows from *Prop 2.07*. And, if $\sigma^{*=}x,0$, then since $\sigma^{*=}0,x$ as well by *Prop 2.07*, $\neg$ *Max*(0) implies $x = 0$ by *Prop 3.14*.

Now let N$n$ & $\sigma n,m$ & $\phi$ & N$m$ & $\neg$ *Max*($m$). Then $\neg$ *Max*($n$) by *Corollary 3.03*. So for some $P$, $\forall x(Px \Leftrightarrow \sigma^{*=}x,n)$. Set $Q$ to ($P \cup \{m\}$). By *Corollary* 3.16, $\forall x(\sigma^{*=}x,m \Leftrightarrow Qx)$. ∎

*Corollary 3.20*. $\forall n$ (N$n \Rightarrow \exists P\, \forall x(Px \Leftrightarrow \sigma^{*=}x,n)$ ).
*Pf:*

Let N$n$. If $\neg$ *Max*($n$), then use the previous *Prop 3.19*. Otherwise, *Max*($n$). Then by *Def 3.01*, $\forall x$(N$x \Leftrightarrow \sigma^{*=}x,n$), so set $P$ to N. ∎

*Def 3.21*. *Neck*($k$) abbreviates

N$k$ & $\neg$ *Max*($k$) & $\exists y$ ( $\sigma k,y$ & *Max*($y$) )

*Chin*($c$) abbreviates

$Max(c)$ & $\exists y$ ( N$y$ & $\sigma y,c$ & $\neg$ $Max(y)$ ) ∎

Obviously, if there exists $k$ such that *Neck*($k$), there exists $c$ such that *Chin*($c$) & $\sigma k,c$; and if there exists $c$ such that *Chin*($c$), then there exists $k$ such that *Neck*($k$) & $\sigma k,c$.

*Examples*. A model with a neck and a chin must be a tadpole or a non-trivial finite segment.

In the tadpole 0 - 1 - 2 - 3 - 4 - 2, the neck is 1 and the chin is 2.

In the trivail tadpole 0 - 1 - 2 - 3 - 4 - 4, the neck is 3 and the chin is 4.

The trivial finite segment 0 does not have a chin or a neck.

In the finite segment 0 - 1 - 2 - 3, the neck is 2 and the chin is 3. ∎

The neck and the chin are unique, if they exist.

*Prop 3.22*. Let *Neck*($k_1$) & *Neck*($k_2$). Then $k_1 = k_2$.
*Pf:*

So N$k_1$ & N$k_2$ & $\neg$ $Max(k_1)$ & $\neg$ $Max(k_2)$ & $\sigma k_1,c_1$ & $\sigma k_2,c_2$ & $Max(c_1)$ & $Max(c_2)$ for some $c_1,c_2$. So $\sigma^{*=}c_1,c_2$ & $\sigma^{*=}c_2,c_1$.

Suppose $\neg$ $k_1 = k_2$. Then by *Prop 2.11,* $\sigma^* k_1,k_2 \vee \sigma^* k_2,k_1$. WLOG suppose the former. So $\sigma k_1,c_1$ & $\sigma^* k_1,k_2$, and thus $\sigma^{*=}c_1,k_2$ by *Prop 2.10*. By Transitivity (*Prop 2.03*), $\sigma^{*=}c_2,k_2$. But $\sigma^{*=}k_2,c_2$ by *Prop 2.02*. $\neg$ $k_2 = c_2$, since $\neg Max(k_2)$ but $Max(c_2)$. So $\sigma^* c_2,k_2$ & $\sigma^* k_2,c_2$. Thus by *Prop 3.14*, $Max(k_2)$, a contradiction.

Thus $k_1 = k_2$. ∎

*Corollary 3.23*. Let *Chin*($c_1$) & *Chin*($c_2$). Then $c_1 = c_2$. ∎

*Prop 3.24*. Let *Neck*($k$) & N$n$. Then $\sigma^{*=}n,k$ if and only if $\neg$ $Max(n)$.
*Pf:*

Let *Chin*($c$) where $\sigma k,c$.

Assume $\sigma^{*=}n,k$. Suppose $Max(n)$. And let N$a$. Then $\sigma^{*=}a,n$, so by *Transitivity* (*Prop 2.03*), $\sigma^{*=}a,k$. Hence $Max(k)$, contradicting *Neck*($k$). Thus $\neg$ $Max(n)$.

Now assume $\neg$ $\sigma^{*=}n,k$. Then by *Prop 2.11*, $\sigma^* k,n$. By *Prop 2.10*, $\sigma^{*=}c,n$. So $Max(n)$.
∎

*Prop 3.25*. Let ¬ *Max*(0) & *Max*(*x*). Then ∃*k Neck*(*k*).
*Pf:*

Suppose ¬ ∃*k Neck*(*k*).
Proceed by induction with $\phi$ as

$$(\mathrm{N}n \Rightarrow \neg\, Max(n)).$$

Then it is the case that $\phi$ when $n = 0$, by assumption.

Now let N*n* & σ*n*,*m* & $\phi$ & N*m*. Then ¬ *Max*(*n*). But then ¬ *Max*(*m*), since ¬ ∃*k Neck*(*k*).

So by induction, ∀*n* (N*n* ⇒ ¬ *Max*(*n*)). But *Max*(*x*) implies N*x*, and so ¬ *Max*(*x*), a contradiction. ∎

*Corollary 3.26*. Let ¬ *Max*(0) & ∃*x Max*(*x*). Then ∃*k*∃*c* (*Neck*(*k*) & *Chin*(*c*) & σ*k*,*c*). ∎

*Prop 3.27*. ∃*P* ∀*x*(*Px* ⇔ *Max*(*x*)).
*Pf*:

If ¬ ∃*x Max*(*x*), then set *P* to ϕ.

Otherwise ∃*x Max*(*x*). If *Max*(0), then by *Corollary 3.04*, ∀*x*(N*x* ⇔ *Max*(*x*)), so in this case set *P* to N.

Otherwise, let ¬ *Max*(0). By *Prop 3.25*, *Neck*(*k*) for some *k*. But ¬ *Max*(*k*), so ∀*x*(*Qx* ⇔ $\sigma^{*=}$*x*,*k*) for some *Q*, by *Prop 3.19*. By *Prop 3.24*, ∀*x*(*Qx* ⇔ N*x* & ¬ *Max*(*x*)). Hence set *P* to (N \ *Q*). ∎

*Prop 3.28*. Let *Max*(*a*) & *Max*(*b*) & ¬ *a* = *b*. Then ∃*p* (*Max*(*p*) & σ*p*,*a*).
*Pf:*

Suppose to the contrary, ¬ ∃*p* (*Max*(*p*) & σ*p*,*a*).

By *Def 3.01*, $\sigma^{*=}$*b*,*a*.

By *Prop 3.27*, ∀*x*(*Px* ⇔ *Max*(*x*)) for some *P*. Set *Q* to *P* \ {*a*}. Then *Qb*.

It is claimed that *Hered*(*Q*). For let N*x* & N*y* & *Qx* & σ*x*,*y*. Then *Px*, so *Max*(*x*), so *Max*(*y*) by *Prop 3.03*. But ¬ ∃*p* (*Max*(*p*) & σ*p*,*a*), so ¬ *y* = *a*. Hence *Qy*. Thus *Hered*(*Q*).

But then *Qa*, a contradiction. ∎

*Corollary 3.29*. Let *Chin*(*c*) & N*x* & σ*c*,*x*. Then ∃*p* (*Max*(*p*) & σ*p*,*c*).
*Pf:*

If *x* = *c*, then set *p* = *c*.

Otherwise, suppose ¬ *x* = *c*. *Max*(*x*) by *Prop 3.03*. Then ∃*p* (*Max*(*p*) & σ*p*,*c*) by *Prop 3.28*. ∎

*Prop 3.30*. Let $\sigma^{*}$*x*,*y*. Then ∃*z*(N*z* & $\sigma^{*=}$*x*,*z* & σ*z*,*y*).
*Pf:*

Suppose $y = 0$. Remark that $\sigma^* x,y$ implies that there are at least two natural numbers, so $\neg$ N $\equiv$ {0}. So by *Corollary 1.03*, $\exists z$(N$z$ & $\sigma 0,z$). Now $\sigma^* x,0$ & $\sigma^* 0,x$, the latter by *Prop 2.07*. So by *Prop 3.14 Max*(0). By *Prop 3.17*, N$z$ & $\sigma z,0$, for some $z$. By *Corollary 3.04*, *Max*($z$). By *Def 3.01*, $\sigma^{*=} x,z$.

Otherwise, let $\neg\, y = 0$.

Suppose *Max*($y$). If *Max*($b$) for any $b$ where $\neg\, b = y$, then by *Prop 3.28*, *Max*($p$) & $\sigma p,y$ for some $p$. By *Def 3.01*, $\sigma^{*=} x,p$. Thus set $z = p$.

Otherwise $\neg$ *Max*($b$) for any $b$ where $\neg\, b = y$. In particular, $\neg$ *Max*($x$). By *Prop 1.04*, N$z$ & $\sigma z,y$ & $\neg\, z = y$, for some $z$. Thus $\neg$ *Max*($z$). Hence *Neck*($z$), by *Def 3.21*. Then $\sigma^{*=} x,z$ by *Prop 3.24*. □

The definition of *Chin* excludes the pure-cycle models, because in these models every number is maximal. Nonetheless, 0 in these models shares many similarities with the chin, which motivates the following definition:

*Defs 3.31*. *Chin0*($c$) if and only if *Chin*($c$) ∨ (*Max*(0) & $c = 0$)

*Tail*($t$) if and only if $\exists k$ (*Neck*($k$) & $\sigma^{*=} t,k$) ∨ *Chin0*($t$). □

*Examples*.

(*a*) In the tadpole 0 - 1 - 2 - 3 - 4 - 2, where 2 is the chin and 1 is the neck, then 0, 1, and 2 compose the tail.

(*b*) In the pure cycle 0 - 1 - 2 - 3 - 0, then 0 is the chin0 and the tail is just 0.

(*c*) In the finite segment 0 - 1 - 2 - 3 - 4, where 4 is the chin and 3 is the neck, then all elements of the segment form the tail. □

Remark that if *Chin0*($x$), then *Max*($x$).

In the case of *Max*(0), obviously *Tail*($t$) if and only if $t = 0$. And if $\neg$ *Max*(0), then *Tail*($t$) if and only if $\sigma^{*=} t,k$ ∨ *Chin*($t$), where *Neck*($k$). So:

*Prop 3.32*. If N0, then *Tail*(0). □

Remark also that any chin0, like the chin, is unique:

*Prop 3.33*. Suppose *Chin0(s)* & *Chin0(t)*. Then $s = t$.
*Pf:*

If $Chin(s)$ & $Chin(t)$, then *Corollary 3.23* implies $s = t$.

Otherwise, WLOG $Max(0)$ & $s = 0$. Then by *Corollary 3.04*, $\forall n$ ($\mathrm{N}n \Rightarrow Max(n)$). So by *Def 3.21*, $\neg Chin(t)$. Hence $Max(0)$ & $t = 0$. So $s = t$. ∎

*Prop 3.34*. Suppose $Tail(t)$ & $\neg Chin0(t)$ & $\sigma^{*=}s,t$. Then $Tail(s)$ & $\neg Chin0(s)$.

*Pf:*

By *Def 3.21*, $Neck(k)$ & $\sigma^{*=}t,k$, for some $k$. By *Transitivity* (*Prop 2.03*), $\sigma^{*=}s,k$. Hence $Tail(s)$.

Suppose $Chin0(s)$. Then $Max(s)$, so by *Prop 3.02*, $Max(k)$. But $\neg Max(k)$ by *Def 3.21*. Hence $\neg Chin0(s)$. ∎

If there is a chin, then every number is either part of the tail or maximal; and the only number which is both, is the chin:

*Prop 3.35*. Let $Chin(c)$ and $\mathrm{N}n$. Then $Tail(n) \vee Max(n)$. Also, $Tail(c)$ & $Max(c)$. And if $Tail(n)$ & $Max(n)$, then $n = c$.

*Pf:*

By *Def 3.21* $Neck(k)$ for some $k$.

Suppose $\neg Max(n)$. By *Prop 3.24*, $\sigma^{*=}n,k$. By *Def 3.31*, $Tail(n)$.

$Tail(c)$ by *Def 3.31*. $Max(c)$ by *Def 3.21*.

Now suppose $Tail(n)$ & $Max(n)$. By *Prop 3.24* $\neg\sigma^{*=}n,k$. By *Def 3.31*, $Chin0(n)$. But $Chin0(c)$, also by *Def 3.31*. By *Prop 3.33* $c = n$. ∎

*Prop 3.36*. Let $Chin0(c)$. Then:

$$\begin{aligned}\forall n(\ \mathrm{N}n \ \&\ \neg Tail(n) \Rightarrow \exists R\ (\ & IsFunction(R)\ \&\ Is1\text{-}1(R)\ \&\ c \in Dom(R)\ \&\ n \in Im(R)\\ & \&\ Dom(R) \setminus \{c\} \equiv Im(R) \setminus \{n\}\ \&\ R \subseteq \sigma\\ & \&\ \forall x(x \in (Dom(R) \cup Im(R)) \Rightarrow Max(x)))).\end{aligned}$$

*Pf:*

Proceed by induction (PA6'), with $\phi$ as

$$\begin{aligned}&(Tail(n) \vee \exists R\ (\ IsFunction(R)\ \&\ Is1\text{-}1(R)\ \&\ c \in Dom(R)\\ &\&\ n \in Im(R)\ \&\ Dom(R) \setminus \{c\} \equiv Im(R) \setminus \{n\}\ \&\ R \subseteq \sigma\\ &\&\ \forall x(x \in (Dom(R) \cup Im(R)) \Rightarrow Max(x)))).\end{aligned}$$

Then $\mathrm{N}0 \Rightarrow \phi$ holds when $n = 0$, since $Tail(0)$ by *Prop 3.32*.

Now let $\mathrm{N}n$ & $\sigma n,m$ & $\phi$ & $\mathrm{N}m$. It needs to be shown that $\phi\ [m\backslash n]$.

If $Tail(m)$, then we are done. Otherwise, $\neg Tail(m)$. So $\neg c = m$.

Suppose $n = c$. Then consider $R$ as $\{(n,m)\}$. Obviously $IsFunction(R)$ & $Is1\text{-}1(R)$ & $c \in Dom(R)$ & $m \in Im(R)$ & $R \subseteq \sigma$. $Max(n)$ since $Chin0(c)$. By *Corollary 3.03*, $Max(m)$. So $\forall x(x \in (Dom(R) \cup Im(R)) \Rightarrow Max(x))$. But also $Dom(R) \setminus \{c\} \equiv \phi \equiv Im(R) \setminus \{m\}$. Hence $\phi\ [m\backslash n]$.

So let $\neg n = c$. Suppose $Tail(n)$. By *Prop 3.33*, $\neg Chin0(n)$. Then by *Defs 3.31*, $Neck(k)$ & $\sigma^{*=}n,k$ for some $k$. Hence $\neg Max(n)$ by *Prop 3.24*. If $\neg Max(m)$, then $\sigma^{*=}m,k$ again by *Prop 3.24*, so $Tail(m)$, a contradiction. Hence $Max(m)$, hence $Chin(m)$ by *Def 3.21*, implying $m = c$ by *Corollary 3.23*, also a contradiction.

Thus $\neg Tail(n)$. Then by the Induction Hypothesis, $IsFunction(R)$ & $Is1\text{-}1(R)$ & $c \in Dom(R)$ & $n \in Im(R)$ & $Dom(R) \setminus \{c\} \equiv Im(R) \setminus \{n\}$ & $R \subseteq \sigma$ & $\forall x(x \in (Dom(R) \cup Im(R) \Rightarrow Max(x))$, for some $R$. Note that $\neg\, n \in Dom(R)$.

Suppose $m \in Im(R)$. We claim that $Hered(Im(R))$. For suppose $x \in Im(R)$ & N$x$ & N$y$ & $\sigma x,y$. If $x = n$, then $y = m$ by (PA3); so $y \in Im(R)$. And if $\neg\, x = n$, then $x \in Dom(R)$. So $Rx,z$ for some $z$. Since $R \subseteq \sigma$, then $\sigma x,z$. By (PA3), $y = z$, and thus $y \in Im(R)$. Thus $Hered(Im(R))$. But $\neg\, c \in Im(R)$, so $\neg\, \sigma^{*=}m,c$, contradicting $Chin0(c)$.

Therefore $\neg\, m \in Im(R)$. Set $S$ to $R \cup \{(n,m)\}$. Since $\neg\, n \in Dom(R)$ & $\neg\, m \in Im(R)$, then $IsFunction(S)$ and $Is1\text{-}1(S)$. Obviously $c \in Dom(S)$ and $m \in Im(S)$ & $S \subseteq \sigma$. $n \in Im(R)$, so $Max(n)$. By *Corollary 3.03*, $Max(m)$. Hence $\forall x(x \in (Dom(R) \cup Im(R) \Rightarrow Max(x))$. Finally,

| | | |
|---|---|---|
| $Dom(S) \setminus \{c\} \equiv (Dom(R) \setminus \{c\}) \cup \{n\}$ | | |
| $\equiv (Im(R) \setminus \{n\}) \cup \{n\}$ | by the Induction Hypothesis | |
| $\equiv Im(R)$ | since $n \in Im(R)$ | |
| $\equiv Im(S) \setminus \{m\}$. | since $\neg\, m \in Im(R)$ | □ |

*Def 3.37*. *ChainMax*($R$,$c$,$n$) abbreviates

$IsFunction(R)$ & $Is1\text{-}1(R)$ & $c \in Dom(R)$ & $n \in Im(R)$ & $Dom(R) \setminus \{c\} \equiv Im(R) \setminus \{n\}$ & $R \subseteq \sigma$ & $\forall x(x \in (Dom(R) \cup Im(R) \Rightarrow Max(x))$. □

Then *Prop 3.36* shows that, if $Chin0(c)$ & N$n$ & $\neg\, Tail(n)$, then $ChainMax(R,c,n)$ for some $R$.

*Prop 3.38*. Let $Chin0(c)$ & $Max(x)$ & $Max(y)$ & $Max(z)$ & $\sigma x,y$ & $\sigma z,y$. Then $x = z$.
*Pf:*

Suppose to the contrary that $\neg\, x = z$. By *Prop 3.28*, there exists $p$ such that $Max(p)$ & $\sigma p,c$. If $c = p$, then $x = z$ by *Prop 3.06*, a contradiction. So $\neg\, c = p$.

Thus $\neg\, Tail(p)$. By *Prop 3.36*, $IsFunction(R)$ & $Is1\text{-}1(R)$ & $c \in Dom(R)$ & $p \in Im(R)$ & $Dom(R) \setminus \{c\} \equiv Im(R) \setminus \{p\}$ & $R \subseteq \sigma$ & $\forall x(x \in (Dom(R) \cup Im(R) \Rightarrow Max(x))$, for some $R$.

It is claimed that $Im(R) \cup \{c\} \equiv \{x : Max(x)\}$, the latter existing by *Prop 3.27*. For suppose $x \in (Im(R) \cup \{c\})$. If $x \in Im(R)$, then $Max(x)$, while if $x = c$, then by *Def 3.21*, $Max(c)$.

On the other hand, assume $Max(x)$. Note that $\sigma^{*=}c,x$.

It is claimed that $Hered(Im(R) \cup \{c\})$. For let N$a$ & N$b$ & $a \in (Im(R) \cup \{c\})$ & $\sigma a,b$. If $a \in (Im(R) \cup \{c\})$, then either $a = p$ or $a \in Dom(R)$. In the former case $b = c$ by (PA3). In the latter case $Ra,u$ for some $u$. So $\sigma a,u$. By (PA3) again, $u = b$, so $b \in Im(R)$. So in both cases $b \in (Im(R) \cup \{c\})$. Hence $Hered(Im(R) \cup \{c\})$.

Obviously $c \in (Im(R) \cup \{c\}$. So $x \in (Im(R) \cup \{c\})$.

Thus $Dom(R) \cup \{p\} \equiv Im(R) \cup \{c\} \equiv \{x : Max(x)\}$.

If $x = p$, then $\neg\, z = p$. So $z \in Dom(R)$. Thus $Rz,v$ for some $v$. Hence $\sigma z,v$. By (PA3), $v = y$. But $x = p$ and (PA3) implies that $y = c$. So $c \in Im(R)$, contrary to assumption. So $\neg\, x = p$. By a similar argument, $\neg\, z = p$. So $x,z \in Dom(R)$. Thus $Rx,s$ & $Rz,t$ for some $s,t$. Hence $\sigma x,s$ & $\sigma z,t$. By (PA3) $s = y$ & $t = y$. But $R$ is one-to-one, so $x = z$, a contradiction. □

*Prop 3.39*. Let *Chin*($c$) & $\exists x$ ($\neg$ $x = c$ & *Max*($x$)). Then $\exists k \exists p$($\neg$ $k = p$ & N$k$ & N$p$ & $\sigma k,c$ & $\sigma p,c$ & $\forall z$(N$z$ & $\sigma z,c \Rightarrow z = k \vee z = p$ )).
*Pf*:

By *Def 3.21, Neck*($k$) & $\sigma k,c$, for some $k$. $\neg$ *Max*($k$) & *Max*($c$), equally by *Def 3.21*. By *Prop 3.28*, $\sigma p,c$ & *Max*($p$) for some $p$. Evidently $\neg$ $k = p$.

Now suppose N$z$ & $\sigma z,c$. If $\neg$ *Max*($z$), then *Neck*($z$) by *Def 3.21*, and so $z = k$ by *Prop 3.22*. And if *Max*($z$), then $z = p$ by *Prop 3.38*. ∎

The chin is maximal and must have one and only one non-maximal predecessor, the neck, by *Def 3.21* and *Prop 3.22*. The previous proposition explains that, when there is another maximal number, different from the chin, σ is not one-to-one at the chin, having precisely two predecessors, a maximal and a non-maximal one (the neck).

For completeness' sake, consider what happens for the special case when there is no other maximal number. The chin either is a predecessor of itself or not. If it is, then (since it cannot be the neck), the chin has precisely two predecessors, itself and the neck, and so σ is not one-to-one at the chin. And if the chin does not precede itself, then the chin has only one predecessor, the neck, and in this case only is σ one-to-one at the chin.

The foregoing paragraphs explain when σ is one-to-one at the chin. The following proposition asserts that elsewhere, i.e. at every other number other than the chin, σ must be one-to-one.

*Prop 3.40*. Suppose N$x$ & $\neg$ *Chin*($x$). Then $\forall y \forall z$(N$y$ & N$z$ & $\sigma y,x$ & $\sigma z,x \Rightarrow y = z$).
*Pf:*

Let N$y$ & N$z$ & $\sigma y,x$ & $\sigma z,x$.

Suppose *Max*($x$). Then *Max*($y$) & *Max*($z$), by *Def 3.21*. By *Prop 3.38*, $y = z$.

Now suppose $\neg$ *Max*($x$). Then by *Corollary 3.15*, $y = z$. ∎

**4. Possible Models**

George Boolos proved the first parts of the next four propositions in [Boolos], following (as he wrote there) Henkin and Frege.

*Prop 4.01*. Let $\neg$ (PA1). Then: (PA2) & (PA4) & (PA5).
*Pf:*

By *Prop 1.01*, N $\equiv \phi$. Then vacuously (PA2) & (PA4) & (PA5). ∎

*Prop 4.02*. Let $\neg$ (PA2). Then: (PA1) & (PA4) & (PA5). Also there exists an $x$ such that all of the following hold:

(*i*) $\neg \exists z$ (N$z$ & $\sigma x,z$)
(*ii*) *Max*($x$)
(*iii*) $\forall y$(*Max*($y$) $\Rightarrow y = x$)

(*iv*) $\forall y$(N$y$ & $\sigma y,x \Rightarrow$ *Neck*($y$) & *Chin*($x$))
(*v*) $x = 0 \vee (\neg\, x = 0$ & *Chin*($x$))

*Pf:*

By ¬ (PA2), there exists an $x$ such that N$x$ & ¬ $\exists z$(N$z$ & $\sigma x,z$).

By *Prop 1.01*, (PA1).

By *Prop 3.07*, *Max*($x$).

By *Corollary 3.12*, $\forall y$(*Max*($y$) $\Rightarrow y = x$). By assumption ¬ $\sigma x,x$,. So, if N$y$ & $\sigma y,x$, then $y$ must be non-maximal, in which case *Chin*($x$) & *Neck*($y$), and so $y$ is the unique predecessor, by *Prop 3.22*. But for every number $z$ different than $x$, $z$ is non-maximal, and so by *Corollary 3.15*, σ is one-to-one at $z$. Thus σ is one-to-one everywhere. Hence (PA4).

Suppose ¬ (PA5). Then N$n$ & $\sigma n$,0, for some $n$. Then *Max*(0) by *Corollary 3.18*. If ¬ $x = 0$, then by *Corollary 3.13*, $\sigma x,m$ & *Max*($m$) for some $m$, contrary to assumption. So $x = 0$. But then by *Corollary 1.03*, N ≡ {0}. Hence $n = 0$, so σ0,0, contrary to assumption. Thus (PA5).

Finally, if ¬ $x = 0$, then N$y$ & $\sigma y,x$ for some $y$ by *Prop 1.04*, which as above must be non-maximal, and thus *Chin*($x$). □

*Prop 4.02* is particularly important for the following reason. If (PA1) & (PA4) & (PA5), then the axioms of **FPA** hold, and all the results of *Arithmetic without the Successor Axiom* hold and may be used.

*Prop 4.03*. Let ¬ (PA4). Then: (PA1) & (PA2) & (PA5). Also: $\exists p,c$ ( $\sigma p,c$ & *Max*($p$) & *Chin*($c$)).
*Pf:*

By assumption there exists $c,k,p$ such that N$c$ & N$k$ & N$p$ & $\sigma k,c$ & $\sigma p,c$ & ¬ $k = p$.

By *Prop 1.01*, (PA1).

By *Prop 3.40*, *Chin*($c$). By *Prop 3.22*, either *Max*($k$) or *Max*($p$), WLOG *Max*($p$). By *Corollary 3.11*, (PA2).

¬ *Max*($k$) by *Prop 3.38*. By *Corollary 3.08*, (PA2). □

*Prop 4.04*. Let ¬ (PA5). Then: (PA1) & (PA2) & (PA4). Also:

(*i*) $\forall n$ (N$n \Rightarrow$ *Max*($n$))
(*ii*) *Chin0*(0)
(*iii*) $\forall x$ ¬ *Chin*($x$)
(*iv*) $\forall n$ (*Chin0*($n$) $\Rightarrow n = 0$)

*Pf:*

By assumption there exists $p$ such that N$p$ & $\sigma p$,0.

By *Prop 1.01*, (PA1).

By *Corollaries 3.18*, *Max*(0). By *Corollary* 3.*04*, *Max*($p$). By *Corollary 3.10*, (PA2).

By *Corollary 3.04*, $\forall n$ (N$n \Rightarrow$ *Max*($n$)).

By *Def 3.21*, $\forall x$ ¬ *Chin*($x$). By *Prop 3.40*, (PA4).

*Chin0*(0) and $\forall n$ (*Chin0*($n$) $\Rightarrow n = 0$) by *Def 3.31*. □

*Prop 4.05*. Let (PA2). Suppose $\exists x$ *Max*($x$). Then (PA1) but not both (PA4) and (PA5).

*Pf:*

By *Def 3.01* and *Prop 1.01*, N0, i.e. (PA1).

Now suppose (PA5).

By (PA2), ∃*z*(N*z* & σ0,*z*). Since (PA5), by *Prop 3.17*, ¬ *Max*(0).

By *Prop 3.25*, *Neck*(*k*) & *chin*(*c*) & σ*k*,*c*, for some *k* and *c*. So ¬ *Max*(*k*) by *Def 3.21*.

Now ∃*z*(N*z* & σ*c*,*z*) by (PA2). Thus, by *Corollary* 3.29, *Max*(*p*) & σ*p*,*c*, for some *p*. So ¬ *k* = *p*. Thus ¬ (PA4). □

*Prop 4.06*. ¬ ∃*x Max*(*x*) if and only if (PA2) & (PA4) & (PA5).

*Pf:*

First, suppose ¬ ∃*x Max*(*x*).

If ¬ N0, then (PA2) & (PA4) & (PA5) by *Prop 4.01*.

Otherwise, let N0. By *Corollary 3.08*, (PA2). By (PA2), ∃*z*(N*z* & σ0,*z*). But ¬ *Max*(0). By *Prop 3.17*, (PA5).

By *Corollary 3.15*, (PA4).

Now let (PA2) & (PA4) & (PA5). Suppose *Max*(*x*) for some *x*. If *Max*(0), then ¬ (PA5), by *Corollary 3.18*. Otherwise, ¬ *Max*(0), so there exist *k*,*c* such that *Neck*(*k*) & *Chin*(*c*). So N*k* & ¬ *Max*(*k*) & σ*k*,*c*. If N*v* & σ*c*,*v* for some *v*, then by *Corollary 3.29*, *c* has a maximal predecessor, so ¬ (PA4). Otherewise, ¬ N*y* & σ*c*,*y*, for any *y*; hence ¬ (PA2). □

So (PA2) & (PA4) & (PA5) can be replaced by "¬ ∃*x Max*(*x*)", and a possible axiomatization for Peano Arithmetic is:

(1) N0, i.e. (PA1)
(2) ¬ ∃*x Max*(*x*).
(3) ∀*n*∀*m*∀*m*' ( N*n* & N*m* & N*m*' & σ*n*,*m* & σ*n*,*m*' ⇒ *m* = *m*' ), i.e. (PA3)
(4) Induction, i.e. (PA6).

It is now possible, again following Boolos, to describe the possible models.

*Case 1*. ¬ (PA1).

By *Prop 1.01*, N ≡ ϕ. This is the vacuous case.

*Case 2*. ¬ (PA2).

Then (PA1), (PA4), and (PA5), by *Prop 4.02*. Let N*x* & ¬ ∃*z*(N*z* & σ*x*,*z*), *x* existing by ¬ (PA2). Then *Max*(*x*) by *Prop 3.07*. So every number other than *x* has a successor, by *Prop 3.09*. This is the case of the finite segments.

*Case 3*. (PA1) & (PA2) & (PA4) & (PA5).

This is the case of the standard model. ¬ ∃*x Max*(*x*), by *Prop 4.06*.

*Case 4*. ¬ (PA4).

Then (PA1) & (PA2) & (PA5) and *Chin*(*c*), for some *c*, by *Prop 4.03*. Then *Neck*(*k*) for some *k*. Both *c* and *k* are the unique chin and neck, respectively. By *Prop 3.39*, *c* has precisely two predecessors. By *Props 1.04* and 3.40, every number other than 0 and *c* has exactly one predecessor. 0 has no predecessor. This is the case of the tadpoles.

*Case 5*. ¬ (PA5).

Then (PA1) & (PA2) & (PA4) and ∀*n* (N*n* ⇒ *Max*(*n*)), by *Prop 4.04*. Every number has a unique successor and a unique predecessor. This is the case of the pure cycles.

By logic these cases are exhaustive. By *Props 4.01*, *4.02*, *4.03*, and *4.04*, they are mutually exclusive.

Let us summarize our findings in the following theorem:

*Theorem 4.07*. One and only one of the following holds:

*Case 1*. ¬ (PA1).

*Case 2*. ¬ (PA2).

*Case 3*. (PA1) & (PA2) & (PA4) & (PA5).

*Case 4*. ¬ (PA4).

*Case 5*. ¬ (PA5). □

∃*x* (N*x* & σ*x*,*x*) defines trivial sub-cases of *Cases 4* and *5*. That is, suppose N*x* & σ*x*,*x*, for some *x*. If *x* = 0, then ¬ (PA5); this is the case of the trivial cycle. And if ¬ *x* = 0, then by *Prop 1.04*, *x* has two distinct predecessors, hence ¬ (PA4).

A trivial sub-case of ¬ (PA2) is: N0 & ¬ ∃*x* σ0,*x*.

*Def 4.08*. Define the following conditions:

(TRV2) N0 & ¬ ∃*x* (N*x* & σ0,*x*).

(TRV4) ∃*x* (N*x* & ¬ *x* = 0 & σ*x*,*x*).

(TRV5) N0 & σ0,0. □

*Prop 4.09*.

*a*. (TRV2) ⇒ ¬ (PA2).

*b*. (TRV2) ⇒ N ≡ {0}.

*c*. (TRV4) $\Rightarrow \neg$ (PA4).

*d*. (TRV4) $\Rightarrow \forall n$ ( N$n \Rightarrow$ *Tail*($n$) ).

*e*. (TRV5) $\Rightarrow \neg$ (PA5).

*f*. (TRV5) $\Rightarrow$ N $\equiv \{0\}$.

*Pf:*

*b*. Use *Corollary 1.03*.

*d*. Suppose (TRV4). Then N$c$ & $\neg\, c = 0$ & $\sigma c,c$. Then by *Prop 1.04*, $c$ has two distinct predecessors. By *Prop 3.40 Chin*($c$). By *Def 3.21 Max*($c$). By *Prop 3.06*, $c$ is the only maximal number. *Tail*($c$) by *Def 3.21*. By *Prop 3.35*, $\forall n$ ( N$n \Rightarrow$ *Tail*($n$) ).

*f*. Use (PA3) and *Prop 1.02*. ∎

The condition $\exists p,c$ ( $\sigma p,c$ & *Max*($p$) & *Chin0*($c$)) has already appeared in *Props 4.02* and *4.03*. It will prove important beginning in the next section, so the next proposition summarizes when the condition holds:

*Prop 4.10*. $\exists p,c$ ($\sigma p,c$ & *Max*($p$) & *Chin0*($c$)) $\Leftrightarrow \neg$ (PA4) $\vee \neg$ (PA5). Furthemore, in the case of $\neg$ (PA4), $\neg\, c = 0$ and so *Chin*($c$); and in the case of $\neg$ (PA5), $c = 0$ and $\neg$ *Chin*($c$). ∎

## 5. σ\

*Def 5.01*. Set σ\ to $\{(x,y) : \sigma x,y$ & (*chin0*($y$) $\Rightarrow \neg$ *Max*($x$))$\}$. ∎

In other words, σ\ is just σ, except at most one link is removed. It there exists a chin0 $c$ and a maximal number $p$ which is its predecessor (i.e. not the predecessor which is the neck), then ($p$,$c$) is removed.

*Examples*. In the minimal representation, if any link is removed, it is the last.

(*a*) In the finite segment 0 - 1 - 2 - 3, σ\ is just the same as σ. For the chin is 3 and 2 is non-maximal.

(*b*) In the trivial tadpole 0 - 1 - 2 - 3 - 3, the link 3 - 3 is removed. For the chin is 3 and 3 is maximal.

(*c*) In the tadpole 0 - 1 - 2 - 3 - 1, the link 3 - 1 is removed. For the chin is 1 and 3 is maximal. Note the link 0 - 1 is retained, since 0 is not maximal.

(*d*) In the trivial pure cycle 0 - 0, the link 0 - 0 is removed, since 0 is the chin0 and it is maximal.

(*e*) In the pure cycle 0 - 1 - 2 - 3 - 0, the link 3 - 0 is removed, since 0 is the chin0 and 3 is maximal. ∎

*Prop 5.02*. $\sigma\backslash \subseteq \sigma$. Indeed, $\sigma\backslash \equiv \sigma \vee \exists x \exists y\, (\sigma\backslash \equiv \sigma \setminus \{(x,y)\}\ \&\ \sigma x,y\ \&\ Max(x)\ \&\ chin0(y))$.
*Pf*:

Suppose N*y* & *chin0*(*y*). Then *Max*(*y*), so there is at most one *x* such that N*x* & σ*x*,*y* & *Max*(*x*), by *Prop 3.38*. ∎

*Prop 5.03*. $\sigma\backslash \equiv \sigma \Leftrightarrow$ (PA4) & (PA5).

*Remark:* By *Theorem 4.07*, the right-hand side is equivalent to
¬ (PA1) ∨ ¬ (PA2) ∨ ( (PA1) & (PA2) & (PA4) & (PA5) ).

*Pf:*

Apply *Prop 4.10* and *Prop 5.02*. ∎

*Prop 5.04*. ¬ (σ\)0,0.
*Pf*:

Suppose (σ\)0,0. Then σ0,0. By *Prop 3.05*, *Max*(0). So *Chin0*(0). Hence ¬ *Max*(0) by *Def 5.01*, a contradiction. ∎

*Prop 5.05* (PA3\) $\forall n \forall m \forall m'$ ( N*n* & N*m* & N*m'* & (σ\)*n*,*m* & (σ\)*n*,*m'* ⇒ *m* = *m'* )
*Pf:*

This follows from (PA3) and $\sigma\backslash \subseteq \sigma$ (*Prop 5.02*). ∎

*Prop 5.06*. (PA4\) $\forall n \forall m \forall n'$ ( N*n* & N*m* & N*n'* & (σ\)*n*,*m* & (σ\)*n'*,*m* ⇒ *n* = *n'* )
*Pf:*

Let N*n* & N*m* & N*n'* & (σ\)*n*,*m* & (σ\)*n'*,*m*. Suppose ¬ *n* = *n'*.

Now σ*n*,*m* & σ*n'*,*m*. By *Prop 3.40*, this forces *Chin*(*m*). But by *Prop 3.22*, either *Max*(*n*) or *Max*(*n'*), contradicting (σ\)*n*,*m* & (σ\)*n'*,*m* and *Def 5.01*. ∎

*Prop 5.07*. (PA5\) $\forall n$ ( N*n* ⇒ ¬ (σ\)*n*,0 )
*Pf:*

Let N*n* & (σ\)*n*,0. Then σ*n*,0. So by *Corollary* 3.*18*, *Max*(0). By *Corollary 3.04*, *Max*(*n*). But then ¬ (σ\)*n*,0, by *Def 5.01*. ∎

*Lemma 5.08*. Let $\phi$ be a well-formed formula (with no appearance of *m*). Suppose $\varphi$ [0*n*] and

$\forall n \forall m$ ( N$n$ & $(\sigma\backslash) n,m$ & $\varphi \Rightarrow \varphi [m\backslash n]$ ). Then $\forall n$ ( *Tail*($n$) $\Rightarrow \varphi$ ).
*Pf:*

Proceed by induction, with $\phi$ as

$$Tail(n) \ \& \ \neg \ chin0(n) \Rightarrow \varphi$$

$\varphi [0\backslash n]$, so $\phi$ holds if $n = 0$.

Now let N$n$ & $\sigma n,m$ & $\phi$ & *Tail*($m$) & ¬ *Chin0*($m$). Then by *Props 2.02* and 3.*34*, *Tail*($n$) & ¬ *Chin0*($n$). By the Induction Hypothesis, $\varphi$. But $(\sigma\backslash) n,m$ since ¬ *Chin0*($m$). So $\varphi [m\backslash n]$.

Hence by Induction, $\forall n$ ( *Tail*($n$) & ¬ *Chin0*($n$) $\Rightarrow \varphi$ ).

Now suppose *Tail*($n$) & *Chin0*($n$). If $n = 0$, then $\varphi [0\backslash n]$. So suppose ¬ $n = 0$. Then by *Defs 3.31* and 3.*21*, *Neck*($k$) & $\sigma k,n$ for some $k$. But evidently *Tail*($k$) and ¬ *Max*($k$), so ¬ *Chin0*($k$). Hence $\varphi [k\backslash n]$. But $(\sigma\backslash) k,n$ since *Neck*($k$). Thus $\varphi$. ▯

*Prop 5.09* (PA6\). Let $\phi$ be a well-formed formula (with no appearance of $m$). Suppose $\varphi [0\backslash n]$ and $\forall n \forall m$ ( N$n$ & $(\sigma\backslash) n,m$ & $\varphi \Rightarrow \varphi [m\backslash n]$ ). Then $\forall n$ ( N$n \Rightarrow \varphi$ ).
*Pf:*

Let N$n$ & $\sigma n,m$ & $\varphi$. If ¬ *Chin0*($m$), then $(\sigma\backslash) n,m$, so $\varphi [m\backslash n]$. And if *Chin0*($m$), then *Tail*($m$), so by *Lemma 5.08*, $\varphi [m\backslash n]$. Hence $\forall n \forall m$ ( N$n$ & $\sigma n,m$ & $\varphi \Rightarrow \varphi [m\backslash n]$ ).
Thus by Induction, $\forall n$ ( N$n \Rightarrow \varphi$ ). ▯

*Propositions 5.05*, *5.06*, *5.07*, and *5.09* show that **GA** proves (PA3\), (PA4\), (PA5\), and (PA6\), i.e. the Peano axioms reformulated in terms of (σ\). But (PA3) + (PA4) + (PA5) + (PA6) is just **FPA**, the system investigated in *Arithmetic without the Successor Axiom*. Now the definitions of addition and multiplication presented below will be slightly different than those for addition and multiplication in *Arithmetic*, but it is a straightfoward exercise to show that they are equivalent in the context of **FPA**. Hence, if **FPA** proves assertion *S* and *S** is the corresponding assertion in the system induced by (σ\), then **GA** proves *S**.

Since all propositions proved hitherto used only (PA3) and (PA6), they can all be transfered to true propositions involving σ\ instead of σ, changing of course definitions appropriately as well. A "\" will be put after the relevant proposition's number to indicate the altered proposition.

*Defs 5.10*. $Hered_{\sigma\backslash}(P)$ abbreviates

$$\forall a,b(\mathrm{N}a \ \& \ \mathrm{N}b \ \& \ Pa \ \& \ (\sigma\backslash)a,b \Rightarrow Pb)$$

$(\sigma\backslash)^{*=}x,y$ abbreviates

$$\mathrm{N}x \ \& \ \mathrm{N}y \ \& \ \forall P \ (Px \ \& \ Hered_{\sigma\backslash}(P) \Rightarrow Py)$$

$(\sigma\backslash)^{*}x,y$ abbreviates

$$(\sigma\backslash)^{*}x,y \ \& \ \neg \ x = y$$

$Max_{\sigma\backslash}(x)$ abbreviates

$$Nx \;\&\; \forall n(Nn \Rightarrow (\sigma\backslash)^{*=}n,x)$$

Similarly we will subscript other predicates, with the obvious meaning. □

*Prop 5.11.* $\forall x\forall y\ ((\sigma\backslash)^{*}x,y \Rightarrow \neg\ (\sigma\backslash)^{*}y,x)$.
*Pf:*

Let $(\sigma\backslash)^{*}x,y\ \&\ (\sigma\backslash)^{*}y,x$. Then $\neg\ x = y$. And by *Prop 3.14*, $Max_{\sigma\backslash}(x)\ \&\ Max_{\sigma\backslash}(y)$.

Suppose $Max_{\sigma\backslash}(0)$. Then either $x$ or $y$ is not equal to 0. So by *Corollary 1.03*, N$z$ & $(\sigma\backslash)0,z$, for some $z$. Hence by *Prop 3.28*, N$p$ & $(\sigma\backslash)p,0$ for some $p$, contradicting (PA5\), i.e. *Prop 5.07*.

Thus $\neg\ Max_{\sigma\backslash}(0)$. By *Corollary 3.26* $Chin_{\sigma\backslash}(c)$ for some $c$. By *Prop 3.39* $\neg\ k = p$ & N$k$ & N$p$ & $(\sigma\backslash)k,c\ \&\ (\sigma\backslash)p,c$, for some $k$ and $p$. But this contradicts (PA4\), i.e. *Prop 5.06*. □

*Corollary 5.12.* $\forall x\ \neg\ (\sigma\backslash)^{*}x,x$. □

*Corollary 5.13.* $\forall x\ \neg\ (\sigma\backslash)x,x$. □

*Prop 5.14.* $\forall x\forall y\ (Max_{\sigma\backslash}(x)\ \&\ Max_{\sigma\backslash}(y) \Rightarrow x = y\ \&\ \neg\ \exists z\ (Nz\ \&\ (\sigma\backslash)x,z))$.
*Pf:*

Let $Max_{\sigma\backslash}(x)\ \&\ Max_{\sigma\backslash}(y)$. Suppose $\neg\ x = y$. Then $(\sigma\backslash)^{*}x,y\ \&\ (\sigma\backslash)^{*}y,x$ by *Def 5.10*, contradicting *Prop 5.11*.

Thus $x = y$.

So suppose N$z$ & $(\sigma\backslash)x,z$, for some $z$. $\neg\ z = x$ by *Corollary 5.13*, so $(\sigma\backslash)^{*}x,z$. But $Max_{\sigma\backslash}(x)$, so $(\sigma\backslash)^{*}z,x$. But this contradicts *Prop 5.11*. □

So, if σ\ has a maximal element, then it has only one maximal element, which does not have a successor. Thus there are no pure cycles and no tadpoles under σ\. Thus:

*Prop 5.15.* $(\sigma\backslash)\backslash \equiv \sigma\backslash$
*Pf:*

Suppose $(\sigma\backslash)x,y\ \&\ chin0_{\sigma\backslash}(y)\ \&\ Max_{\sigma\backslash}(x)$. Then $Max_{\sigma\backslash}(y)$ by *Def 3.31*. So $x = y$ and $\neg\ \exists z$ ( N$z$ & $(\sigma\backslash)x,z)$, by *Prop 5.14*, contradicting $(\sigma\backslash)x,y$. Hence by *Def 5.01*, $(\sigma\backslash)\backslash \equiv \sigma\backslash$ □

*Prop 5.16.* $\forall n\forall x\ ((\sigma\backslash)^{*=}n,x \Rightarrow \sigma^{*=}n,x)$.
*Pf:*

By induction on σ\,, with $\phi$ as

$$\forall x\ ((\sigma\backslash)^{*=}n,x \Rightarrow \sigma^{*=}n,x).$$

Suppose $(\sigma\backslash)^{*=}0,x$. Then N$x$, so $\sigma^{*=}0,x$.

Now let N$n$ & (σ\)$n,m$ & $\phi$ & $(\sigma\backslash)^{*=}m,x$. By *Corollary 2.04(b)*\, $(\sigma\backslash)^{*=}n,x$. By the Induction Hypothesis, $\sigma^{*=}n,x$. σ$n,m$ by *Def 5.01* of σ\.

By *Corollary 5.13*, ¬ $n = m$.

Suppose $n = x$. Then ¬ $m = x$. So $(\sigma\backslash)^{*}m,x$ and $(\sigma\backslash)^{*}x,m$, the latter by *Prop 2.02*\. But this contradicts *Prop 5.11*.

Thus ¬ $n = x$. So $\sigma^{*}n,x$. By *Prop 2.10*, $\sigma^{*=}m,x$. ∎

*Corollary 5.17*. ∀$x$ ($Max_{\sigma\backslash}(x) \Rightarrow Max(x)$). ∎

*Corollary 5.18*. ∃$x$ $Max_{\sigma\backslash}(x)$ ⇔ ∃$x$ $Max(x)$.

*Pf:*

If $Max_{\sigma\backslash}(x)$, then $Max(x)$ by *Corollary 5.17*.

Now let $Max(x)$. If σ\ ≡ σ, then the result follows immediately. Otherwise, by *Prop 5.02*, σ\ ≡ σ \ {($p,y$)} for some $p,y$ where σ$p,y$ & $Max(p)$ & $chin0(y)$.

We claim that $Max_{\sigma\backslash}(p)$. For suppose N$z$ & (σ\)$p,z$ for some $z$. Then σ$p,z$, by *Def 5.01*. By (PA3) $z = y$. But this contradicts σ\ ≡ σ \ {($p,y$)}. So ¬ ∃$z$ (N$z$ & (σ\)$p,z$). By *Prop 3.07* $Max_{\sigma\backslash}(p)$. ∎

Other than the trivial (PA1\), (PA2\) is the sole axiom which may not hold in the system induced by (σ\), so it is worthwhile to consider exactly when it does and does not hold:

*Prop 5.19*. (PA2\) ⇔ ¬ (PA1) ∨ ( (PA1) & (PA2) & (PA4) & (PA5) ), i.e.
¬ (PA2\) ⇔ ¬ (PA2) ∨ ¬ (PA4) ∨ ¬ (PA5).

*Pf:*

First, note that by *Theorem 4.07*,

(PA2\) ⇔ ¬ (PA1) ∨ ( (PA1) & (PA2) & (PA4) & (PA5) )

and

¬ (PA2\) ⇔ ¬ (PA2) ∨ ¬ (PA4) ∨ ¬ (PA5)

are equivalent.

Suppose ¬ (PA1). Then (PA2\) vacuously.

Suppose (PA1) & (PA2) & (PA4) & (PA5). Then by *Prop 4.06*, ¬ ∃$x$ $Max(x)$. By *Prop 5.02*, σ\ ≡ σ, so (PA2\).

Suppose ¬ (PA2). Then ¬ ∃$p,c$ (σ$p,c$ & $Max(p)$ & $Chin0(c)$) by *Prop 4.10* and *Theorem 4.07*; so by *Prop 5.02*, σ\ ≡ σ, so ¬ (PA2\).

Suppose ¬ (PA4) ∨ ¬ (PA5). Then by *Prop 4.10*, σ$p,c$ & $Max(p)$ & $Chin0(c)$ for some $p,c$. We claim that $p$ does not have a σ\-successor. For ¬ (σ\)$p,c$ by *Def 5.01*. So if N$z$ & (σ\)$p,z$, then σ$p,z$ by *Prop 5.02*, forcing by (PA3) $z = c$, contradicting ¬ (σ\)$p,c$. Hence

$\neg$ (PA2\). ∎

So, in the cases of $\neg$ (PA2), $\neg$ (PA4), or $\neg$ (PA5), *Prop 4.02* applies, and there exists a unique $x$ such that all of the following hold:

(*i*) $\neg \exists z$ (N$z$ & ($\sigma$\)$x,z$)
(*ii*) $Max_{\sigma\backslash}(x)$
(*iii*) $\forall y(Max_{\sigma\backslash}(y) \Rightarrow y = x)$
(*iv*) $\forall y$(N$y$ & ($\sigma$\)$y,x \Rightarrow Neck_{\sigma\backslash}(y)$ & $Chin_{\sigma\backslash}(x)$)
(*v*) $x = 0 \vee (\neg\, x = 0$ & $Chin_{\sigma\backslash}(x))$.

## 6. Ordering

($\sigma$\) allows the definition of a well ordering on N.

*Defs 6.01*. $x \leq y$ if and only if ($\sigma$\)$^{*=}x,y$.
$x < y$ if and only if $x \leq y$ & $\neg\, x = y$. ∎

$\leq$ is the normal ordering. We state without proof:

*Prop 6.02*.

(*a*) $\forall n$ (N$n \Rightarrow 0 \leq n$)

(*b*) $\forall n$ ($n \leq 0 \Leftrightarrow$ N$n$ & $n = 0$)

(*c*) $\forall n$ (N$n \Rightarrow n \leq n$)

(*d*) $\forall n \forall m \forall k$ ($n \leq m$ & $m \leq k \Rightarrow n \leq k$)

(*e*) $\forall n \forall m \forall k$ ($n \leq m$ & $m \leq n \Rightarrow n = m$)

(*f*) $\forall n \forall m$ (($\sigma$\)$n,m \Rightarrow n < m$)

(*g*) $\forall n \forall m$ (($\sigma$\)$n,m \Rightarrow \neg\, m \leq n$)

(*h*) $\forall n \forall m$ (($\sigma$\)$n,m \Rightarrow \forall x(x < m \Leftrightarrow x \leq n)$)

(*i*) $\forall n \forall m$ ($n < m \Rightarrow \exists x$ (($\sigma$\)$n,x$ & $x \leq m$) & $\exists y$ (($\sigma$\)$y,m$ & $n \leq y$)) ∎

Also:

*Prop 6.03*. Let N$n$. Then $\{x : x \leq n\}$ and $\{x : x < n\}$ exist.
*Pf:*
By *Corollary 3.20*, $\{x : x \leq n\}$ exists. But then $\{x : x \leq n \ \& \ \neg\, x = n\}$ exists, by *Predicative Comprehension.* □

*Theorem 6.04*. Let $S \subseteq$ N, where $\neg\, S \equiv \phi$. Then $S$ has a minimum element (wrt to the ordering $\leq$). □

*Prop 6.05*. Suppose ¬ (PA2) ∨ ¬ (PA4) ∨ ¬ (PA5). Then there exists a unique $t$ such that $\forall n$ (N$n \Rightarrow n \leq t$).
*Pf:*
¬ (PA2\), by *Prop 5.19*. *Prop 4.02* assures us the existence of such a unique $t$. □

*Def 6.06*. If there exists $t$ such that N$t$ & $\forall n$ (N$n \Rightarrow n \leq t$), then we write *top*($t$) and say that $t$ is the *top*. □

By *Prop 6.02*(*e*) any top must be unique.

*Prop 6.07*. Suppose ¬ (PA4) ∨ ¬ (PA5), and let *top*($t$) & *chin0*($c$). Then $\sigma t,c$.
*Pf:*
By *Prop 4.02* ¬ ∃$z$ (N$z$ & (σ\)$t,z$). By *Prop 4.03* and *4.04*, (PA2), so N$v$ & $\sigma t,v$ for some $v$. *Prop 5.02* forces $v = c$. □

*Theorem 6.08*. (*Pigeon Hole Principle*) Let N$n$, $A \subseteq \{x : x \leq n\}$. Suppose for some $R$, *Is1-1*($R$) & *Dom*($R$) $\equiv \{x : x \leq n\}$ & *Im*($R$) $\equiv A$. Then $A \equiv \{x : x \leq n\}$.
*Pf:*
Use the standard induction proof. □

*Prop 6.09*. Suppose ¬ (PA2) ∨ ¬ (PA4) ∨ ¬ (PA5). Let $A \subseteq$ N, and suppose for some $R$, *Is1-1*($R$) & *Dom*($R$) ≡ N & *Im*($R$) ≡ $A$. Then $A$ ≡ N.
*Pf:*
Apply *Theorem 6.08*, *Prop 5.19*, and *Prop 4.02*. □

*Corollary 6.10*. Suppose ¬ (PA2) ∨ ¬ (PA4) ∨ ¬ (PA5). Let $A \subseteq$ N \ {0}, and suppose for some $R$, $Is1\text{-}1(R)$ & $Dom(R) \equiv$ N N \ {0} & $Im(R) \equiv A$. Then $A \equiv$ N \ {0}. ▯

Finally, note that, by *Prop 5.15*, (σ\) induces exactly the same ordering as ≤. That is, (≤\) ≡ ≤,, so we will always write ≤, even in the cases where (≤\) might be more appropriate.

## 7. Addition.

In this section it is shown that there exists an addition formula which satisfies (ADD0) through (ADD4), and it will be shown to have most of the standard properties.

*Def 7.01*. $+(a,b,c)$ if and only if N$a$ & N$b$ & N$c$ &

$$\exists R\ (\ R0,a\ \&\ Rb,c\ \&\ IsFunction(R)\ \&\ Dom(R) \equiv \{z : z \leq b\}$$
$$\&\ \forall u \forall u' \forall w \forall w'\ (\mathrm{N}w'\ \&\ (\sigma\backslash)u,u'\ \&\ \sigma w,w'\ \&\ u' \leq b\ \&\ Ru,w \Rightarrow Ru',w')$$
$$\&\ \forall u \forall u' \forall w \forall w'\ (\mathrm{N}w'\ \&\ (\sigma\backslash)u,u'\ \&\ Ru',w' \Rightarrow \exists w(\mathrm{N}w\ \&\ \sigma w,w'\ \&\ Ru,w)))$$

▯

*Example*. Suppose N ≡ {0,1,2,3,4}. Of the five cases of *Theorem 4.07*, this excludes *Case 1* (the vacuous model) and *Case 3* (the standard model). Let "U" stand for the addition being undefined.

*Case 2*. ¬ (PA2). The finite segments.

| **+** | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 |
| **1** | 1 | 2 | 3 | 4 | U |
| **2** | 2 | 3 | 4 | U | U |
| **3** | 3 | 4 | U | U | U |
| **4** | 4 | U | U | U | U |

*Case 4*. ¬ (PA4). The tadpoles.

(*i*) 0 - 1 - 2 - 3 - 4 - 1.

| **+** | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 |
| **1** | 1 | 2 | 3 | 4 | 1 |
| **2** | 2 | 3 | 4 | 1 | 2 |
| **3** | 3 | 4 | 1 | 2 | 3 |
| **4** | 4 | 1 | 2 | 3 | 4 |

(*ii*) 0 - 1 - 2 - 3 - 4 - 2.

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 |
| **1** | 1 | 2 | 3 | 4 | 2 |
| **2** | 2 | 3 | 4 | 2 | 3 |
| **3** | 3 | 4 | 2 | 3 | 4 |
| **4** | 4 | 2 | 3 | 4 | 2 |

(*iii*) 0 - 1 - 2 - 3 - 4 - 3.

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 |
| **1** | 1 | 2 | 3 | 4 | 3 |
| **2** | 2 | 3 | 4 | 3 | 4 |
| **3** | 3 | 4 | 3 | 4 | 3 |
| **4** | 4 | 3 | 4 | 3 | 4 |

(*iii*) 0 - 1 - 2 - 3 - 4 - 4.

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 |
| **1** | 1 | 2 | 3 | 4 | 4 |
| **2** | 2 | 3 | 4 | 4 | 4 |
| **3** | 3 | 4 | 4 | 4 | 4 |
| **4** | 4 | 4 | 4 | 4 | 4 |

*Case 5.* ¬ (PA5). The pure cycle.

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 |
| **1** | 1 | 2 | 3 | 4 | 0 |
| **2** | 2 | 3 | 4 | 0 | 1 |
| **3** | 3 | 4 | 0 | 1 | 2 |
| **4** | 4 | 0 | 1 | 2 | 3 |

Remark that these are the only addition tables which satisfy conditions (ADD0) through (ADD4). □

Recall the properites of addition introduced in Section 1.

(ADD0) $\forall k \forall n \forall m\ (\ +(k,n,m) \Rightarrow \mathrm{N}k\ \&\ \mathrm{N}n\ \&\ \mathrm{N}m\ )$
(ADD1) $\forall n\ (\ \mathrm{N}n \Rightarrow +(n,0,n)\ )$
(ADD2) $\forall n \forall m\ (\ +(n,0,m) \Rightarrow n = m\ )$
(ADD3) $\forall k \forall n \forall m \forall n' \forall m'\ (\ \mathrm{N}n'\ \&\ \mathrm{N}m'\ \&\ +(k,n,m)\ \&\ \sigma n,n'\ \&\ \sigma m,m' \Rightarrow +(k,n',m')\ )$
(ADD4) $\forall k \forall n \forall n' \forall m'\ (\ \mathrm{N}n\ \&\ +(k,n',m')\ \&\ \sigma n,n' \Rightarrow \exists m\ (\ \sigma m,m'\ \&\ +(k,n,m)\ )$

It will be shown that + as defined in 7.01 satisfies these five conditions. First, and evidently, (ADD0) holds.

*Prop 7.02*. Let $\mathrm{N}c\ \&\ (\sigma\backslash)n,m\ \&\ \sigma b,c\ \&\ +(a,n,b)$. Then $+(a,m,c)$.
*Pf:*
Since $+(a,n,b)$,

$$R0,a \ \& \ Rn,b \ \& \ IsFunction(R) \ \& \ Dom(R) \equiv \{z : z \leq b\}$$
$$\& \ \forall u \forall u' \forall w \forall w' \ (\mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ \sigma w,w' \ \& \ u' \leq b \ \& \ Ru,w \Rightarrow Ru',w')$$
$$\& \ \forall u \forall u' \forall w \forall w' \ (\mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ Ru',w' \ \Rightarrow \exists w(\mathrm{N}w \ \& \ \sigma w,w' \ \& \ Ru,w) \ )$$

for some $R$, by *Def 7.01*.

Define $S$ as $R \cup \{(m,c)\}$. $S0,a$ since $R0,a$. $Sm,c$ by definition. $\neg\, m \leq n$ by *Prop 6.02*($g$), so $\neg\, m \in Dom(R)$. Thus $IsFunction(S)$ follows from $IsFunction(R)$. Also

$$\begin{aligned} Dom(S) &\equiv Dom(R) \cup \{m\} \\ &\equiv \{z : z \leq n\} \cup \{m\} \\ &\equiv \{z : z \leq m\} \qquad \text{by } \textit{Prop 6.02}(h). \end{aligned}$$

Next, suppose $\mathrm{N}w \ \& \ \mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ \sigma w,w' \ \& \ u' \leq m \ \& \ Su,w$.

Let $u' = m$. Then by *Prop 5.06* (PA4\), $u = n$. Hence $Ru,w$. But $Rn,b$. Thus $w = b$, since $IsFunction(R)$. But then by (PA3), $c = w'$. So $Su',w$.

Otherwise, suppose $\neg\, u' = m$. Then by *Prop 6.02*($h$), $u' \leq n$. So $u \leq n$, hence $Ru,w$. Thus $Ru',w'$, hence $Su',w'$.

Finally, suppose $\mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ Su',w'$. If $\neg\, u' = m$, then $Ru',w'$. So by assumption, $\mathrm{N}w \ \& \ \sigma w,w' \ \& \ Ru,w$, for some $w$, hence $Su,w$.

On the other hand, suppose $u' = m$. Then $w' = c$ and, by (PA4\), $u = n$. $Sn,b$ since $Rn,b$; hence $\mathrm{N}b \ \& \ \sigma b,w' \ \& \ Sn,b$. $\square$

*Prop 7.03*. Let $(\sigma\backslash)n,m \ \& \ +(a,m,c)$. Then there exists $v$ s.t. $\sigma v,c \ \& \ +(a,n,v)$.

*Pf:*

Since $+(a,m,c)$,

$$R0,a \ \& \ Rm,c \ \& \ IsFunction(R) \ \& \ Dom(R) \equiv \{z : z \leq m\}$$
$$\& \ \forall u \forall u' \forall w \forall w' \ (\mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ \sigma w,w' \ \& \ u' \leq m \ \& \ Ru,w \Rightarrow Ru',w')$$
$$\& \ \forall u \forall u' \forall w \forall w' \ (\mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ Ru',w' \ \Rightarrow \exists w(\mathrm{N}w \ \& \ \sigma w,w' \ \& \ Ru,w) \ )$$

for some $R$, by *Def 7.01*. Now $\mathrm{N}c \ \& \ (\sigma\backslash)n,m \ \& \ Rm,c$, so $\mathrm{N}v \ \& \ \sigma v,c \ \& \ Rn,v$, for some $v$.

We claim that $+(a,n,v)$.

Set $S$ to $R \setminus \{(m,c)\}$.

$\neg\, m = 0$ by *Prop 5.07*. So $S0,a$.

$\neg\, n = m$ by *Corollary 5.13*. So $Sn,v$.

Obviously, $IsFunction(S)$.

Since $IsFunction(R)$, $\neg\, m \in Dom(S)$. Thus

$$\begin{aligned} Dom(S) &\equiv Dom(R) \setminus \{m\} \\ &\equiv \{z : z \leq m\} \setminus \{m\} \\ &\equiv \{z : z \leq n\} \qquad \text{by } \textit{Prop 6.02}(h). \end{aligned}$$

Next, let $\mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ \sigma w,w' \ \& \ u' \leq n \ \& \ Su,w$. So $u' \leq m \ \& \ Ru,w$. Hence, $Ru',w'$. $\neg\, u' = m$ since $u' \leq n$. Thus $Su',w'$.

Finally, let $\mathrm{N}w' \ \& \ (\sigma\backslash)u,u' \ \& \ Su',w'$. Then $Ru',w'$, so $\mathrm{N}w \ \& \ \sigma w,w' \ \& \ Ru,w$, for some $w$. But $u' \in Dom(S)$, so $u' \leq n$, and so $u \leq n$. But $n < m$, hence $\neg\, u = m$. And so $Su,w$.

But then $+(a,n,v)$. $\square$

*Prop 7.04*. (ADD2) Let $+(a,0,b)$. Then $a = b$.

*Pf:*

By *Def 7.01*, $R0,a$ & $R0,b$ & *IsFunction*($R$), for some $R$. Hence $a = b$. □

*Prop 7.05*. $\forall n \forall c\ (+(0,n,c) \Rightarrow n = c)$
*Pf:*

By induction via (σ\), with $\phi$ as

$$\forall c\ (+(0,n,c) \Rightarrow n = c).$$

Then $\phi$ holds when $n = 0$, by *Prop 7.04*.

Now assume N$n$ & (σ\)$n,m$ & $\phi$, i.e. $\forall c\ (+(0,n,c) \Rightarrow n = c)$. And suppose $+(0,m,c)$. By *Prop 7.03* there exists $v$ s.t. σ$v,c$ & $+(0,n,v)$. By the induction hypothesis, $n = v$. By (PA3) $m = c$. □

*Prop 7.06*. $\forall a\ (\mathrm{N}a \Rightarrow +(0,a,a))$
*Pf:*

Let N$a$. Note that N0 by *Prop 1.01*.

Set $R$ to $\{x,y : x \le a\ \&\ x = y\}$, which exists by *Prop 6.03* and *Predicative Comprehension*. Then $R0,0$ & $Ra,a$ & *IsFunction*($R$) & *Dom*($R$) ≡ $\{z : z \le a\}$.

Suppose N$w'$ & (σ\)$u,u'$ & σ$w,w'$ & $u' \le a$ & $Ru,w$. Then $u = w$. By (PA3) $u' = w'$. Thus $Ru',w'$.

Finally, suppose N$w'$ & (σ\)$u,u'$ & $Ru',w'$. Then $u' \le a$, so $u \le a$. Hence $Ru,u$. But σ$u,w'$ since $u' = w'$.

Thus $+(0,a,a)$. □

*Prop 7.07*. (ADD1) $\forall a\ (\mathrm{N}a \Rightarrow +(a,0,a))$
*Pf:*

Let N$a$. Note that N0 by *Prop 1.01*.

Set $R$ to $\{x,y : x = 0\ \&\ y = a\}$. Then $R0,a$ & $R0,a$ & *IsFunction*($R$) & *Dom*($R$) ≡ {0} ≡ $\{z : z \le 0\}$.

Suppose N$w'$ & (σ\)$u,u'$ & σ$w,w'$ & $u' \le 0$ & $Ru,w$. Then $u' = 0$, so by *Prop 4.07* (PA5\), ¬ (σ\)$u,u'$. Hence $Ru',w'$ follows vacuously.

Finally, suppose N$w'$ & (σ\)$u,u'$ & $Ru',w'$. Then again $u' = 0$, so ¬ (σ\)$u,u'$. Thus vacuously, ∃$w$(N$w$ & σ$w,w'$ & $Ru,w$).

Thus $+(a,0,a)$. □

It will be convenient to prove (ADD2) and (ADD3) with the help of commuativity of addition, so our first objective is to prove commutativity.

*Prop 7.08*. Let N$c'$ & (σ\)$a,a'$ & σ$c,c'$ & $+(a,n,c)$. Then $+(a',n,c')$.
*Pf:*

By induction via (σ\), with $\phi$ as

$$\forall a \forall a' \forall c \forall c'\ (\ \mathrm{N}c'\ \&\ (\sigma\backslash)a,a'\ \&\ \sigma c,c'\ \&\ +(a,n,c) \Rightarrow +(a',n,c')\ ).$$

Assume N$c'$ & (σ\)$a,a'$ & σ$c,c'$ & $+(a,0,c)$. Then $a = c$, by *Prop 7.04*. By (PA3)

$a' = c'$. By *Prop* 7.07 $+(a',0,c')$.

Now assume N$n$ & $(\sigma\backslash)n,m$ & $\phi$. And suppose N$c'$ & $(\sigma\backslash)a,a'$ & $\sigma c,c'$ & $+(a,m,c)$. Note by the definition of +, N$a$ & N$c$ & N$m$. By *Prop* 7.03, $\sigma v,c$ & $+(a,n,v)$, for some $v$. By the induction hypothesis $+(a',n,c)$. By *Prop* 7.02, $+(a',m,c')$. ∎

*Prop 7.09. (Commutative Law of Addition)* $\forall n \forall a \forall b\ (+(a,n,b) \Rightarrow +(n,a,b))$.
*Pf:*

By induction via $(\sigma\backslash)$, with $\phi$ as

$$\forall a \forall b\ (+(a,n,b) \Rightarrow +(n,a,b)).$$

Suppose $+(a,0,b)$. Then $a = b$ by *Prop* 7.04, so $+(0,a,b)$ by *Prop 7.06*.

Now assume N$n$ & $(\sigma\backslash)n,m$ & $\phi$. And suppose $+(a,m,b)$. Then by *Prop* 7.02 $+(a,n,c)$ & $\sigma c,b$, for some $c$. By the induction hypothesis, $+(n,a,c)$. By *Prop* 7.08 $+(m,a,b)$. ∎

It is now possible to prove (ADD3) and (ADD4).

*Prop 7.10*. (ADD3) $\forall n \forall a \forall b \forall a' \forall b'$ (N$a'$ & N$b'$ & $\sigma a,a'$ & $\sigma b,b'$ & $+(n,a,b) \Rightarrow +(n,a',b')$).
*Pf:*

By induction via $(\sigma\backslash)$, with $\phi$ as

$$\forall a \forall b \forall a' \forall b'\ (\text{N}a' \ \& \ \text{N}b' \ \& \ \sigma a,a' \ \& \ \sigma b,b' \ \& \ +(n,a,b) \Rightarrow +(n,a',b')).$$

Suppose N$a'$ & N$b'$ & $\sigma a,a'$ & $\sigma b,b'$ & $+(0,a,b)$. By *Prop 7.05, $a = b$*. By (PA3) $a' = b'$. By *Prop 7.06*, $+(0,a',b')$.

Now assume N$n$ & $(\sigma\backslash)n,m$ & $\phi$. Let N$a'$ & N$b'$ & $\sigma a,a'$ & $\sigma b,b'$ & $+(m,a,b)$. By *Commutativity* (*Prop 7.09*), $+(a,m,b)$. By *Prop 7.03*, $\sigma v,b$ & $+(a,n,v)$ for some $v$. By *Commutativity* again, $+(n,a,v)$. By the Induction Hypothesis, $+(n,a',b)$. By *Commutativity* again, $+(a',n,b)$. By *Prop 7.02*, $+(a',m,b')$. By a fourth use of *Commutativity*, $+(m,a',b')$. ∎

*Prop 7.11*. (ADD4) $\forall n \forall a \forall a' \forall b'$ ( N$a$ & $\sigma a,a'$ & $+(n,a',b') \Rightarrow \exists b\ (\sigma b,b'$ & $+(n,a,b))$ ).
*Pf:*

By induction via $(\sigma\backslash)$, with $\phi$ as

$$\forall a \forall a' \forall b'\ (\ \text{N}a \ \& \ \sigma a,a' \ \& \ +(n,a',b') \Rightarrow \exists b\ (\sigma b,b' \ \& \ +(n,a,b))\ ).$$

Suppose N$a$ & $\sigma a,a'$ & $+(0,a',b')$. Then $a' = b'$ by *Prop 7.05*. Set $b = a$. Then $+(0,a,b)$ by *Prop 7.06*.

Now assume N$n$ & $(\sigma\backslash)n,m$ & $\phi$. Let N$a$ & $\sigma a,a'$ & $+(m,a',b')$. Then $+(a',m,b')$ by *Commutativity* (*Prop 7.09*). By *Prop 7.03*, $\sigma b,b'$ & $+(a',n,b)$ for some $b$. By *Commutativity* again, $+(n,a',b)$. By the Induction Hypothesis, $\sigma c,b$ & $+(n,a,c)$ for some $c$. By *Commutativity,* $+(a,n,c)$. By *Prop 7.02*, $+(a,m,b)$. ∎

*Prop 7.12*. $\forall n \forall a \forall b \forall a' \forall b'$ ($\sigma a,a'$ $+(n,a,b)$ & $+(n,a',b') \Rightarrow \sigma b,b'$ ).
*Pf:*

By induction, with $\phi$ as

$$\forall a \forall b \forall a' \forall b' \, (\sigma a,a' \;\; +(n,a,b) \,\&\, +(n,a',b') \Rightarrow \sigma b,b' \,).$$

Let $\sigma a,a'$  $+(0,a,b)$ & $+(0,a',b')$. Then $a = b$ & $a' = b'$ by *Prop 7.05*, so $\sigma b,b'$.
Now assume N$n$ & $\sigma n,m$ & $\phi$. Let $\sigma a,a'$  $+(m,a,b)$ & $+(m,a',b')$. By *Commutativity* (*Prop 7.09*), $+(a,m,b)$ & $+(a',m,b')$. By *Prop 7.03*, $\sigma v,b$ & $+(a,n,v)$ & $\sigma v',b'$ & $+(a',n,v')$, for some $v$ and $v'$. By the Induction Hypothesis, $\sigma v,v'$. By (PA3), $b = v'$. Hence $\sigma b,b'$. □

As one would expect based on the functionality of σ, + is also functional.

*Prop 7.13*. Let $+(a,n,b)$ & $+(a,n,c)$. Then $b = c$.
*Pf:*
By induction via (σ\), with $\phi$ as

$$\forall a \forall b \forall c \, (+(a,n,b) \,\&\, +(a,n,c) \Rightarrow b = c).$$

Suppose $+(a,0,b)$ & $+(a,0,c)$. Then by *Prop* 7.04, $a = b$ and $a = c$. So $b = c$.
Now assume N$n$ & (σ\)$n,m$ & $\phi$. And suppose $+(a,m,b)$ & $+(a,m,c)$. Then there exists $v,u$ s.t. $\sigma v,b$ & $+(a,n,v)$ and N$u$ & $\sigma u,c$ & $+(a,n,u)$, by *Prop 7.03*. By the induction hypothesis $u = v$. By (PA3) $b = c$. □

*Def 7.14*. If $+(n,m,k)$, then use $(n{+}m)$ to refer to $k$. □

By stipulation, when $(n{+}m)$ appears in an atomic proposition, it may be infered that $(n{+}m)$ exists.

*Lemma 7.15*. $\forall n \forall a \forall b \, ( \, \exists x \, ((a + n) + b) = x \Rightarrow \exists x \, (a + b) = x \, )$.
*Pf:*
By induction, with $\phi$ as

$$\forall a \forall b \, ( \, \exists x \, ((a + n) + b) = x \Rightarrow \exists x \, (a + b) = x).$$

Suppose $((a + 0) + b) = x$. Then N$a$ by *Def 7.01*, so $(a + 0) = a$ by *Prop 7.07*.
Now assume N$n$ & $\sigma n,m$ & $\phi$. Suppose $((a + m) + b) = x$. Then $(a + m) = y$, for some $y$ where N$y$. By *Prop 7.03*, $\sigma z,y$ & $(a + n) = z$, for some $z$. By *Commutativity* (*Prop 7.09*), $(b + (a + m)) = x$. By *Prop 7.03*, $\sigma u,x$ & $(b + (a + n)) = u$. By *Commutativity*, $((a + n) + b) = u$. By the Induction Hypothesis, $\exists x \, (a + b) = x$. □

*Prop 7.16*. (*Associative Law of Addition*).
$\forall n \forall a \forall b \forall c \, ( \, ((a + b) + n) = c \Leftrightarrow (a + (b + n)) = c)$.
*Pf:*
First, we prove

$$\forall n \forall a \forall b \forall c \; ( \, ((a + b) + n) = c \Rightarrow (a + (b + n)) = c) \qquad (*).$$

by induction via (σ\), with $\phi$ as

$$\forall a \forall b \forall c \; ( \, ((a + b) + n) = c \Rightarrow (a + (b + n)) = c).$$

Let $((a + b) + 0) = c$. Then $(a + b) = c$ and $(b + 0) = b$, so $(a + (b + 0)) = c$.

Now assume N$n$ & (σ\)$n,m$ & $\phi$. And let $((a + b) + m) = c$. By *Prop 7.03*, $((a + b) + n) = v$ for some $v$ where σ$v,c$. By the Induction Hypothesis, $(a + (b + n)) = v$. By *Lemma 7.15*, $(b + m)$ exists. By *Prop 7.12*, σ$(b + n),(b + m)$. By *Prop 7.12* again, σ$(a + (b + n)),(a + (b + m))$, that is σ$v,(a + (b + m))$. By (PA3), $(a + (b + m)) = c$.

Hence (*).

To see that $\forall n \forall a \forall b \forall c \; ( \, (a + (b + n)) = c \Rightarrow ((a + b) + n) = c)$, use (*) and apply *Commutativity*. □

*Def 7.17*. Suppose ∃$j$ (N$j$ & σ0,$j$). Then, by (PA3), this $j$ is unique. Use "1" to refer to this number, should it exist. □

Remark that 1 may equal 0. If this is the case, N ≡ {0}.

*Prop 7.18*. ∀$n$∀$m$ (N$n$ & N$m$ & σ$n,m$ ⇔ $(n + 1) = m$).
*Remark:* The proposition holds even in the case when $0 = 1$.
*Pf:*

Let N$n$ & N$m$ & σ$n,m$. By *Prop 1.05* 1 exists. By *Prop 7.07* $(n + 0) = n$. By *Prop 7.10*, $(n + 1) = m$.

Now let $(n + 1) = m$. Then N$n$ & N$m$ by *Def 7.01*. By *Prop 7.04*, $(n + 0) = n$. By *Prop 7.12*, σ$n,m$. □

*Corollary 7.19*. Suppose ¬ (PA4) ∨ ¬ (PA5), and let *top*($t$) & *chin0*($c$). Then $t + 1 = c$. In the case of ¬ (PA5), $t + 1 = 0$.
*Pf:*

By *Prop 6.07*, σ$t,c$. By *Defs 6.06* and *3.31*, N$t$ & N$c$. Hence by *Prop 7.18*, $t + 1 = c$. In the case of ¬ (PA5), $c = 0$ by *Prop 4.04*. □

As we have just seen, it is possible that non-zero numbers sum to 0, in the case when ¬ (PA5). The following proposition offers a characterization of sums to 0.

*Prop 7.20*.

(*a*) ¬ (PA1) ∨ (PA5) ∨ (TRV2) ∨ (TRV5) ⇔ $\forall x \forall y \; ( \, (x + y) = 0 \Rightarrow x = 0 \;\&\; y = 0 \, )$.

(*b*) ¬ (PA1) ∨ ¬ (PA5) ∨ (TRV2) ∨ (TRV5) ⇔ ∀$n$ ( N$n$ ⇒ ∃$c$ $(n + c) = 0$ ).

*Pf:*

(*a*) If ¬ (PA1) ∨ (TRV2) ∨ (TRV5), then trivially $\forall x \forall y$ ( $(x + y) = 0 \Rightarrow x = 0$ & $y = 0$ ).

Assume (PA5) and let $(x + y) = 0$. Suppose ¬ $x = 0$ ∨ ¬ $y = 0$. WLOG suppose ¬ $x = 0$. Then by *Prop 1.04*, N*k* & σ*k*,*x* for some *k*. By *Prop 7.18* $(k + 1) = x$. Substituting and manipulating, $((k + y) + 1) = 0$, hence by *Prop 7.18*, ¬ (PA5), a contradiction. Therefore $x = 0$ & $y = 0$.

Now suppose $\forall x \forall y$ ( $(x + y) = 0 \Rightarrow x = 0$ & $y = 0$ ) & (PA1) & ¬ (TRV2) & ¬ (TRV5). By the latter three conjuncts, N0 & σ0,*u* for some *u* where ¬ $u = 0$. That is, 1 exists and ¬ $1 = 0$. We need to prove (PA5). So suppose N*n* & σ*n*,0. By *Prop 7.18*, $(n + 1) = 0$. By the first conjunct, $1 = 0$, a contradiction.

(*b*) If ¬ (PA1) ∨ (TRV2) ∨ (TRV5), then trivially $\forall n$ ( N*n* ⇒ $\exists c\ (n + c) = 0$ ).

Suppose ¬ (PA5). Then N*k* & σ*k*,0 for some *k*. By *Prop 7.18*, $(k + 1) = 0$. Proceed by induction (PA6'), with $\phi$ as $\exists c\ (n + c) = 0$.

N0 ⇒ $\phi$ holds when $n = 0$ since $(0 + 0) = 0$ by *Prop 7.06*.

Now assume N*n* & N*m* & σ*n*,*m* & $\phi$. $(n + c) = 0$ by the induction hypothesis. If $c = 0$, then $n = 0$, by *Prop 7.04*, so $m = 1$ by (PA3); but then $(m + k) = 0$. Otherwise, suppose ¬ $c = 0$. By *Prop 1.04*, N*w* & σ*w*,*c* for some *w*. Thus by *Prop 7.04*, $(w + 1) = c$. So $(n + (w + 1)) = 0$. Re-organizing, $(m + w) = 0$.

Finally, suppose (PA1) & (PA5) & ¬ (TRV2) & ¬ (TRV5). By the first, third, and fourth conjuncts, 1 exists and ¬ $1 = 0$. By (*a*) ¬ $\exists c\ (1 + c) = 0$. □

*Def 7.21*. (+-TOT) Suppose $\forall n \forall m$ (N*n* & N*m* ⇒ $\exists k\ (n + m) = k$). Then we say + is *total.*

*Prop 7.22*. (PA2) ∨ (TRV2) ⇔ (+-TOT).
*Pf*:

Suppose ¬ (PA2) & ¬ (TRV2). For some *n*, N*n* & ¬ ∃*m* (N*m* & σ*n*,*m*). By *Prop 1.01*, N0. By ¬ (TRV2), ∃*x* (N*x* & σ0,*x*). Hence 1 exists. But then by *Prop 7.18*, $(n + 1)$ does not exist, whence ¬ (+-TOT).

Suppose (TRV2). Then N ≡ {0} by *Prop 4.09*(*b*). But $(0 + 0) = 0$ by *Prop 7.06*. So + is total.

Finally suppose (PA2). Proceed by induction, with $\phi$ as

$$\forall x\ (\mathrm{N}x \Rightarrow \exists k\ (x + n) = k).$$

Suppose N*x*. Then N0, and $(x + 0) = x$ by (ADD1), *Prop 7.07*. Hence $\phi$ [0*n*].

Now assume N*n* & σ*n*,*m* & $\phi$. And let N*x*. Then $(x + n) = k$, for some *k*, by the Induction Hypothesis. By (PA2), σ*k*,*z*, for some *z* with N*z*. By (ADD3) aka *Prop 7.10*, $(x + m) = z$. □

*Corollary 7.23*. Suppose ¬ (PA1) ∨ ¬ (PA4) ∨ ¬ (PA5). Then (+-TOT).
*Pf:*

By *Props 4.01*, *4.03*, and *4.04*, ¬ (PA1) ∨ ¬ (PA4) ∨ ¬ (PA5) implies (PA2). Now use *Prop 7.22*. □

An important property which + may not have, is Cancellation. Additive cancellation has a right and a left version:

(+-RCANC) $\forall a \forall b \forall c \forall d$ ( $+(a,b,c)$ & $+(a,d,c) \Rightarrow b = d$)
(+-LCANC) $\forall a \forall b \forall c \forall d$ ( $+(a,b,c)$ & $+(d,b,c) \Rightarrow a = d$)

Because additive commutativity holds in **GA**, these two are equivalent in **GA**, and indeed both are equivalent to:

(+-CANC) $\forall a \forall b \forall c$ ( $(a + b) = (a + c) \Rightarrow b = c$).

Additive cancellation does not necessarily hold in **GA**, because of the maximal elements in the tadpole model. Now the tadpole models can be excluded by assuming that the successor relationship is an injection, i.e. by assuming (PA4). In fact:

*Prop 7.24*. (+-CANC) ⇔ (PA4).
*Pf*:
Suppose (PA4). We proceed by induction, with $\phi$ as

$\forall b \forall c$ ( $(n + b) = (n + c) \Rightarrow b = c$).

For, assume $(0 + b) = (0 + c)$. Then $b = c$ by two uses of *Prop 7.05*.
Now assume N$n$ & $\sigma n,m$ & $\phi$. Let $(m + b) = (m + c)$. By (ADD4) and *Commutativity of Addition* (*Prop 7.09*), $\sigma k_1,c$ & $(n + b) = k_1$ & $\sigma k_2,c$ & $(n + b) = k_2$, for some $k_1,k_2$. By (PA4), $k_1 = k_2$. By the induction hypothesis, $b = d$.
For the other direction, suppose ¬ (PA4). Then N$n$ & N$m$ & N$n'$ & $\sigma n,m$ & $\sigma n',m$ & ¬ $n = n'$, for some $n,n',m$. By *Prop 7.18*, $(n + 1) = m$ and $(n' + 1) = m$. Hence ¬ (+-CANC).
∎

σ\ induces an addition. That is, we may define $(+\backslash)(a,b,c)$ if and only if N$a$ & N$b$ & N$c$ &

$\exists R$ ( $R0,a$ & $Rb,c$ & *IsFunction*($R$) & *Dom*($R$) ≡ $\{z : z \leq b\}$
& $\forall u \forall u' \forall w \forall w'$ (N$w'$ & $(\sigma\backslash)u,u'$ & $(\sigma\backslash)w,w'$ & $u' \leq b$ & $Ru,w \Rightarrow Ru',w'$)
& $\forall u \forall u' \forall w \forall w'$ (N$w'$ & $(\sigma\backslash)u,u'$ & $Ru',w'$ $\Rightarrow \exists w$(N$w$ & $(\sigma\backslash)w,w'$ & $Ru,w$))).

Remark that we should have written $((\sigma\backslash)\backslash)u,u'$ instead of $(\sigma\backslash)u,u'$ but didn't because of *Prop 5.15*. A similar comment applies to ≤.

*Example*. Suppose N ≡ {0,1,2,3,4}. Then:

| + \ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | U |
| 2 | 2 | 3 | 4 | U | U |
| 3 | 3 | 4 | U | U | U |
| 4 | 4 | U | U | U | U |

∎

+\ has all the properties of +, since σ\ has all the properties of σ. As before, we will put a "\" after the proposition number when we are citing a proposition for +\ instead of +.

+ coincides with +\, the addition induced by σ\, when the latter is defined:

*Prop 7.25.* ∀*j*∀*k*∀*n* ( (*j* + *n*) = *k* & ∃*i* (*j* (+\) *n*) = *i* ⇔ (*j* (+\) *n*) = *k*).

*Pf:*

Proceed by induction via (σ\), with $\phi$ as

∀*j*∀*k* ( (*j* + *n*) = *k* & ∃*i* (*j* (+\) *n*) = *i* ⇔ (*j* (+\) *n*) = *k*).

Suppose (*j* + 0) = *k*. Then *k* = *j* by *Prop 7.04*. N*j* by *Def 7.01*. So (*j* (+\) 0) = *k* by *Prop 7.07*.

Now suppose (*j* (+\) 0) = *k*. Then *k* = *j* and N*j*, by *Prop 7.04*. So (*j* + 0) = *k*, by *Prop 7.07*. Obviously, ∃*i* (*j* (+\) 0) = *i*).

Hence the base case holds.

Now assume N*n* & (σ\)*n*,*m* & $\phi$. Suppose (*j* + *m*) = *k* & (*j* (+\) *m*) = *i*, for some *i*. Then (*j* (+\) *n*) = *v*, for some *v* with (σ\)*v*,*i*, by *Prop 7.03*. By the induction hypothesis, (*j* + *n*) = *v*. By *Prop 7.02*, (*j* + *m*) = *i*. Thus *i* = *k*.

For the other direction, suppose (*j* (+\) *m*) = *k*. Then (*j* (+\) *n*) = *v*, for some *v* where (σ\)*v*,*k*, by *Prop 7.03*. By the induction hypothesis, (*j* + *n*) = *v*. By *Prop 7.02*, (*j* + *m*) = *k*. Obviously, ∃*i* (*j* (+\) *m*) = *i*. ▯

Although + and +\ cannot be assumed to be relationships, and are officially symbols within an abbreviation, we will abuse the "⊆" and "≡" notation in the obvious way because it allows for a more concise and clearer expression of the proposition we would like to assert. For instance, officially the next corollary asserts, "∀*j*∀*k*∀*n* ( (*j* (+\) *n*) = *k* ⇒ (*j* + *n*) = *k*)":

*Corollary 7.26.* +\ ⊆ +. ▯

*Prop 7.27.* +\ ≡ + ⇔ σ\ ≡ σ ∨ (TRV5).

*Pf:*

(⇒) Suppose +\ ≡ + & ¬ (TRV5). By *Prop 5.02*, it suffices to show that ¬ ∃*x*∃*y* such that σ*x*,*y* & *Max*(*x*) & *chin0*(*y*). For, suppose to the contrary, that there are such *x*,*y*. Then N*x* & N*y*, by *Defs 3.01* and *3.21*. So N0 by *Prop 1.01*. If ¬ ∃*j* (N*j* & σ0,*j*), then N ≡ {0} by *Prop 1.02*; but σ*x*,*y*, so σ0,0, a contradiction. Hence ∃*j* (N*j* & σ0,*j*), i.e. 1 exists. By *Prop 7.18*, (*x* + 1) = *y*. By assumption (*x* (+\) 1) = *y*. Suppose N*u* & (σ\)0,*u* for some *u*. Then *u* = 1 by *Prop 5.02* and (PA3). And thus (*x* (+\) *u*) = *y*. So by *Prop 7.18*, (σ\)*x*,*y*. But this contradicts the *Def 5.01*. So there does not exists any *u* such that N*u* & (σ\)0,*u*. By *Prop 1.02*, N ≡ {0}, which forces (TRV5), a final contradiction.

(⇐) By *Corollary 7.26*, it suffices to show ∀*j*∀*k*∀*n* ( (*j* + *n*) = *k* ⇒ (*j* (+\) *n*) = *k*).

Suppose σ\ ≡ σ. Proceed by induction, with $\phi$ as

$\forall j \forall k$ ( $(j + n) = k \Rightarrow (j$ (+\) $n) = k$).

Suppose $(j + 0) = k$. Then $j = k$ by *Prop 7.04*. N$j$ by *Def 7.01*. So ($j$ (+\) 0) = $k$ by *Prop 7.07*.

Now assume N$n$ & (σ\)$n$,$m$ & $\phi$. Suppose $(j + m) = k$. Then there exists $v$ s.t. σ$v$,$k$ & $(j + n) = v$, by *Prop 7.03*. By assumption (σ\)$v$,$k$. By induction, ($j$ (+\) $n$) = $v$. By *Prop* 7.10\ ($j$ (+\) $m$) = $k$.

Suppose (TRV5). Then N ≡ {0} by *Prop 4.09(f)*. So, suppose $(j + n) = k$. Then by *Def 7.01*, N$j$ & N$n$ & N$k$, so $j = n = k = 0$. By *Prop 7.02*, ($j$ (+\) $n$) = $k$. □

*Corollary 7.28.*
+\ ≡ + ⇔ ¬ (PA1) ∨ ¬ (PA2) ∨ ( (PA1) & (PA2) & (PA4) & (PA5) ) ∨ (TRV5)
*Pf:*

Apply *Prop 5.03* to *Prop 7.27*. □

*Corollary 7.29.* (+\)\ ≡ +\
*Pf:*

(PA4\) by *Prop 5.06* and (PA5\) by *Prop 5.07*.
So by *Theorem 4.07*, ¬ (PA1\) ∨ ¬ (PA2\) ∨ ( (PA1\) & (PA2\) & (PA4\) & (PA5\) ).
Now use *Corollary 7.28*. □

*Prop 7.30.* $\forall n \forall x$ ( $x \le n \Leftrightarrow \exists c$ ($x$ (+\) $c$) = $n$ )
*Pf:*

Proceed by induction via (σ\), with $\phi$ as

$\forall x$ ( $x \le n \Leftrightarrow \exists c$ ($x$ (+\) $c$) = $n$ ).

$x \le 0 \Leftrightarrow x = 0$ & N0 by *Prop 6.02*. N0 ⇒ 0 (+\) 0 = 0 by *Prop* 7.07. Now suppose $x$ (+\) $c$ = 0. By *Prop 7.20(a)*, $x = 0$, which implies $x \le 0$ by *Prop 6.02(b)*.

Now assume N$n$ & (σ\)$n$,$m$ & $\phi$.

Suppose $x \le m$, i.e. (σ\)$^{*=}x,m$. By *Prop 2.06*, (σ\)$^{*=}x,n$ ∨ $x = m$. If $x = m$, then set $c = 0$ and use *Prop 7.07*. If ¬ $x = m$, then (σ\)$^{*=}x,n$, so $x \le n$. By the induction hypothesis, $x$ (+\) $d = n$ for some $d$. By the induction hypothesis, $d \le n$, hence $d < m$ by *Props 6.02(d)* and *6.02(e)*. By *Prop 6.02(i)*, (σ\)$d$,$e$ & N$e$ for some $e$. By *Prop 7.10* (ADD3\), $x$ (+\) $e = m$.

In the other direction, suppose ($x$ (+\) $c$) = $m$. If $c = 0$, then $x = m$ by *Prop 7.04* (ADD2\). Otherwise, ¬ c = 0, so by *Prop 1.04*, (σ\)$p$,$c$ & N$p$ for some $p$. Now $x$ (+\) $p = u$ & (σ\)$u$,$m$, for some $u$ where by *Prop 7.11* (ADD4\). By (PA4\) *aka Prop 5.06*, $u = n$. Thus $x \le n$ by the induction hypothesis. So $x \le m$ by *Props 6.02(f)* and 6.02($d$). □

*Corollary 7.31.* Suppose $x \le n$. Then $\exists c$ $(x + c) = n$. □

*Corollary 7.32.* $\forall x \forall y$ ( N$x$ & N$y \Rightarrow x \le$ ($x$ (+\) $y$) ) □

In the case of ¬ (PA5) the premise in *Corollary 7.31* can be dropped.

*Prop 7.33*. Suppose ¬ (PA5). Then ∀*x*∀*y*(N*x* & N*y* ⇒ ∃*c* (*x* + *c*) = *y*).
*Pf:*
Let N*x* & N*y*. By *Prop 7.20*(*b*) (*x* + *z*) = 0 for some *z*. By *Prop 7.06* (0 + *y*) = *y*. Substituting, and then by *Associativity*, (*x* + (*z* + *y*)) = *y*. ▯

*Prop 7.34*. Suppose *a* ≤ *b* and (*b* + *c*) exists. Then (*a* + *c*) exists.
*Pf:*
*a* (+\) *e* = *b*, for some *e*, by *Prop 7.30*. *a* + *e* = *b* by *Prop 7.25*. So ((*a* + *e*) + *c*) exists. Apply *Commutativity* and *Associativity* to see that ((*a* + *c*) + *e*) exists. Thus clearly *a* + *c* exists. ▯

## 8. Multiplication.

*Def 8.01*. *(*a*,*b*,*c*) if and only if N*a* & N*b* & N*c* &

∃*R* ( *R*0,0 & *Rb*,*c* & *IsFunction*(*R*) & *Dom*(*R*) ≡ {*z* : *z* ≤ *b*}
& ∀*u*∀*u*'∀*w*∀*w*' ((σ\)*u*,*u*' & *u*' ≤ *b* & *Ru*,*w* & (*a* + *w*) = *w*' ⇒ *Ru*',*w*')
& ∀*u*∀*u*'∀*w*∀*w*' ( N*w*' & (σ\)*u*,*u*' & *Ru*',*w*'
⇒ ∃*w*((*a* + *w*) = *w*' & *Ru*,*w*) ) ) ▯

*Example*. Suppose N ≡ {0,1,2,3,4}. Again, of the five cases of *Theorem 4.07*, this excludes *Case 1* (the vacuous model) and *Case 3* (the standard model). Let "U" stand for the multiplication being undefined.

*Case 2*. ¬ (PA2). The finite segments.

| * | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | U | U |
| 3 | 0 | 3 | U | U | U |
| 4 | 0 | 4 | U | U | U |

*Case 4*. ¬ (PA4). The tadpoles.

(*i*) 0 - 1 - 2 - 3 - 4 - 1.

| * | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| **1** | 0 | 1 | 2 | 3 | 4 |
| **2** | 0 | 2 | 4 | 2 | 4 |
| **3** | 0 | 3 | 2 | 1 | 4 |
| **4** | 0 | 4 | 4 | 4 | 4 |

(*ii*) 0 - 1 - 2 - 3 - 4 - 2.

| * | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 |
| **2** | 0 | 2 | 4 | 3 | 2 |
| **3** | 0 | 3 | 3 | 3 | 3 |
| **4** | 0 | 4 | 2 | 3 | 4 |

(*iii*) 0 - 1 - 2 - 3 - 4 - 3.

| * | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 |
| **2** | 0 | 2 | 4 | 4 | 4 |
| **3** | 0 | 3 | 4 | 3 | 4 |
| **4** | 0 | 4 | 4 | 4 | 4 |

(*iii*) 0 - 1 - 2 - 3 - 4 - 4.

| * | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 |
| **2** | 0 | 2 | 4 | 4 | 4 |
| **3** | 0 | 3 | 4 | 4 | 4 |
| **4** | 0 | 4 | 4 | 4 | 4 |

*Case 5*. ¬ (PA5). The pure cycle.

| * | **0** | **1** | **2** | **3** | **4** |
|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 |
| **2** | 0 | 2 | 4 | 1 | 3 |
| **3** | 0 | 3 | 1 | 4 | 2 |
| **4** | 0 | 4 | 3 | 2 | 1 |

□

As with addition, consider the following properties of multiplication:

(MULT0) $\forall k \forall n \forall m\ (\ *(k,n,m) \Rightarrow \mathrm{N}k\ \&\ \mathrm{N}n\ \&\ \mathrm{N}m\ )$
(MULT1) $\forall n\ (\ \mathrm{N}n \Rightarrow *(n,0,0)\ )$
(MULT2) $\forall n \forall m\ (\ *(n,0,m) \Rightarrow m = 0\ )$
(MULT3) $\forall k \forall n \forall m \forall n' \forall m'\ (\ \mathrm{N}n'\ \&\ \mathrm{N}m'\ \&\ *(k,n,m)\ \&\ \sigma n,n'\ \&\ +(m,k,m') \Rightarrow *(k,n',m')\ )$
(MULT4) $\forall k \forall n \forall n' \forall m'\ (\ \mathrm{N}n\ \&\ *(k,n',m')\ \&\ \sigma n,n' \Rightarrow \exists m\ (+(m,k,m')\ \&\ *(k,n,m))\ )$

We willl show that our * satisfies these properties. Obviously (MULT0) holds by *Def 8.01*.

*Prop 8.02*. Let $(\sigma\backslash)n,m$ & $(a + b) = c$ & $*(a,n,b)$. Then $*(a,m,c)$.
*Pf:*

Since $*(a,n,b)$,

$$R0,0 \ \&\ Rb,c \ \&\ IsFunction(R) \ \&\ Dom(R) \equiv \{z : z \le b\}$$
$$\&\ \forall u \forall u' \forall w \forall w' \ ((\sigma\backslash)u,u' \ \&\ u' \le b \ \&\ Ru,w \ \&\ (a + w) = w' \ \Rightarrow Ru',w')$$
$$\&\ \forall u \forall u' \forall w \forall w' \ (\mathrm{N}w' \ \&\ (\sigma\backslash)u,u' \ \&\ Ru',w' \ \Rightarrow \exists w((a + w) = w' \ \&\ Ru,w))$$

for some $R$.

Now use $R \cup \{(m,c)\}$ to prove $*(a,m,b)$ ▯

*Prop 8.03*. Let $(\sigma\backslash)n,m$ & $*(a,m,c)$. Then there exists $v$ s.t. $(a + v) = c$ & $*(a,n,v)$.
*Pf:*

Since $*(a,m,c)$,

$$R0,0 \ \&\ Rb,c \ \&\ IsFunction(R) \ \&\ Dom(R) \equiv \{z : z \le m\}$$
$$\&\ \forall u \forall u' \forall w \forall w' \ ((\sigma\backslash)u,u' \ \&\ u' \le m \ \&\ Ru,w \ \&\ (a + w) = w' \ \Rightarrow Ru',w')$$
$$\&\ \forall u \forall u' \forall w \forall w' \ (\mathrm{N}w' \ \&\ (\sigma\backslash)u,u' \ \&\ Ru',w' \ \Rightarrow \exists w((a + w) = w' \ \&\ Ru,w))$$

for some $R$. Now $(\sigma\backslash)n,m$ & $Rm,c$, so $(a + v) = c$ & $Rn,v$, for some $v$. Use $R \setminus \{(m,c)\}$ to prove that $*(a,n,\mathrm{v})$. ▯

*Prop 8.04* (MULT2). Let $*(a,0,b)$. Then $b = 0$.
*Pf:*

By *Def 8.01*, $R0,0$ & $R0,b$ & $IsFunction(R)$, for some $R$. Hence $b = 0$. ▯

*Prop 8.05*. $\forall n \forall c \ (*(0,n,c) \Rightarrow c = 0)$
*Pf:*

By induction via $(\sigma\backslash)$, with $\phi$ as

$$\forall c \ (*(0,n,c) \Rightarrow c = 0).$$

Then $\phi$ holds when $n = 0$, by *Prop* 8.*04*.

Now assume N$n$ & $(\sigma\backslash)n,m$ & $\phi$, i.e. $\forall c \ (*(0,n,c) \Rightarrow c = 0)$. And suppose $*(0,m,c)$. By *Prop* 8.*03* there exists $v$ s.t. $(0 + v) = c$ & $*(0,n,v)$. By the induction hypothesis, $v = 0$. By *Prop 7.05* , $c = 0$. ▯

*Prop 8.06*. $\forall a \ (\mathrm{N}a \Rightarrow *(0,a,0))$
*Pf:*

Let N$a$. Note that N0 by *Prop 1.01*.

Set $R$ to $\{x,y : x \le a \ \&\ y = 0\}$. Then $R0,0$ & $Ra,0$ & $IsFunction(R)$ & $Dom(R) \equiv \{z : z \le a\}$.

Suppose $(\sigma\backslash)u,u'$ & $u' \le a$ & $Ru,w$ & $(0 + w) = w'$. Then $w = 0$. By *Prop 7.05*, $w' = 0$. Thus $Ru',w'$.

Finally, suppose N$w'$ & $(\sigma\backslash)u,u'$ & $Ru',w'$. Then $w' = 0$. Also, $u' \le a$, so $u \le a$. Hence $(0 + 0) = 0$ & $Ru,0$.

Thus $*(0,a,0)$. ▯

*Prop 8.07* (MULT1). $\forall a$ (N$a \Rightarrow *(a,0,0)$)
*Pf:*

Let N$a$. Then N0 by *Prop 1.01*.

Set $R$ to $\{x,y : x = 0 \;\&\; y = 0\}$. Then $R0,0$ & $R0,0$ & *IsFunction*($R$) & *Dom*($R$) $\equiv \{z : z \leq 0\}$.

Suppose ($\sigma\backslash$)$u,u'$ & $u' \leq 0$ & $Ru,w$ & $(a + w) = w'$. By *Prop 6.02*(*b*), $u' = 0$. But by (PA5\) $\neg\, u' = 0$. Thus vacuously, $Ru',w'$.

Finally, suppose N$w'$ & ($\sigma\backslash$)$u,u'$ & $Ru',w'$. Then $u' = 0$. But by (PA5\) $\neg\, u' = 0$. Thus vacuously, $\exists w((a + w) = w' \;\&\; Ru,w)$.

Thus $*(a,0,0)$. ∎

*Prop 8.08*. $\forall a$ (N$a \Rightarrow *(a,0,a)$)
*Pf:*

Let N$a$. Note that N0 by *Prop 1.01*.

Set $R$ to $\{x,y : x = 0 \;\&\; y = a\}$. Then $R0,a$ & $R0,a$ & *IsFunction*($R$) & *Dom*($R$) $\equiv \{0\} \equiv \{z : z \leq 0\}$.

Suppose N$w'$ & ($\sigma\backslash$)$u,u'$ & $\sigma w,w'$ & $u' \leq 0$ & $Ru,w$. Then $u' = 0$ by *Prop 6.02*(*b*), so by *Prop 5.07* (PA5\), $\neg$ ($\sigma\backslash$)$u,u'$. Hence $Ru',w'$ follows vacuously.

Finally, suppose N$w'$ & ($\sigma\backslash$)$u,u'$ & $Ru',w'$. Then again $u' = 0$, so $\neg$ ($\sigma\backslash$)$u,u'$. Thus vacuously, $\exists w$(N$w$ & $\sigma w,w'$ & $Ru,w$).

Thus $+(a,0,a)$. ∎

*Prop 8.09*. Let ($\sigma\backslash$)$a,a'$ & $(n + c) = c'$ & $*(a,n,c)$. Then $*(a',n,c')$.
*Pf:*

By induction via ($\sigma\backslash$), with $\phi$ as

$$\forall a \forall a' \forall c \forall c' \,(\,(\sigma\backslash)a,a' \;\&\; (n + c) = c' \;\&\; *(a,n,c) \Rightarrow *(a',n,c')\,).$$

Assume ($\sigma\backslash$)$a,a'$ & $(0 + c) = c'$ & $*(a,0,c)$. Then $c = 0$, by *Prop 8.04*. By *Prop 7.04*, $c = c'$, so $c' = 0$. By *Prop* 8.07 $*(a',0,c')$.

Now assume N$n$ & ($\sigma\backslash$)$n,m$ & $\phi$. And suppose ($\sigma\backslash$)$a,a'$ & $(m + c) = c'$ & $*(a,m,c)$. By *Prop* 8.03, $(a + v) = c$ & $*(a,n,v)$, for some $v$. By *Prop 7.18*, $m = (n + 1)$ and $a' = (a + 1)$. Thus,

| | |
|---|---|
| $c' = (m + c)$ | |
| $= ((n + 1) + (a + v))$ | by Substitution |
| $= ((a + 1) + (n + v))$ | by *Associativity* and *Commutativity of Addition* |
| $= (a' + (n + v))$ | by Substitution |

Now $*(a',n,(n + v))$ by the Induction Hypothesis. So by *Prop 8.02*, $*(a',m,c')$. ∎

*Prop 8.10*. *(Commutative Law of Multiplication)* $\forall n \forall a \forall b\, (*(a,n,b) \Rightarrow *(n,a,b))$.
*Pf:*

By induction via ($\sigma\backslash$), with $\phi$ as

$$\forall a \forall b\, (*(a,n,b) \Rightarrow *(n,a,b)).$$

Let $*(a,0,b)$. By *Prop 8.04*, $b = 0$. By 8.06, $*(0,a,0)$.

Now assume N$n$ & $(\sigma\backslash)n,m$ & $\phi$. And suppose $*(a,m,b)$. Then $(v + n) = b$ & $*(a,n,v)$, for some $v$. By the Induction Hypothesis, $*(n,a,v)$. By *Prop 8.09*, $(m,a,b)$. ▯

As with addition, it is now possible to prove small but important generalizations of *Props 8.02* and 8.*03*.

*Prop 8.11* (MULT3). $\forall n \forall a \forall b \forall a' \forall b'$ (N$a'$ & $\sigma a,a'$ & $(b + n) = b'$ & $*(n,a,b) \Rightarrow *(n,a',b')$).

*Pf:*

By induction via $(\sigma\backslash)$, with $\phi$ as

$$\forall a \forall b \forall a' \forall b' \text{ (N}a' \text{ \& } \sigma a,a' \text{ \& } (b + n) = b' \text{ \& } *(n,a,b) \Rightarrow *(n,a',b')).$$

Let N$a'$ & $\sigma a,a'$ & $(b + 0) = b'$ & $*(0,a,b)$. Then $b = 0$ by *Prop 8.05*, so $b' = 0$ by *Prop 7.04*. Thus $*(0,a',b')$.

Now assume N$n$ & $(\sigma\backslash)n,m$ & $\phi$. Let N$a'$ & $\sigma a,a'$ & $(b + m) = b'$ & $*(m,a,b)$. Then by *Prop 8.10*, $*(a,m,b)$. By *Prop 8.03*, $(a + v) = b$ & $*(a,n,v)$ for some $v$. By *Commutativity of Multiplication*, (*Prop 8.10*), $*(n,a,v)$. By *Prop 7.18*, $m = (n + 1)$ and $a' = (a + 1)$. Thus

$$\begin{aligned} b' &= (b + m) \\ &= ((a + v) + (n + 1)) \\ &= ((a + 1) + (v + n)) && \text{by } \textit{Associativity} \text{ and } \textit{Commutativity of Addition} \\ &= (a' + (v + n)) \end{aligned}$$

By the Induction Hypothesis, $*(n,a',(v + n))$. By *Commutativity of Multiplication*, $*(a',n,(v + n))$. By *Prop 8.02* and *Commutativity of Mutliplication*, $*(m,a',b')$. ▯

*Prop 8.12* (MULT4). $\forall n \forall a \forall a' \forall b'$ (N$a$ & $\sigma a,a'$ & $*(n,a',b') \Rightarrow \exists b$ $((n + b) = b'$ & $*(n,a,b))$).

*Pf:*

By induction via $(\sigma\backslash)$, with $\phi$ as

$$\forall a \forall a' \forall b' \text{ ( N}a \text{ \& } \sigma a,a' \text{ \& } *(n,a',b') \Rightarrow \exists b \ ((n + b) = b' \text{ \& } *(n,a,b)) \text{ ).}$$ ▯

*Prop 8.13*. $\forall n \forall a \forall b \forall a' \forall b'$ ($\sigma a,a'$ & $*(n,a,b)$ & $*(n,a',b') \Rightarrow (n + b) = b'$).

*Pf:*

By induction, with $\phi$ as

$$\forall a \forall b \forall a' \forall b' \ (\sigma a,a' \text{ \& } *(n,a,b) \text{ \& } *(n,a',b') \Rightarrow (n + b) = b').$$

Let $\sigma a,a'$ & $*(0,a,b)$ & $*(0,a',b')$. Then $b = 0$ and $b' = 0$ by *Prop 8.05*. By *Prop 7.06*, $(0 + b) = b'$.

Now assume N$n$ & $\sigma n,m$ & $\phi$. Let $\sigma a,a'$ & $*(m,a,b)$ & $*(m,a',b')$. By *Commutativity* (*Prop 8.10*), $*(a,m,b)$ & $*(a',m,b')$. By *Prop 8.12*, $(v + a) = b$ & $*(a,n,v)$ & $(v' + a') = b'$ & $*(a',n,v')$, for some $v$ and $v'$. By *Commutativity* and the Induction Hypothesis, $(n + v) = v'$. By *Prop 7.18*, $a' = (a + 1)$. So $((n + v) + (a + 1)) = b'$. By *Associativity* and *Commutativity of Addition* (*Props 7.09* and 7.*16*), $((n + 1) + (v + a)) = b'$. By *Prop 7.18* again, $(n + 1) = m$. Thus $(m + b) = b'$. ▯

*Prop 8.14*. Let $*(a,n,b)$ & $*(a,n,c)$. Then $b = c$.
*Pf:*

By induction via (σ\), with $\phi$ as

$$\forall a \forall b \forall c\ (*(a,n,b)\ \&\ *(a,n,c) \Rightarrow b = c).$$

Suppose $*(a,0,b)$ & $*(a,0,c)$. Then by *Prop* 8.*04*, $b = 0$ and $c = 0$. So $b = c$.
Now assume N$n$ & (σ\)$n,m$ & $\phi$. And suppose $*(a,m,b)$ & $*(a,m,c)$. Then there exists $v,u$ s.t. $(a + v) = b$ & $*(a,n,v)$ and $(a + u) = c$ & & $*(a,n,u)$, by *Prop 8.03*. By the induction hypothesis $u = v$. Thus $b = c$. ∎

Henceforth we will write $(a*b)$ for $c$ when $*(a,b,c)$, which is guaranteed to be well-defined by the previous proposition. If $(a*b)$ appears in an atomic proposition, then it will be assumed that $*(a,b,c)$ for some $c$.

*Prop 8.15*. Suppose $\exists j$ (N$j$ & σ$0,j$), i.e. 1 exists. Then $\forall n$(N$n \Rightarrow (1*n) = n$)
*Remark:* The proposition holds even when $1 = 0$.
*Pf:*

Let N$n$. Then

| | |
|---|---|
| $(1*n) = ((0*n) + n)$ | by *Props 8.11* and *8.10* |
| $= (0 + n)$ | by *Prop 8.06* |
| $= n$ | by *Prop 7.06* |

∎

*Prop 8.16*. *(Distributive Laws)*

($a$) $\forall n \forall a \forall b \forall c\ (\ (n*(a + b)) = c \Rightarrow ((n*a) + (n*b)) = c\ ).$

($b$) $\forall n \forall a \forall b \forall c\ (\ ((n*a) + (n*b)) = c\ \&\ \neg\ n = 0 \Rightarrow (n*(a + b)) = c\ ).$

($c$) $\forall n \forall a \forall b \forall c \forall d\ (\ ((n*a) + (n*b)) = c\ \&\ (a + b) = d \Rightarrow (n*(a + b)) = c\ ).$

*Remark:* $\forall n \forall a \forall b \forall c\ (\ ((n*a) + (n*b)) = c \Rightarrow (n*(a + b)) = c)$ does not hold in general. For in the case $n = 0$, $((n*a) + (n*b))$ may exist but not $(a + b)$.

*Pf:*

($a$) By induction, with $\phi$ as

$$\forall a \forall b \forall c\ (\ (n*(a + b)) = c \Rightarrow ((n*a) + (n*b)) = c\ ).$$

Let $(0*(a + b)) = c$. Then $c = 0$ by *Prop 8.05* & $(0*a) = 0$ & $(0*b) = 0$ by *Prop 8.06*, so $((0*a) + (0*b)) = c$ by *Prop 7.06*.
Now assume N$n$ & σ$n,m$ & $\phi$. And suppose $(m*(a + b)) = c$.

| | |
|---|---|
| $c = ((n*(a + b)) + (a + b))$ | by *Props 8.11* and *8.10* |
| $= (((n*a) + (n*b)) + (a + b))$ | by the Induction Hypothesis |

| | |
|---|---|
| $= (((n*a) + a) + ((n*b) + b))$ | by *Associativity and Commutativity of Addition* |
| $= ((m*a) + (m*b))$ | by *Props 8.11* and *8.10* |

(*b*) By induction, with $\phi$ as

$$\forall a\forall b\forall c\ (\ ((n*a) + (n*b)) = c\ \&\ \neg\ n = 0 \Rightarrow (n*(a + b)) = c\ ).$$

$\phi$ holds vacuously when $n = 0$.
Now assume N$n$ & $\sigma n,m$ & $\phi$. And let $((m*a) + (m*b)) = c$ & $\neg\ m = 0$. Suppose $n = 0$. Then $m = 1$. By *Prop 8.15* $c = (a + b) = (m*(a + b))$.
Otherwise suppose $\neg\ n = 0$.

| | |
|---|---|
| $c = (((n*a) + a) + ((n*b) + b))$ | by *Props 8.11* and *8.10* |
| $= (((n*a) + (n*b)) + (a + b))$ | by *Associativity and Commutativity of Addition* |
| $= ((n*(a + b) + (a + b))$ | by the Induction Hypothesis |
| $= (m*(a + b))$ | by by *Props 8.11* and *8.10* |

(*c*) As (*b*). ▯

*Prop 8.17*. (*Associative Laws of Multiplication*).

(*a*) $\forall n\forall a\forall b\forall c\ (\ ((n*a)*b) = c\ \&\ \neg\ n = 0 \Rightarrow (n*(a*b)) = c\ ).$

(*b*) $\forall n\forall a\forall b\forall c\forall d\ (\ ((n*a)*b) = c\ \&\ (a*b) = d \Rightarrow (n*(a*b)) = c\ ).$

*Pf:*
(*a*) By induction, with $\phi$ as

$$\forall a\forall b\forall c\ (\ ((n*a)*b) = c\ \&\ \neg\ n = 0 \Rightarrow (n*(a*b)) = c).$$

$\phi$ holds vacuously when $n = 0$.
Now assume N$n$ & $\sigma n,m$ & $\phi$. And let $((m*a)*b) = c$ & $\neg\ m = 0$.
Suppose $m = 1$. Then $c = (a*b) = (1*(a*b))$ by *Prop 8.15*.
Otherwise, suppose $\neg\ m = 1$. Then $\neg\ n = 0$. So

| | |
|---|---|
| $c = (((n*a) + a)*b)$ | by *Prop 8.12* |
| $= (((n*a)*b) + (a*b))$ | by *Prop 8.16*(*a*) and *Commutativity of Multiplication* |
| $= ((n*(a*b)) + (a*b))$ | by the Induction Hypothesis |
| $= ((a*b)*m)$ | by *Prop 8.11* and *Commutativity of Multiplication* |

(*b*) As (*a*). ▯

*Prop 8.18*. (*Division Algorithm - Existence*) Let N$n$ & $k > 0$. Then there exists $q$, $r$ such that $n = ((q * k) + r)$ & $r < k$.
*Pf:*
Let $S$ be $\{\ s : \exists t\ (\ n = ((t * k) + s))\ \}$. $Sn$, since $n = ((0 * k) + n)$. Hence $S$ has a minimum element $r$, by *Theorem 6.04*. So $n = ((q * k) + r)$, for some $q$.
Suppose $\neg\ r < k$. By *Prop 2.11*\, $k \leq r$. By *Prop 7.30*, $(k\ (+\backslash)\ c) = r$ for some $c$. By

*Prop 1.02*, 1 exists. By *Prop 8.15*, (1 * *k*) = *k*. Hence

$$n = ((q * k) + (1 * k)) + c \qquad \text{using the } \textit{Associative Law of Addition}$$
$$= ((q + 1) * k) + c \qquad \text{by } \textit{Prop 8.16(b)}$$

Hence *Sc*. $c \leq r$ by *Prop 7.30*. Since *r* is the minimum, $c = r$. But then $k = 0$ by *Prop 7.24*\, a contradiction. ∎

*Def 8.19*. (*-TOT) Suppose $\forall n \forall m$ (N*n* & N*m* $\Rightarrow \exists k$ ($n * m$) = *k*). Then we say * is *total*. ∎

In order to characterize totality of multiplication, it is helpful to consider a new trivial case, where the only natural numbers are 0 and 1:

*Def 8.20*. Let (TRV*) be the condition:

$$\forall n \text{ ( N}n \Rightarrow n = 0 \vee \sigma 0,n \text{ ).}$$ ∎

*Prop 8.21*. (TRV*) if and only if one of these cases:

(*i*) 0
(*ii*) 0 - 0
(*iii*) 0 - 1
(*iv*) 0 - 1 - 0
(*v*) 0 - 1 - 1 ∎

*Prop 8.22*. ¬ (TRV*) ⇔ $\exists y$ (N*y* & σ1,*y* & ¬ *y* = 0 & ¬ *y* = 1).
*Pf:*

Suppose ¬ (TRV*). Then there exists *x* such that N*x* & ¬ *x* = 0 & ¬ σ0,*x*. By *Prop 1.04*, N*p* & σ*p*,*x* for some *p*. By *Prop 1.05*, 1 exists. By *Prop 2.14* , $1 \leq x$, so indeed $1 < x$. By *Prop 6.02*(*i*) σ1,*y*, for some *y*. *Prop 1.02* implies that ¬ *y* = 0. If *y* = 1, then an easy induction implies that N ≡ {0,1}, a contradiction. Hence ¬ *y* = 1.

The other direction is obvious, using (PA3). ∎

*Prop 8.23*. (PA2) ∨ (TRV*) ⇔ (*-TOT).
*Pf:*

If (PA2), then (+-TOT) by *Prop 7.22*, so a simple induction show that (*-TOT).

If (TRV*), then either N ≡ {0} or N ≡ {0,1}, and in both cases (*-TOT).

Now suppose (*-TOT) & ¬ $\forall n$ ( N*n* ⇒ *n* = 0 ∨ σ0,*n* ). By *Prop 8.22* N*y* & σ1,*y*, for some *y*. We want to show (PA2). So let N*n*. It has already been shown that 0 and 1 have a successor, so it may be supposed that ¬ *n* = 0 and ¬ *n* = 1. By *Prop 2.14*, $1 < n$. ($n * y$) exists by (*-TOT). By *Prop 7.18*, $y = 1 + 1$, so

$$(n * y) = (n * (1 + 1))$$

$= (n * 1 + n * 1)$ by *Prop 8.16*(*a*)
$= (n + n)$ by *Prop 8.15*

So $(n + n)$ exists. By *Prop 7.34*, $(n + 1)$ exists. By *Prop 7.18* $n$ has a successor. □

*Corollary 8.24*. Suppose ¬ (PA1) ∨ ¬ (PA4) ∨ ¬ (PA5). Then (*-TOT). □

As with addition, σ\ induces a multiplication, which we will symbolize by (*\). An easy induction shows:

*Prop 8.25*. *\ ⊆ * □

Now it can be seen that **FPA** proves the assertion (originally proved by Lagrange) that "Every natural number can be expressed as the sum of four squares." Thus, **GA** proves:

*Prop 8.26*. Let N*n*. Then ∃*a*∃*b*∃*c*∃*d* such that

$n = (a\ (*\backslash)\ a)\ (+\backslash)\ (b\ (*\backslash)\ b)\ (+\backslash)\ (c\ (*\backslash)\ c)\ (+\backslash)\ (d\ (*\backslash)\ d)$ □

But the previous proposition immediately implies the following:

*Prop 8.27*. (*Lagrange*) Let N*n*. Then ∃*a*,*b*,*c*,*d* such that

$n = (a * a) + (b * b) + (c * c) + (d * d)$

*Pf:*
Use *Props 8.26*, *7.25*, and *8.25*. □

We willl state without proof:

*Prop 8.28*. *\ ≡ * ⇔ σ\ ≡ σ ∨ (TRV*). □

*Prop 8.29*. ∀*x*∀*y* ( N*x* & N*y* & ¬ $x + 0$ & ¬ $y = 0$ & $(x\ (*\backslash)\ y)$ exists ⇒ $x \leq (x\ (*\backslash)\ y)$ ) □

## 9. Division

*Def 9.01*. If $\exists b\ (a * b) = c$, then we write $a \mid c$ and say that *a divides c*. □

*Prop 9.02*.

(*a*) $\forall a\ (\mathrm{N}a \Rightarrow a \mid 0)$

(*b*) $\forall a\ (\ 0 \mid a \Rightarrow a = 0\ )$

(*c*) $\forall a\ (\ \mathrm{N}a\ \&\ 1$ exists $\Rightarrow 1 \mid a\ )$

(*d*) $\forall a\ (\ \mathrm{N}a \Rightarrow a \mid a\ )$

(*e*) (*Transitivity*) $\forall a \forall b \forall c\ (\ a \mid b\ \&\ b \mid c \Rightarrow a \mid c\ )$

(*f*) $\forall a \forall b \forall c\ (\ a \mid b\ \&\ a \mid c\ \&\ (b + c)$ exists $\Rightarrow a \mid (b + c)\ )$

(*g*) $\forall a \forall b \forall c\ (\ a \mid b\ \&\ (b * c)$ exists $\Rightarrow a \mid (b * c)\ )$

(*h*) $\forall a \forall b \forall c\ (\ a \mid b\ \&\ (b * c)$ exists $\Rightarrow (a * c) \mid (b * c)\ )$

*Pf:*

(*a*) Let N$a$. Then $(a * 0) = 0$ by *Prop 8.07*.

(*b*) Let $0 \mid a$. Then $(0 * b) = a$ for some $b$. So $a = 0$ by *Prop 8.05*.

(*c*) Let N$a$ and suppose 1 exists. Then $(1 * a) = a$ by *Prop 8.15*. So $1 \mid a$.

(*d*) Let N$a$. If 1 exists, then $(a * 1) = a$ by *Prop 8.15*, so $a \mid a$. If 1 does not exist, then $a = 0$ by *Prop 1.02*, and so $a \mid a$ by (*a*).

(*e*) Suppose $a \mid b\ \&\ b \mid c$. If $a = 0$, then $b = c = 0$ by (*b*), and thus $0 \mid c$ by (*a*).
Otherwise $\neg\, a = 0$. $(a * x) = b$ and $(b * y) = c$, for some $x,y$. So $((a * x) * y) = c$. By the *Associative Law of Multiplication* (*Prop 8.17*(*a*)), $(a * (x * y)) = c$, hence $a \mid c$.

(*f*) Let $a \mid b\ \&\ a \mid c$, and suppose $(b + c)$ exists. If $a = 0$, then $b = c = 0$ by (*b*), so $(b + c) = 0$ by *Prop 7.06*, and thus $0 \mid (b + c)$ by (*a*).
Otherwise, suppose $\neg\, a = 0$. $(a * x) = b$ and $(a * y) = c$, for some $x,y$. So $((a * x) + (a * y)) = (b + c)$, since the latter exists. By the *Distributive Law* (*Prop 8.16*(*b*)), $(a * (x + y)) = (b + c)$, so $a \mid (b + c)$.

(*g*) Let $a \mid b$ and suppose $(b * c)$ exists. Obviously, $b \mid (b * c)$, so apply (*e*).

(*h*) Apply (*e*) and (*g*). □

*Prop 9.03*. Suppose $\neg$ (PA5). Let $a \mid b$ & $a \mid (b + c)$. Then $a \mid c$.
*Pf:*

If $a = 0$, then $b = 0$ and $(b + c) = 0$ by *Prop 9.02*(*b*), so $c = 0$ by *Prop 7.04*, and so $a \mid c$ by *Prop 9.02*(*a*).

Otherwise, suppose $\neg\, a = 0$.

$a * u = b$ & $a * v = (b + c)$, for some $u$,$v$. By *Prop 7.33*, there exists $c$ such that $u + c = v$. Substituting and re-arranging, $a * u + a * c = a * u + c$. By *Cancellation* (*Prop 7.24*), $a * u = c$. Thus $a \mid c$. ∎

*Prop 9.04*. Suppose $\neg$ (PA5), and let *top*($t$). Then $\forall n$ (N$n \Rightarrow t \mid n$ ).
*Pf:*

$t + 1 = 0$, by *Corollary 7.19*.

Let N$n$. Then $n + v = 0$ for some $v$, by *Prop 7.20*(*b*). If $v = 0$, then $n = 0$ by *Prop 7.04*, and so $t \mid n$ by *Prop 9.02*(*a*).

Otherwise $\neg\, v = 0$.

$v * (t + 1) = v * 0 = 0$, by *Prop 8.07*.

$v * t + v = 0$, by *Props 8.16*(*a*) and *8.15*.

So $v * t = n$ by *Cancellation* (*Prop 7.24*). ∎

As usual ($\sigma\backslash$) induces a division relationship, which will be written ($\mid\backslash$). So $x$ ($\mid\backslash$) $y$ if and only if ($x$ ($*\backslash$) $z$) = $y$, for some $z$.

*Prop 9.05*. ($\mid\backslash$) $\subseteq \mid$
*Pf:*

Use *Prop 8.25*. ∎

In **PA** the smallest multiple of a number is itself. This need not be the case in **GA**.

*Def 9.06*. Let N$n$. The *least multiple* of $n$, written *lm(n)*, is:

0 when $n = 0$, and
the least number $a > 0$ such that $n \mid a$ when $n > 0$. ∎

*Prop 9.07*.

(*a*) $\forall a$ ( N$a \Rightarrow a \mid lm(a)$ )

(*b*) $\forall a$ ( N$a$ & $\neg\, a = 0 \Rightarrow 0 < lm(a)$ )

(*c*) $\forall a$ ( N$a \Rightarrow lm(a) \leq a$ )

(*d*) 1 exists $\Rightarrow lm(1) = 1$

(*e*) $a \mid 1 \Leftrightarrow lm(a) = 1$

(*f*) If $lm(a) = 1$, then $\forall n\ (\mathrm{N}n \Rightarrow a \mid n)$ $\square$

*Def 9.08*. Let $\mathrm{N}n \ \&\ \neg\, n = 0$. Then $n$ is a *zero divisor*, written $zd(n)$, if there exists an $x$ such that $\neg\, x = 0$ and $(n * x) = 0$. $\square$

Zero divisors only occur in the case $\neg$ (PA5):

*Prop 9.09*. If there exists a zero divisor, then $\neg$ (PA5).
*Pf:*
Suppose $(x * y) = 0 \ \&\ \neg\, x = 0 \ \&\ \neg\, y = 0$. By *Prop 1.04* $\mathrm{N}u \ \&\ \sigma u,y \ \&\ \mathrm{N}v \ \&\ \sigma v,x$ for some $u,v$. *Prop 7.18*, $(u + 1) = y$ and $(v + 1) = x$. By *Distribution* (*Prop 8.16*(*i*)) and *Prop 8.15*, $(x * u) + x = 0$. Substituting and rearranging, $((x * u) + v) + 1 = 0$. By *Prop 7.18*, $\mathrm{N}((x * u) + v) \ \&\ \sigma((x * u) + v)$. Thus $\neg$ (PA5). $\square$

Remark the converse does not hold, as there may be no zero divisors even when $\neg$ (PA5). For instance, there are no zero divisors in the case 0 - 1 - 2 - 0.

*Prop 9.10*. Suppose $n \mid 1$. Then $\neg\, zd(n)$.
*Pf:*
By assumption $n * x = 1$ for some $x$.
Suppose $n * y = 0$, where $\neg\, n = 0 \ \&\ \neg\, y = 0$. $n * x * y = y$, by *Prop 8.15*, so re-arranging (*Commutativity and Associativity of Multiplication*), $(n * y) * x = y$. By *Prop 8.06*, $0 = y$, a contradiction. Hence $n$ is not a zero divisor. $\square$

*Corollary 9.11*. $\neg\, zd(1)$. $\square$

*Prop 9.12*. Suppose $\neg$ (PA5), and let $\mathrm{N}n \ \&\ \neg\, n = 0 \ \&\ \neg\, n = 1$. Then $lm(n) = 1 \vee zd(n)$.
*Pf:*
Suppose $x * n = y * n \ \&\ \neg\, x = y$, for some $x,y$. By *Prop 7.33*, $(x + c) = y$, for some $c$. By *Prop 7.04* $\neg\, c = 0$. By *Distribution* (*Prop 8.16*(*i*)) and *Cancellation* (*Prop 7.24*), $c * n = 0$. Thus $zd(n)$.
Otherwise each $x * n$ must be unique, so by *Prop 6.09*, there exists some $x$ such that $x * n = 1$. Then $lm(n) = 1$. $\square$

## 10. Prime Numbers

*Def 10.01*. Let *p* > 1. *p* is *prime* if and only if ∀*n* (*n* | *p* ⇒ *n* = *p* ∨ *n* = 1 ). □

Remark that (σ\), as usual, induces a notion of primality, which will be called \-prime; that is, *p* is \-prime if and only if ∀*n* (*n* (|\) *p* ⇒ *n* = *p* ∨ *n* = 1 ). Obviously:

*Prop 10.02*. Suppose *p* is prime. Then *p* is \-prime.
*Pf:*
Use *Prop 9.05*. □

We will show that if ¬ (PA5), then there are no primes, except for a few special cases.

*Prop 10.03*. Suppose ¬ (PA5), and suppose *p* is \-prime. Then *lm*(*p*) = *p* ∨ *lm*(*p*) = 1.
*Pf:*
By the *Division Algorithm* (*Prop 8.18*\), there exist *q*,*r* such that

*p* = *lm*(*p*) (*\) *q* (+\) *r* & *r* < *lm*(*p*).

By *Props* 7.25 and *8.25*, *p* = *lm*(*p*) * *q* + *r*. Now *p* | *lm*(*p*) by *Prop 9.07*(*a*), so *p* | *r* by *Prop 9.03*. But *r* < *lm*(*p*), so this forces *r* = 0. Since *p* is \-prime, *lm*(*p*) = *p* ∨ *lm*(*p*) = 1. □

*Prop 10.04*. Suppose ¬ (PA5) and let *p* be prime. Then *top*(*p*).
*Pf:*
Use *Prop 9.04*. □

So, if ¬ (PA5), there can be at most one prime, and this must be the top. The question now becomes when the top can be prime.

*Prop 10.05*. Suppose ¬ (PA5).

(*a*) In the case 0 - 1 - 2 - 0, there is a unique prime, 2.

(*b*) In the case 0 - 1 - 2 - 3 - 0, there is a unique prime, 3.

(*c*) In the case 0 - 1 - 2 - 3 - 4 - 5 - 0, there is a unique prime, 5.

In any other case there are no primes.

*Pf:*
Manual calculation of the multiplications tables in the three cases reveals that 2, 3, and 5 are primes and uniquely so. (Alternatively, *Prop 9.04* proves uniqueness as well.)
*Prop 10.04* says that only a top can be a prime. By *Prop 10.02*, any prime must be a \-prime. This means, in the case of 0 - 1 - 2 -3 - 4 - 0, there is no prime.

So let *top*(*t*) and suppose *t* is a prime. It may be assumed that $t \geq 6$.

By *Prop 9.12* and *9.07*(*f*), it suffices to show that there is one number between (exclusive) 1 and *t* which is not a zero divisor.

By Bertrand's Postulate (which can be proven in **FPA**, see [Boucher] *Proving Bertrand's Postulate*), there exists a \-prime *p* such that $s < p < t$ , where 2 (*\) _s_ (+\) 1 = _t_. We are done unless *p* is a zero divisor, so assume it is. We claim that 2 * _p_ = 0.

Now _p_ (+\) _x_ = _t_ for some *x* by *Prop 7.30*, where ¬ _x_ = 0 by *Prop 7.04*\. _p_ ≥ _s_ (+\) 1 by *Prop 6.02*(*h*)\ and *Prop 7.18*\, so _s_ (+\) 1 (+\) _c_ = _p_, for some *c*, by *Prop 7.30*. Equating two expressions of *t* (and allowing re-arrangements using *Associativity* or *Commutativity*)

2 (*\) _s_ (+\) 1 = _p_ (+\) _x_<br>
_s_ (+\) _s_ (+\) 1 = _s_ (+\) 1 (+\) _c_ (+\) _x_<br>
_s_ (+\) 1 = _c_ (+\) _x_ (+\) 1 by *Cancellation* (*Prop 7.24*\)

Thus _s_ (+\) 1 ≥ _x_ (+\) 1 by *Prop 7.30*, so _p_ ≥ _x_ (+\) 1 by *Prop 6.02*(*d*). By *Prop 7.30* again, _p_ = _x_ (+\) 1 (+\) _e_, for some *e*. $e < p$ by *Prop 7.24*\ and *Prop 7.20*(*a*)\. But then

$$
\begin{aligned}
2 * p &= p + p \\
&= p + x + 1 + e && \text{by } \textit{Prop 7.25} \\
&= t + 1 + e && \text{by } \textit{Prop 7.25} \\
&= 0 + e && \text{by } \textit{Corollary 7.19} \\
&= e && \text{by } \textit{Prop 7.06}
\end{aligned}
$$

If $e > 0$, then $lm(p) \leq e < p$, which by *Prop 10.03* forces $lm(p) = 1$, in which case *p* is not a zero divisor by *Prop 9.10*, a contradiction. Hence $e = 0$.

Now $t \geq 6$, so $s \geq 3$, so $p > 3$. Thus ¬ _p_ = 3. And so ¬ 3 (|\) _p_ since *p* is \-prime. By the *Division Algorithm* (*Prop 8.18*\), _p_ = 3 (*\) _q_ (+\) _r_, where _r_ = 1 or 2. So $p = 3 * q + r$, and

$$
\begin{aligned}
0 &= p + p \\
&= 3 * q + 3 * q + 2 * r
\end{aligned}
$$

So by *Prop 9.03*, 3 | (2 * *r*). So 3 | 2 or 3 | 4. But 3 | 4 implies 3 | 1 by *Prop 9.03* again, so in both case $lm(3) < 3$, which forces $lm(3) = 1$. But then 3 | *t* by *Prop 9.07*(*f*), and *t* is not prime.

□

Note in passing that a similar analysis shows that, when ¬ (PA4), i.e. the tadpole models, the prime numbers occur only in the tail (and are the usual primes), except for a few special cases.

## 11. Gaussian Arithmetic

Consider a final addition to our hierarchy of arithmetic theories. Define **GSA** (*Gaussian Arithmetic*) to be the sub-theory of **PA** consisting of **GA** + (PA4). So:

**SLA** ⊆ **IND** ⊆ **GA** ⊆ **GSA** ⊆ **FPA** ⊆ **PA**,

where

**IND** = **SLA** + (PA6)<br>
**GA** = **IND** + (PA3)

**GSA** = **GA** + (PA4)
**FPA** = **GSA** + (PA5) = **PA** \ {(PA1),(PA2)}.

In this section we will be working exclusively in **GSA**.

By assuming (PA4) the tadpoles have been excluded, so the possible models are: the vacuous; the standard; the finite segments; and the cycles.

**GSA** can be broken down into two mutually exclusive cases: (PA5), in which case the theory is just **FPA**; and ¬ (PA5), in which case any proposition proved in **GA** by supposing ¬ (PA5) holds. As we shall see, a reasoning by cases will allow us to tackle certain well-known arithmetic assertions.

As *Prop 7.24* says, the addition of (PA4) gives us the cancellation law for addition, and thus a well-defined subtraction.

*Prop 9.03* already proved part of the next proposition, but a self-contained proof is still provided here, for the sake of completeness:

(**GSA**) *Prop 11.01.* $\forall a \forall b \forall c\ ( a \mid b \ \&\ a \mid (b + c) \Rightarrow a \mid c\ )$

*Remark*: The proposition does not hold in **GA**. For instance, consider 0 - 1 - 2 - 3 - 2. Then (3 + 1) = 2, so 2 | 3 and 2 | (3 + 1), but ¬ 2 | 1.

*Pf:*

Suppose $a \mid b$ & $a \mid (b + c)$. Then $(a * x) = b$ and $(a * y) = (b + c)$, for some $x,y$. So $(a * y) = ((a * x) + c)$.

By *Prop 2.11*\, $x \le y$ or $y \le x$.

Suppose $x \le y$. Then by *Corollary 7.31*, $x + z = y$ for some $z$. Thus

| | |
|---|---|
| $(a * (x + z)) = ((a * x) + c)$ | |
| $((a * x) + (a * z)) = ((a * x) + c)$ | by *Distribution* (*Prop 8.16*(*a*)) |
| $(a * z) = c$ | by *Cancellation* (*Prop 7.24*) |

So $a \mid c$.

Now suppose $y \le x$. Then by *Corollary 7.31*, $y + u = x$, for some $u$. So

| | |
|---|---|
| $(a * y) = ((a * (y + u)) + c)$ | |
| $(a * y) = ((a * y) + (a * u)) + c)$ | by *Distribution* (*Prop 8.16*(*a*)) |
| $0 = (a * u) + c$ | by *Cancellation* (*Prop 7.24*). |

Suppose (PA5). Then by *Prop 7.20*(*a*) $c = 0$, so $a \mid c$ by *Prop 9.02*(*a*). On the other hand, suppose ¬ (PA5). Then $(u + d) = 0$ for some $d$. So $a * d = ((a * (u + d)) + c) = c$, and $a \mid d$. ∎

(**GSA**) *Prop 11.02*. Let N$a$. Then $lm(a) \mid a$.

*Remark:* The proposition does not hold in **GA**. For instance, consider

0 -1 - 2 - 3 - 4 - 5 - 6 - 7- 4.

Then $lm(6) = 4$, but ¬ 4 | 6.

*Pf:*

If $a = 0$, then the result holds since $0 \mid 0$ by *Prop 9.02*(*a*).

Otherwise, suppose $\neg\, a = 0$. Then $\neg\, lm(a) = 0$, so by the *Division Algorithm* (*Prop 8.18*), there exist $q,r$ such that $a = (q * lm(a) + r)$ & $r < lm(a)$. By *Prop 11.01*, $a \mid r$, which forces $r = 0$ given the leastness of $lm(a)$. Thus $lm(a) \mid a$. ∎

(**GSA**) *Corollary 11.03*.

(*a*) $\forall a \forall b$ ( $a \mid b \Leftrightarrow lm(a) \mid b$ ).

(*b*) $\forall a \forall b$ ( $a \mid b \Leftrightarrow a \mid lm(b)$ ).

(*c*) $\forall a \forall b$ ( $\mathrm{N}a \Rightarrow lm(lm(a)) = lm(a)$ ). ∎

(**GSA**) *Corollary 11.04*. Let N*a* & N*b*. Then $(a * x) = b$ has a solution if and only if $lm(a) \mid b$. ∎

*Prop 9.13* shows that the previous corollary is equivalent in **FPA** to the assertion:

Let N*a* & N*b*. Then $(a * x) = b$ has a solution if and only if $a \mid b$.

In the case of ¬ (PA5), the corollary is effectively equivalent to the assertion that $(a * x) \equiv b$ $(mod\ n)$ if and only if $gcd(a,n) \mid b$.

(**GSA**) *Prop 11.05*. Let $lm(x) \mid y$. Then $lm(x)$ (|\) $y$.

*Pf:*

If $lm(x) = 0$, then $y = 0$, and so clearly $lm(x)$ (|\) $y$.

Otherwise, suppose $\neg\, lm(x) = 0$.

By the *Division Algorithm* (*Prop 8.18*\), there exist $q,r$ such that $y = q$ (*\) $lm(x)$ (+\) $r$ and $r < lm(x)$. By *Props 7.25* and *8.25*, $y = q * lm(x) + r$. But $lm(x) \mid lm(y)$ by *Transitivity* (*Prop 9.02*(*e*)), so by *Prop 11.01*, $lm(x) \mid r$. By *Transitivity* again, $x \mid r$. By the assumption of the leastness of $lm(x)$, $r = 0$. Hence $lm(x)$ (|\) $y$. ∎

(**GSA**) *Corollary 11.06*. $\forall x \forall y$ ( $lm(x) \mid y \Leftrightarrow lm(x)$ (|\) $y$ ) ∎

It has been shown in *Arithmetic without the Successor Axiom* that the *Law of Quadratic Reciprocity* can be proved in **FPA**. Recall that this Law says, "Suppose *p* and *q* are distinct odd primes. Then..." But *Prop 10.05* shows that, in **GA** + ¬ (PA5), there are no distinct primes, much less distinct odd primes. Hence, the *Law of Quadratic Reciprocity* can be proved in **GSA**, with one proof beginning ,"Let *p* and *q* are distinct odd primes. By *Prop 10.05*, (PA5)." The rest of the proof could then be the one furnished in *Arithmetic*. This still leaves open the question, of course, whether a more natural proof exists.

**FPA** proves the following version of *Fermat's Little Theorem*:

Let $a$ be a natural number, $p$ be a prime, where $\neg\, p \mid a$, and suppose $a^{p-1}$ exists. Then $p \mid (a^{p-1} - 1)$.

In the special cases mentioned in *Prop 10.05*, the unique prime *p* divides *every* number, by *Prop 9.04*. So, **GA** + ¬ (PA5) proves *Fermat's Little Theorem*. Since **FPA** does as well, **GSA** proves it.

It is perhaps instructive to consider *Fermat's Little Theorem* in the context of **GA**. First, unlike with **GSA**, subtraction is not well-defined in **GA**, and a natural number may have two predecessors. So *Fermat's Little Theorem* would need to be modified in some way from the form we have just given it. Its most natural variant might be:

Let $a$ be a natural number, $p$ be a prime, where $\neg\, p \mid a$, and suppose $Nn$ & $\sigma n,p$. Further suppose $Nx$ & $\sigma x,a^n$. Then $p \mid x$.

However, this assertion is not true in **GA**. Consider for example the tadpole 0 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - ... - 14 - 15 - 6. Set $p = 3$, $a = 4$, $n = 2$, and $x = 5$. Although it has not been defined, $a^n = 4^2$ should be equal to 4 * 4 = 6, so we have σ2,3 and σ5,6. But ¬ 3 | 5. There are weaker assertions than the natural variant which can be proved in **GA**, but in the author's opinion it would be an abuse to call any of them *Fermat's Little Theorem*.

## 12. Henkin's "On Mathematical Induction"

There is much in the present paper already covered in "On Mathematical Induction" [Henkin], so it is worthwhile to write a brief survey of that paper here. Henkin considers a base system which assumes induction, successoring's functionality, and successoring's totality, that is (PA2) + (PA3) + (PA6). This, of course, is different from **GA** in the inclusion of (PA2), which, in the hierarchy we have considered, is the last axiom added. The results of Henkin, however, do not depend essentially on (PA2) - that is, they can be reframed so that the axiom need not be assumed and Henkin's proofs go through - so this difference is less important than it might seem at first.

Henkin proves that addition and multiplication can be defined in his base system, and he characterizes the pure cycle and tadpole models in terms of congruences. He knew that the Commutative, Associative, and Distributive Laws hold, but he does not say anything about propositions further along, and in particular it does not seem he considered whether more advanced number-theoretic results can be demonstrated, or at the least there is no mention of such. The development in his paper is more abstract and phrased model-theoretically, whereas the development here is more in the classic mathematical tradition of definition and proof.

## 13. Conclusion

The surface has only been scratched in terms of an understanding of what can and what cannot be demonstrated in General or Gaussian Arithmetic. It would be an interesting exercise to classify all the theorems, say which appear in Hardy and Wright, according to the hierarchy presented here. It would be especially interesting to get natural proofs. Further investigation is needed, and further investigation will yield new insights into the structure of the natural numbers.

**BIBLIOGRAPHY**

Boolos, George. *Frege's Theorem and the Peano Postulates* in ***Logic, Logic, and Logic***. Harvard University Press, 1998.

Boucher, Andrew. *Arithmetic without the Successor Axiom*, 2006. On web site, www.andrewboucher.com/papers.

Boucher, Andrew. *Proving Bertrand's Postulate*, 2007. On web site, www.andrewboucher.com/papers.

Hardy, G.H. & Wright, E.M. ***An Introduction to the Theory of Numbers***. Clarendon Press, 1938.

Henkin, Leon. "On Mathematical Induction," *The American Mathematical Monthly*, Vol. 67, No. 4 (Apr. 1960), pp. 323-338.